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The Quest for Better Attitudes[1]

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For Paul Davenport, who showed the way

Prologue

This article is adapted from my Brouwer lecture of February 2001 [1]. It has been shortened by about one quarter and updated. Given the occasion (in 2001) I had thought it inappropriate to give a talk in my usual style, which is rich in equations and mathematical derivation. Instead, I attempted to give a talk rich in perspective and personal anecdotes (and maybe a few equations). Since the original lecture, my appreciation of the subject matter has grown and some addenda and corrigenda have become necessary.

Since the word *quest* appears in the title, it might be assumed that I will devote part of this article to the QUEST algorithm [2, 3], certainly my best known work. What is not known generally, however, is that the QUEST algorithm was the subject of my very first task in spacecraft attitude determination and that the work was accomplished almost entirely in my very first year in Engineering. QUEST was not, as many believe, created by a "renowned expert" on spacecraft attitude applying his considerable knowledge and experience. Rather, it was the lucky creation of a newcomer who had no training and no experience in spacecraft attitude determination or in any part of Astronautics, someone who simply stumbled along obstinately until he reached his goal. That this unlikely creation would become one of the most widely-used spacecraft attitude determination algorithms in the world today has surprised no one more than its creator.

It is hard from these remarks to escape the conclusion (not necessarily a happy one for me) that my Engineering career must have peaked very early, in its first year, in fact, and then for the next 28 years has been in constant decline. You might

[1]This article is dedicated also to the memory of Roger D. Werking (1941–2006), who was a strong force behind the development of QUEST.

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expect, therefore, that the QUEST algorithm is the last thing I would want to talk about. So, of course, the QUEST algorithm and how I came to develop it will occupy almost all of this article. But I do not wish to spend an entire article deriving QUEST. Instead, I wish to talk about the circumstances of QUEST's birth, its adolescence, and its adult life. I wish also to talk a little about QUEST's recent competitors. And I would like to talk about how my early work on QUEST has influenced so much of what I have done in the past quarter century. Mostly, however, I just want to tell you the story of how QUEST came to be. If I am still known best for my first year's work in Spacecraft Attitude Determination, it is because that work has been remarkably fruitful.

Dramatis Personae

To appreciate the development of the QUEST algorithm fully, one must know something of the development of its creator at the time. My work on QUEST began only a few months after I entered the world of Engineering. This took place in May 1977, when I joined the Attitude Systems Operation of the Computer Sciences Corporation in Silver Spring, Maryland. Before then I had been a theoretical nuclear physicist. I was, in fact, a pretty good nuclear physicist. Many of my journal articles in Physics are still cited regularly, and occasionally someone even sends me a Physics Ph.D. thesis to read. My personal life as a physicist had been pretty interesting as well. For one thing I got to change my official country of residence six times. In Paris I led the life of a yuppie bohemian zipping around in my red convertible and surviving the dubious pleasure of having a knife held to my throat in the Paris Métro. In Germany I was forced to resign my university position by the Third Reich, not an easy accomplishment in 1973. In Israel I was nearly gunned down by the bodyguard of the then Minister of Defense, Shimon Peres. In Pittsburgh, my last stop as a physicist, I dated the estranged wife of a local drug lord. It's hard to imagine that anyone would want to abandon such a life for the more staid life of an engineer.[3] Nonetheless, I had many reasons in 1977 for wanting to make a career change. And so, at 4:30 p.m. on Friday, May 14, 1977, I bid farewell to my life as a nuclear physicist, and at 8:30 a.m. on Monday, May 17, 1977, I suddenly found myself employed as a rocket scientist.

In the beginning, obviously, I was a very deficient rocket scientist. The only Engineering course I had taken previously was a sophomore course on Electronic Circuits, which I failed the first time and had to repeat.[4] Even worse than that, as a theoretical nuclear physicist I had become very proficient at Quantum Mechanics, but, except for the undergraduate Physics courses that I had taught, I had had very little contact with Classical Physics. I was far less comfortable, in fact, with Rigid Body Mechanics than I was with Relativistic Quantum Field Theory, a fact that will resurface repeatedly in this work.

The Good Old Days—Computing in the Early Space Age

The young engineer today can hardly imagine what it was like to carry out computations in the late 1970s. In the 1950s a computer was most often a human being

[3]The transition to a more normal personal life has been gradual. A month and a half after I entered the world of Engineering, I managed to acquire Soviet citizenship, and part of my personal life as an engineer has been classified "confidential" by the FBI. I have also been robbed at gunpoint twice; not as chic, perhaps, as being nearly gunned down by an Israeli uzi, but it will have to do.

[4]Of the 162 students enrolled in this course only four, I among them, received a grade of "F."

with a Marchant or Frieden calculator, noisy electrically driven mechanical contrivances only one step removed from an abacas. What few electronic computers existed were exceedingly slow and unreliable. Only in 1960 did IBM, with its 1400 series, introduce computers which relied on transistors rather than vacuum tubes.[5] The IBM 360 series, which debuted in 1964, was the first to use integrated circuits. Microchips were still a long way off.

The effective clock frequency of the IBM-360 series was somewhere between 500 kHz and 1 MHz. The top-of-the-line model, the IBM-360 Model-91, had a whopping 4 MB of RAM, which was called (magnetic) core in those days. Disk drives were the size of a home washing machine and had a capacity of only about 5 MB. Tape was the frequent medium for long-term and short-term storage. Computation on such a computer system was arduous. A trivial 200-state vibration analysis which I carried out on an IBM-360 Model-91 computer in 1980 required 3.5 MB of core, three disc drives and six tape drives. In order for me to have access to that much core it was necessary to shut down all systems except the operating system while I monopolized the computer from midnight until 6:00 a.m. The same task might be accomplished in a few minutes today (2006) on a student's notebook computer boasting a clock frequency of 4 GHz, 512 MB of RAM, and a 100 GB disc drive, computer power undreamed of in a mainframe only 25 years ago.

Operating systems in those days were yet another ordeal. For its users IBM had created Operating System 360 Job Control Language, one of the minor cruelties perpetrated against humankind. There was no virtual memory allocation in the IBM-360 series, that is, the computer would not automatically swap data between core and the disc drives. Hence, the movement of data from core to disc drives or tape drives or back had to be programmed explicitly by the user in OS-360 JCL. Every array in the program had to be specified in advance. Instructions had to be sent via JCL to the system operator to mount or dismount tapes. Writing OS-360 JCL was an arcane art, as JCL was slightly more difficult to interpret than Sumerian cuneiform. And the few available computers were overworked. Long programs often waited for days before they were executed. Twenty years into the Space Age computing was a highly frustrating task.

The Good Old Days—Attitude Determination

What was attitude[6] determination like in those days? For all practical purposes there were only two methods: Batch Least-Square Estimation and the TRIAD Algorithm [3–5], which at that time was known more commonly as the Algebraic Method [4].

In the TRIAD method, invented in 1964 by Harold D. Black [5], one is given two unit vectors, the observation vectors, $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{W}}_2$, which are two directions measured in the spacecraft body frame. These correspond to two unit vectors in the inertial reference frame, the reference vectors, denoted by $\hat{\mathbf{V}}_1$ and $\hat{\mathbf{V}}_2$. Ideally, in the absence of measurement noise, these satisfy

$$\hat{\mathbf{W}}_1 = A\,\hat{\mathbf{V}}_1 \quad \text{and} \quad \hat{\mathbf{W}}_2 = A\,\hat{\mathbf{V}}_2 \tag{1}$$

[5]At MIT in 1964–65, my undergraduate thesis research required the use of a home-grown vacuum-tube-based computer, the TX-0, in order to transfer my data from paper tape to 9-track magnetic tape. Apparently, it was then the only device in the entire Institute capable of this function.

[6]By attitude, without any qualifying adjectives, I will always mean three-axis attitude.

where A is the attitude matrix, a 3×3 proper orthogonal matrix [6], for which one wishes to solve. In general, a solution will not exist, because the observation vectors are corrupted by measurement noise. But we can always force a solution by defining first the two orthonormal triads

$$\hat{\mathbf{r}}_1 = \hat{\mathbf{V}}_1, \qquad \hat{\mathbf{r}}_2 = \frac{\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2}{|\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2|}, \qquad \hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \tag{2a}$$

$$\hat{\mathbf{s}}_1 = \hat{\mathbf{W}}_1, \qquad \hat{\mathbf{s}}_2 = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|}, \qquad \hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \tag{2b}$$

and then setting

$$A = [\hat{\mathbf{s}}_1 \quad \hat{\mathbf{s}}_2 \quad \hat{\mathbf{s}}_3][\hat{\mathbf{r}}_1 \quad \hat{\mathbf{r}}_2 \quad \hat{\mathbf{r}}_3]^{\mathrm{T}} \tag{3}$$

where the brackets denote two matrices labeled by their column vectors and T denotes the matrix transpose. The matrix A is always proper orthogonal and satisfies the first of equations (1) exactly. If there is no measurement noise, the second of equations (1) will also be satisfied.

The TRIAD algorithm is of limited use because: (1) it assumes that the measurements are unit vectors, and (2) it can make use of only two unit-vector measurements. The first limitation is not very damaging in practice, because most attitude sensors do furnish a direction, usually that of the Sun, one or more stars, the magnetic field, or the nadir. The second restriction is more of a problem, since it limits the accuracy of the attitude estimates.

When one doesn't have unit-vector measurements or one has more than two of them, one needed to resort to a least-square algorithm, often called an optimal method. In this case, one writes the measurements as

$$\mathbf{z}_k = \mathbf{f}_k(A) + \mathbf{v}_k, \qquad k = 1, \ldots, N \tag{4}$$

where $\mathbf{z}_k$ is a measurement vector, $\mathbf{f}_k(\cdot)$ is some known vectorial function of the attitude, and $\mathbf{v}_k$ is the noise vector, assumed to have zero mean. The optimal attitude matrix is then taken to minimize a cost function of the form

$$J_A(A) = \frac{1}{2} \sum_{k=1}^{N} [\mathbf{z}_k - \mathbf{f}_k(A)]^{\mathrm{T}}\, W_k [\mathbf{z}_k - \mathbf{f}_k(A)] \tag{5}$$

where W_k is a weight matrix, necessarily positive semi-definite. For vector measurements, such as we use in the TRIAD algorithm,

$$\mathbf{f}_k(A) = A\hat{\mathbf{V}}_k \tag{6}$$

and $\mathbf{z}_k$ is just the observed direction $\hat{\mathbf{W}}_k$. Estimation Theory [7] tells us how to choose the weight matrices W_k as well, but that will not concern us here.

We cannot optimize the cost function directly in terms of the nine elements of A, because only three of them can be independent. Hence, we write A as a function, say, of the 3-1-3 Euler angles,

$$A = R_{313}(\varphi, \vartheta, \psi) = \begin{bmatrix} c\psi\, c\varphi - s\psi\, c\vartheta\, s\varphi & c\psi\, s\varphi - s\psi\, c\vartheta\, c\varphi & s\psi\, s\vartheta \\ -s\psi\, c\varphi - c\psi\, c\vartheta\, s\varphi & -s\psi\, s\varphi + c\psi\, c\vartheta\, c\varphi & c\psi\, s\vartheta \\ s\vartheta\, s\varphi & -s\vartheta\, c\varphi & c\vartheta \end{bmatrix} \tag{7}$$

The new cost function $J_{\varphi,\vartheta,\psi}(\varphi, \vartheta, \psi) \equiv J_A(R_{313}(\varphi, \vartheta, \psi))$ is now minimized by an iterative procedure such as the Newton-Raphson method. From the complexity of equation (7) it is obvious that $J_{\varphi,\vartheta,\psi}(\varphi, \vartheta, \psi)$ is a very ugly function and that such a minimization must be very tedious. Thus, if one could not use the TRIAD algorithm, the computation of spacecraft attitude, given the computational resources of the times, was very slow. Spacecraft Attitude Determination in the late 1970s was not for the faint-hearted.

The Gathering Storm

The two methods just discussed were adequate for ground-based spacecraft attitude determination before the late 1970s. However, a trend was developing in which mission requirements were becoming more demanding both in terms of the required attitude accuracy and in terms of the required attitude computation rate. When a mission required that attitude be computed only once per minute and with an accuracy of only one degree per axis, the current algorithms and computational resources were more than adequate. That situation was about to change.

Entr'acte

My first undistinguished efforts at the Computer Sciences Corporation weren't in attitude determination at all, but in attitude dynamics and control. When I first walked through the door at CSC in February 1977 for an interview, I knew nothing about either and had learned only a few weeks earlier that the attitude of a spacecraft did not refer to its emotional posture. Supposedly, I did know something about dynamics, because I had spent the previous half-dozen years as an assistant professor of Physics, more or less, which inspired some trust in me. That trust was exaggerated, but, as I was looking for a job, I did my best to encourage it. Fortunately, I got some lucky breaks.

I was allowed to spend additional time at CSC around my interview, so that I could try the job on for size, the consequence of having a friend, Jerry Lerner, who was a manager there. So for two days there I tried to understand some puzzling simulation results—not puzzling to me, since I had no idea what to expect—on the steady-state pitch rate during attitude acquisition of a spacecraft then under construction for NASA. By the second day, still without anything to show, I was in my usual state of panic before a deadline and cursing myself that I had been so foolish as to expose my unsuitability for the work. Then, by a stroke of luck, using a trick from Quantum Scattering Theory, I was able to arrive at an easily calculable infinite series[7] for the steady-state pitch rate, which could be computed for the entire range of control system parameters in much less time than would be required for repeated simulation of the attitude dynamics. My result agreed with the simulations (and years later with real mission data) but did not really explain what was going on any better than the simulations themselves. It did, however, increase confidence in the simulation results, which, I guess, was worth something. As a result, soon after, before I had received an offer of employment, CSC added my name to a conference paper [8] in which my expression and its derivation appeared as an appendix. My greater achievement was in fooling the company that I really knew something. At the very least I knew that I would not be completely lost in industry.

[7]A Fourier series whose coefficients were modified Bessel functions of the first kind.

One fact that influenced my decision to join CSC was that I found a lot of old Physics friends there. Jerry Lerner, who had coaxed me to apply for a position there, had shared an apartment with me during graduate school. Jim Wertz had been my neighbor in our undergraduate dormitory. Landis Markley had been a post-doctoral fellow at the University of Maryland while I was a graduate student there. There were a few others too. These three would all leave CSC within 18 months of my arrival. I hope it was not because of me.

When I arrived for work at CSC in May 1977 I was assigned the task of determining whether the Magsat spacecraft could meet its attitude determination accuracy requirement. This was not really a problem in attitude determination but rather in attitude dynamics. The important question was: would the spacecraft without pitch-rate gyros be able to maintain its angular velocity within appropriate limits (± 200 arcsec/sec) and for sufficient time to carry out star identification? If one could identify stars, then one could process the star tracker data, in which case it was clear that the attitude determination accuracy requirement would be satisfied. My lack of experience was also clear. Fortunately, as a collaborator on this study I was able to work with Dave Gottlieb, a former astronomer and the creator of SKYMAP, a computerized star catalogue still a frequent component of spacecraft attitude work. Dave really understood star trackers and was the angel on my shoulder.

Well, this problem too had a quantum-mechanical analogy, which was similar to the maximum-time problems that Physics graduate students often are forced to solve using semi-classical approaches to the Heisenberg Uncertainty Principle. So once more after some initial panic I was able to use my background in Quantum Mechanics to solve an attitude problem. Of course, when I presented my results, I made no reference to Quantum Mechanics or to the Heisenberg Uncertainly Principle, so it looked as if I had come up with this (admittedly clunky) method all by myself. What we were able to show finally at the end of two months was that without the pitch-rate gyros, the control system wouldn't always maintain the pitch rate within appropriate limits for sufficient time, and the attitude determination accuracy requirement wouldn't be met. Consequently, The Johns Hopkins University Applied Physics Laboratory, the prime contractor for Magsat, put the gyros back into the spacecraft design. In addition, Dave convinced APL that the boresights of the two star trackers should not be parallel. This seems obvious today, and it shows you how little most people knew about attitude determination in the late 1970s.

CSC was convinced now that I could walk on water. I, on the other hand, was unconvinced that I could even swim in it for long. I still knew very little about attitude dynamics and control[8] and nothing at all yet about attitude determination. At night I was working anxiously through the nearly 900 pages of Jim Wertz' book in progress [4], of which I had been given a manuscript copy, trying to learn enough so that I wouldn't fall flat on my face too dishonorably. CSC was expecting me now (August 1977) to find a faster way to determine attitude for the Magsat mission, which would have a very high attitude-computation rate and a very tight attitude determination accuracy requirement. For the Attitude Determination and Control Section of NASA Goddard Space Flight Center, it would be the most challenging mission operations task to date. I, of course, had not the faintest idea of what to do.

[8]From this point on I would never work on problems of attitude dynamics and control again, except in the classroom.

One factor in my favor, which I didn't realize right away, was that no one really knew very much about three-axis attitude determination. Jim's book, for example, contains only about eight pages on three-axis attitude determination methods. Journal publications in any area of attitude determination were also very rare. The emphasis until then, and most of the CSC experience, was on spinning spacecraft, for which, typically, one determined only the direction of the spin axis. As it turned out, however, Jim's book contained the germ of the faster attitude determination method we needed, but it would take me a while to discover it.

MAGSAT

The Magsat spacecraft [4, 9, 10], to be launched on October 30, 1979, would measure the geomagnetic field with the then unprecedented accuracy of 6γ (= 6 nT). To meet this requirement one needed to know the orientation of the magnetometer payload with an accuracy of 20 arcsec/axis (1σ). Other spacecraft flown by NASA had had similarly high accuracy requirements. But for Magsat, which would create a magnetic field map, one needed magnetic field measurements spaced very closely, which meant taking measurements very frequently, in this case at intervals of 0.25 sec. To carry out this task, Magsat was to be provided with an Adcole Fine Sun Sensor and two Ball Brothers CT-401 fixed-head star trackers, all of which would have accuracies higher than 20 arcsec/axis. That the required attitude accuracy could be achieved in theory was not in doubt. That it could be computed quickly enough at that accuracy was. Would it take a week to process one day's worth of data? If it did, then the anticipated six months of Magsat data would require more than three years to process, which would be an unacceptable expense and delay. NASA had assumed that processing six months of Magsat fine attitude data would require a full year. No one, however, really knew how long it would take.[9] For daily attitude mission operations, of course, one could not wait months or even days for attitude estimates, so a second system of more typical sensors had to be in place on the spacecraft to provide attitude data of the usual, more modest kind.

The data processing algorithms had been largely specified for every Magsat attitude determination software system (there were three) with one exception: we had no idea yet what the fine attitude determination algorithm would be. In August 1977, Dave Gottlieb and I constituted the Magsat Fine Attitude Determination task, with Dave as task leader. Dave was busy with the software specification, particularly for star identification, in which he was one of the world's experts, and I started looking for a faster way to compute attitude.

Constraints

One constraint that needed to be addressed in the development of a new attitude determination algorithm for Magsat was the great dislike of quaternions frequently expressed by Roger Werking (another physicist), who was in charge of attitude operations activities at Goddard Space Flight Center, and who was also the head of the section at NASA/GSFC responsible for the attitude determination software for Magsat.[10] For "hands-on" people like Roger, who didn't do mathematics for fun,

[9]In actual fact, the Magsat spacecraft survived for eight months, so the processing time would have been even longer.

[10]I am grateful to the late Roger Werking, racecar driver and duck modeler extraordinaire, for confirming, with much good humor, my memories of his "iron" rule of NASA/GSFC Attitude Operations.

quaternions were regarded as unphysical and confusing, because they could not be visualized in the same way that Euler angles could. I, of course, loved quaternions. Fortunately, I also respected Roger Werking, which was a good thing, because he had much better common sense than the analysts, myself included. Nonetheless, quaternions could not be avoided.

Another possibility for attitude computation that was also greatly disfavored by Roger was the Kalman Filter. Here Roger stood on firm ground. The Kalman filter up to 1977 had not been a spectacular performer for attitude estimation, certainly not for real spacecraft, and the computational burden was very high. Worse, Roger had on previous occasions acceded to requests from GSFC analysts to implement a Kalman filter in an attitude determination system and had been badly burned. He was, therefore, wary of risking disaster again. In any event, except for the name, I knew very little about the Kalman filter at the time, so it was never really an option for me.

In October 1977, I was made the leader of the Magsat Coarse Attitude Analysis task. This meant that just about all my time would be spent on analysis, software specification, validation, and verification for the Magsat coarse and near-real-time attitude determination systems, as well as the design, development and testing of the Magsat attitude system simulator, and directing typically a half-dozen people in these activities. In other words: *real work.* Developing a faster attitude determination algorithm (for the *fine* attitude determination system) now had a much lower priority and wasn't even my job anymore. Much to my astonishment, CSC essentially put the problem on the shelf. But one of my less endearing traits has always been an unwillingness to let go of anything I have started. Thus, finding a faster attitude determination algorithm became my hobby and holy grail if not my official duty and responsibility, and I pursued it tenaciously in my spare moments and in the evening. No one else was going to do it, and I really felt it had to be done, but mostly I just refused to let go of the problem.

Wahba, Davenport, and the HEAO Mission

Fortunately, other people had been working on new ways of computing spacecraft attitude. After spinning my wheels for a month and getting nowhere I began investigating CSC's other missions for NASA. The HEAO (High Energy Astronomical Observatory) Mission algorithm, briefly described by Jerry Lerner in Jim Wertz' book [4], provided the necessary missing link that I needed.

This missing link had its origin in 1965. In that year Grace Wahba, an employee at IBM Federal Systems Division in Palo Alto, California,[11] was working on attitude determination and posed a problem in *SIAM Review* [11], to wit: How would one calculate the proper orthogonal matrix A which minimized the cost function

$$J(A) = \frac{1}{2} \sum_{k=1}^{N} a_k |\hat{\mathbf{W}}_k - A\, \hat{\mathbf{V}}_k|^2 \tag{8}$$

[11]The earlier conference publication [1] gave incorrect information for Dr. Wahba's professional location at the time she posed her famous problem, and even stated incorrectly that she was only a summer employee. In fact, Dr. Wahba worked full-time for IBM Federal Systems, from 1962 until 1966, in Maryland and in California, while she was a graduate student at the University of Maryland and at Stanford University. Graciously, Dr. Wahba did not inform me of my error until I asked her for further historical details.

with $\hat{\mathbf{W}}_k$ and $\hat{\mathbf{V}}_k$ as before, and a_k, $k = 1, \ldots, N$, a set of positive weights?[12] Several authors responded to this problem, all of them offering interesting but not very practical solutions. These were fine for mathematicians but not for mission support. Numerous other solutions were proposed before 1977, which were also of little help. References and further information on solutions to the Wahba problem can be found in the review by Markley and Mortari [12]. After the summer of 1966, Grace pursued a career in Statistics and, to the misfortune of Astronautics, never worked on problems of spacecraft attitude determination again. She is now a very distinguished professor of Statistics at the University of Wisconsin, unaware, except for our infrequent communications, that her first and last publication on spacecraft attitude is the cornerstone of so much important work.

The most intriguing solution to the Wahba problem came from Paul Davenport, a mathematician working at NASA/GSFC. (Are there no engineers in this story?) Paul was the NASA/GSFC monitor for attitude analysis for the HEAO mission. One of the great heroes of spacecraft attitude determination, he is also one of the most brilliant and innovative thinkers and tinkerers in attitude determination that I have ever known. As a manipulator of equations, I think his skills may exceed even those of Markley. Paul made the next significant step leading to a faster algorithm. What he observed was that if one defined the attitude profile matrix B according to

$$B = \sum_{k=1}^{N} a_i \, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^{\mathrm{T}} \tag{9}$$

and one defined further the quantities

$$s = \operatorname{tr} B, \quad S = B + B^{\mathrm{T}} \quad \text{and} \quad \mathbf{Z} = \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix} \tag{10}$$

as well as the 4×4 *Davenport matrix K*

$$K = \begin{bmatrix} S - sI & \mathbf{Z} \\ \mathbf{Z}^{\mathrm{T}} & s \end{bmatrix} \tag{11}$$

then $\bar{q}^*$, the quaternion equivalent to the attitude matrix which minimizes Wahba's cost function above, must satisfy

$$K\bar{q}^* = \lambda_{\max} \bar{q}^* \tag{12}$$

where $\lambda_{\max}$ is the largest eigenvalue of the Davenport matrix. This brilliant result is the starting point for almost all modern work on the Wahba problem. Since K is a real-symmetric matrix, to find the optimal quaternion, one need only construct K and then determine the largest eigenvalue and the associated eigenvector using Householder's method. This would not do for Magsat, because Householder's method was too slow given the computer resources of the time, but for the HEAO mission [13], in which attitude would be calculated infrequently, it was perfectly adequate.

Paul never published his q-method nor his earlier Y-method and R-method, which also solved the Wahba problem. At the time it was developed, the q-method

[12]The notation here is that of Davenport not Wahba.

was documented along with the R-method only in a CSC report [14][13] to NASA/GSFC. The first archival publication was in Wertz [4]. The Y-method appeared only in NASA reports [15, 16]. Paul's work on the q-method had been done, in fact, only within a year of my arrival at CSC, and the CSC company report was issued just as I was being interviewed for my job there. The timing could not have been more fortunate for me.

QUEST Is Born

While the HEAO algorithm wasn't the solution needed by the Magsat mission, because it was still not fast enough given the computers of the day, it was the gateway to finding a faster algorithm. As you may expect, my approach to the problem was once more that of the quantum physicist.

For our further discussion let us define λ_o according to[14]

$$\lambda_o \equiv \sum_{k=1}^{N} a_k \tag{13}$$

Then we can write Wahba's original cost function as

$$J(\bar{q}) = \bar{q}^{\mathrm{T}}[\lambda_o I_{4\times 4} - K]\bar{q} \tag{14}$$

Let us now make the following notational changes:

$$\bar{q} \rightarrow \Psi, \quad \lambda_o I_{4\times 4} - K \rightarrow H, \quad \text{and} \quad \lambda_o - \lambda_{\max} \rightarrow E \tag{15}$$

We note also that since $\bar{q}$ is real, its transpose is the same as the transpose of its complex conjugate, otherwise known as its Hermitian conjugate, written $\bar{q}^{\dagger}$. Likewise, since K is real-symmetric, so is H. Hence, H is necessarily Hermitian as well ($H^{\dagger} = H$). Noting all these facts and substitutions, we can write Davenport's result as finding the value of Ψ which minimizes $\Psi^{\dagger}H\Psi$, with H Hermitian, subject to the constraint that $\Psi^{\dagger}\Psi = 1$. This is just the variational principle of Quantum Mechanics! The optimization leads straightforwardly to

$$H\Psi = E\Psi \tag{16}$$

otherwise known as the energy representation of the Schrödinger Equation [17], where H is the Hamiltonian. This last result is the same as equation (12) except now E is the *smallest* eigenvalue of H. (In Quantum Mechanics one is usually interested in finding the ground state, the state of lowest energy.) I had found my way home once more.

The mad nuclear physicist strikes again!

Having now transformed the optimal attitude problem into the problem of finding the ground-state wave function and ground-state energy of a very simple system, I began to look through my catalog of Quantum Mechanics methods in search of a neat way to solve the problem. At first I tried Rayleigh-Schrödinger perturbation theory, which turned out to be a waste of time. I even developed a diagrammatic language for my perturbative expansion, essentially Feynman diagrams,

[13]It is from this report that I learned about the Wahba problem and Davenport's q-method.

[14]In references [2] and [3], I chose, without loss of generality, $\lambda_o = 1$, and that choice was also made in the Magsat QUEST code. Davenport's derivation, as reported by Keat [14], does not make this specialization. We will write our formulas for general λ_o even though QUEST was not developed that way.

which proved to be an even greater waste of time, although it elicited the admiration of fellow theoretical physicist Landis Markley. Then I tried various non-perturbative methods to solve for the ground-state energy.[15]

What I came to realize from my non-perturbative studies was that E must be very close to zero, or, equivalently, $\lambda_{\max}$ must be very close to λ_o. Had I been smarter, I would have realized this right away, because

$$\lambda_{\max} = \lambda_o - J(\bar{q}^*) \tag{17}$$

and we expect J to be very small compared to λ_o at the optimal attitude. Now, there is a ready-made equation for $\lambda_{\max}$, which is just the characteristic equation of the Davenport matrix. Thus, $\lambda_{\max}$ must be the largest solution of

$$f(\lambda) \equiv \det[K - \lambda\, I_{4\times4}] = 0 \tag{18}$$

the characteristic equation for K, which has four roots. However, we know that $\lambda_{\max}$ must be close to λ_o. Thus, taking unity as a starting value, we can apply the Newton-Raphson method to equation (18).

The Newton-Raphson method applied to the characteristic equation is usually not a good approach to computing an eigenvalue, but we had a very good starting value, and if the attitude were observable, this method should be all right.[16] If one were lazy, one could just set $\lambda_{\max} = \lambda_o$ and substitute this into equation (12) to obtain the optimal attitude with all of the accuracy one needed. However, one would give up a very great advantage that comes from knowing the value of $\lambda_o - \lambda_{\max}$, as we shall see below. Now that I had a very fast way to compute $\lambda_{\max}$, the calculation of $\bar{q}^*$ was simple, and the new attitude computation algorithm was now essentially complete.

And what about Roger Werking and his interdiction of quaternions? Well, I had several points in my favor. First, since October I had been working on the algorithm entirely on my own time, so he could hardly complain. Secondly, he was eager for a way to avoid the fast approaching possible debacle, and a desperate man often makes compromises. Roger was, in fact, quite happy that I was working on a faster batch attitude determination method, even if it used quaternions, and probably had more faith in me than I deserved.

I would not have you believe that once I had converted the Wahba problem into a nuclear physics problem, it was suddenly smooth sailing. I had to repeat derivations three or four times by different paths before I could have confidence in them. For a while I would obtain results for the attitude by different methods which were the inverses of one another. This was resolved only when I finally came to understand vectors properly, particularly the difference between an abstract physical vector and its representation with respect to an orthonormal basis. Slowly, with little to guide me, I was teaching myself the general theory of attitude, rederiving every attitude relation I came upon in my reading.[17] It would take some time before I had confidence in what I was doing.

[15]These excursions into Quantum Mechanics are described in slightly more detail in reference [18].
[16]If the attitude were only marginally observable, we would expect more than one eigenvalue of the Davenport matrix to be close to λ_o.
[17]These exercises became the core of the survey paper on the attitude representations [6], that I published sixteen years later.

The Method of Sequential Rotations

There was still another hurdle to be overcome. Recall that the quaternion is related to the Rodrigues vector **Y** according to

$$\bar{q} = \frac{1}{\sqrt{1 + |\mathbf{Y}|^2}} \begin{bmatrix} \mathbf{Y} \\ 1 \end{bmatrix} \tag{19}$$

The Rodrigues vector, within QUEST, is given by

$$\mathbf{Y} = \mathbf{X}/\gamma \tag{20}$$

where **X** and γ are an intermediate vector and scalar, respectively, calculated by the algorithm. Hence, the quaternion is given by

$$\bar{q} = \frac{1}{\sqrt{|\mathbf{X}|^2 + \gamma^2}} \begin{bmatrix} \mathbf{X} \\ \gamma \end{bmatrix} \tag{21}$$

When the angle of rotation is 180 deg, γ must vanish, because $|\mathbf{Y}|$ is infinite then. This can happen only because of cancellations within the expression for γ, so there must be a loss of numerical significance when the angle of rotation is close to 180 deg. For Magsat and the IBM-360 in double precision, simulations showed that the angle could be as close as 180 ± 10^{-9} deg before the loss of numerical significance became greater than 1 arcsec. At four attitude computations per second, this might happen once every 20,000 years [2]. A reasonable man would have stopped at this point and said that the algorithm was good enough. I was not reasonable. Like a true inventor, I wanted my creation to be perfect beyond any practical requirement. In particular, I wanted a general algorithm for any situation, and a different mission might have an attitude for which the angle of rotation were always close to 180 deg.

One way to avoid this problem was to separate the (unknown) attitude into the sequence of a given 180-deg rotation about one of the coordinate axes followed by a second rotation (to be determined) through an angle significantly smaller than 180 deg. The 180-deg rotation can be accomplished just by changing the signs of two columns of the attitude profile matrix B. This new B is then input to QUEST. The desired estimated quaternion can then be obtained from the QUEST quaternion from this new B just by shuffling a few components and changing a few signs. This is the Method of Sequential Rotations. I argued (incorrectly, as it turned out—see below) that for one of the four choices of the first rotation (no rotation or a rotation of 180 deg about one of the three coordinate axes), the angle of the second rotation had to be less than 90 deg. One tested the angle of rotation simply by putting a lower bound on acceptable values of γ.

It was now February 1978 and I had, for the moment, run out of ideas for things to do to make the algorithm better. I began writing a company report [19]. In April, after having done a lot more simulation I gave a seminar at CSC entitled "Application of the Methods of Theoretical Nuclear Physics to Optimal Attitude Estimation." Our seminar room, which could seat sixty people, was packed to overflowing. Even the president of the CSC's Systems Sciences Division, which then employed over 900 analysts and programmers, mostly in NASA mission support activities, showed up. I would like to think that this enthusiasm was due entirely to CSC's deep confidence in and deep appreciation of my work. Unfortunately, I think it may have been due in reality to the fact that in the politically incorrect 1970s

I advertised that my talk would be preceded by a short subject: "Girls of Tel-Aviv Beach." To my relief, the audience for the featurette slide show stayed for the seminar as well, even the division president (a geophysicist). It was at this seminar that I unveiled the name of the algorithm, QUEST, for QUaternion ESTimator. QUEST was the third and last time that I tried to solve an attitude problem by analogy with Quantum Mechanics.

How fast was QUEST? Early tests showed that using QUEST was 1000 times faster than calculating the optimal quaternion from the Davenport matrix using Householder's method and orders of magnitude faster still than applying a least-square minimization based on the Euler angles.

Speed Is Not Enough

I now began to add extra features to QUEST. First, I wanted a formally correct way to calculate the weights a_k in order to obtain the most accurate attitude estimate. My knowledge of Estimation Theory at the time was limited to only the vaguest notions of minimum-variance estimation (MVE). Therefore, I reasoned, in order to find the best choice for the a_k I must minimize the attitude-error covariance matrix as a function of these weights. What I needed was a simple expression for the attitude-error covariance matrix for the Wahba problem, something I could differentiate. A more experienced person would have known how foolhardy a task this was. I, however, was blissfully unaware of this and proceeded with a boldness that comes only from ignorance and naïvety.

The starting point in deriving a simple expression for the attitude-error covariance matrix was obviously a simple model for the covariance matrix of the attitude sensor measurements. Now, the weighting of each vector measurement in the Wahba problem was characterized by only a single parameter, the weight a_k. Hence, I reasoned, the measurement model also should have only a single parameter. I proposed

$$\hat{\mathbf{W}}_k = A\,\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k, \qquad k = 1, \ldots, N \tag{22}$$

where $\Delta\hat{\mathbf{W}}_k$, the measurement error, is assumed to be zero-mean, white and Gaussian, which couldn't be true exactly but was close to the truth, and had the covariance matrix[18] [3]

$$E\{\Delta\hat{\mathbf{W}}_k \Delta\hat{\mathbf{W}}_k^{\mathrm{T}}\} = \sigma_k^2\,[I_{3\times3} - \hat{\mathbf{W}}_k^{\mathrm{true}}\hat{\mathbf{W}}_k^{\mathrm{true\,T}}] \tag{23}$$

Thus, σ_k^2 is the variance of a component of $\hat{\mathbf{W}}_k$ along any axis perpendicular to $\hat{\mathbf{W}}_k^{\mathrm{true}} = A^{\mathrm{true}}\hat{\mathbf{V}}_k$. This was equivalent to assuming that the measurements had to lowest order a circle of error rather than the more general (and more correct) ellipse of error. This might be a poor approximation for an infrared horizon scanner, in which the errors don't have a very symmetrical distribution (nonetheless, it is frequently used nowadays for that sensor), but for the Magsat fine attitude sensors, it should be a reasonably realistic representation of the truth. I eventually called this the QUEST Measurement Model.

Given my simple measurement model and the expression for the optimal quaternion as provided by QUEST, I was now able to calculate an analytical expression for the attitude-error covariance matrix as a function of the weights a_k,

[18]The negative term in equation (21) accounts for the fact that the length of a unit vector is perfectly known, so the variance in the measured unit vector must be zero along its direction.

$k = 1, \ldots, N$, and the measurement error parameters σ_k, $k = 1, \ldots, N$, an expression that was not very complicated. Nonetheless, the MVE condition on the a_k turned out to be hopelessly complicated, and for some time I was very disheartened.

Fortunately, all was not lost. While there was no easy way to minimize the attitude-error covariance matrix, I did find, however, from my analytical covariance calculations that as a function of the weights, the cost function (optimized over attitude) would be smallest if I chose

$$a_k = \lambda_o \frac{\sigma_{\text{tot}}^2}{\sigma_k^2} \tag{24}$$

where σ_{tot}^2 was defined as[19]

$$\frac{1}{\sigma_{\text{tot}}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{25}$$

When I chose these optimum values of the weights, which gave greater weight to the more accurate data, I obtained an even simpler result for the attitude-error covariance matrix, namely

$$P_{\theta\theta} = \left[\sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left(I_{3\times 3} - (A^{\text{true}}\hat{\mathbf{V}}_k)(A^{\text{true}}\hat{\mathbf{V}}_k)^{\text{T}} \right) \right]^{-1} \tag{26}$$

I was definitely on a roll.

Equation (24) was very significant. Without knowing it, I had just reinvented maximum-likelihood estimation (MLE) [7], which shows the limitations of my knowledge of Estimation Theory at the time. We will return to this later.

It should be obvious that my approach to attitude problems had changed by this point. For most of a year I had been been lost and confused, cautiously feeling my way, and taking much too much time to discover the obvious (such as equation (17)). Now I had begun to be in control of what I was doing and even enjoying the work. New obstacles became challenges rather than defeats. Finally, I was working entirely within the context of spacecraft attitude without the aid of Quantum Mechanics. My internal transition from nuclear physicist to astronauticist took place, I suppose, sometime in the spring of 1978. My knowledge of Engineering and spacecraft attitude was still very limited, but from this point on I was at least working on Engineering problems from the inside. What a difference a year makes, but a very strenuous year, to be sure.

The TASTE of QUEST

If this simple model for the attitude-error covariance matrix was not enough, QUEST also provided an easily calculable figure of merit for data checking which turned out to be a far greater time saver than the speed of the attitude computations themselves. Equation (17) shows that the cost function evaluated at the optimal attitude is just $\lambda_o - \lambda_{\text{max}}$. Hence, λ_{max}, whose calculation is central to the attitude computation, also tells us how well we optimized.

By now I had become skilled at calculating statistical quantities with the QUEST measurement model and was able to show that for N direction measurements with N large the random variable

[19]Generally, one chooses $\lambda_o = 1$ or $\lambda_o = 1/\sigma_{\text{tot}}^2$. Equation (24) illustrates the arbitrariness of λ_o.

$$\text{TASTE} \equiv \frac{2(\lambda_o - \lambda_{\text{max}})}{\lambda_o \sigma_{\text{tot}}^2} \tag{27}$$

would have an approximately χ^2 distribution with $2N$ degrees of freedom. In practice this was a good approximation for the statistical distribution of TASTE even for $N = 3$. Not long after, I was able to show that TASTE, in fact, to much better approximation a χ^2 distribution with $2N - 3$ degrees of freedom.[20] Thus, typically, TASTE would have a mean of $(2N - 3)$ and a variance of $2(2N - 3)$. For three vector measurements, the ideal case for Magsat, this would mean that TASTE $=$ 3.0 ± 2.45. If something were wrong with the data (for example, if a star were misidentified, so that its assumed direction might be wrong by about one degree,[21] then TASTE would have a humongous value on the order of 10^5. Glints in the star trackers might result in even greater values for TASTE. So by examining TASTE, one could validate the data very quickly.

Some background is needed to understand the value of this data-checking method. An important part of NASA/GSFC attitude ground support was the removal of outliers from the data. The way this was done prior to QUEST was primitive and time-consuming. Essentially, one computed the attitude estimates for a data segment and converted them to roll, pitch and yaw, in this case the Euler angles with respect to a local vertical coordinate system. An eighth-order expansion in Tschebyscheff polynomials was then fit to each of these angles and displayed on a graphics device. An analyst would then examine every curve by eye and eliminate by hand with a light pen any data points that were far from the fitted curve. The coefficients of the Tschebyscheff polynomial expansion would then be recalculated and the values of the fit curve would become the accepted values of the attitude estimate for the missing data, in fact for any time. This primitive smoother was a very time-consuming method of data validation. The TASTE test, which was fast and could be automated, was clearly superior.

QUEST Goes Public

QUEST was presented to the outside world for the first time at the AIAA Guidance and Control Conference[22] in Palo Alto, California, in August 1978 [2], just fifteen months after my joining CSC. The covariance analysis, the optimal prescription for the weights, and the TASTE test had all been finished too late to be included in the conference paper. A month later, the QUEST work, but not the name, had the honor of being one of the last things to be included in Jim Wertz' book [4], although it receives only a single sentence, just after the presentation of the HEAO algorithm, at the bottom of page 428. Missing from the conference report, from the brief remark in Jim's book, and from any succeeding publication (except the present one) was any mention of Quantum Mechanics.

The MAGSAT QUEST Code

My official responsibility in the Magsat mission was for the coarse and near-realtime attitude determination systems, none of which I coded myself except for the

[20]A derivation of this result can be found in reference [20].
[21]The star density of the Magsat star catalog was on the order of one star per square degree.
[22]In the same session as my QUEST paper, in which Landis Markley was also a presenter, Jim Murrell [21] presented his results on the attitude Kalman filter for the Landsat mission, which became the starting point for Landis' and my work on the attitude Kalman filter [22].

TRIAD subroutine, which I made look as much like QUEST as possible, including a newly derived attitude-error covariance matrix for TRIAD based, of course, on the QUEST measurement model [3]. Since I already had working software, I was asked to code the QUEST algorithm for the fine attitude determination system. However much sense this made, it was a error in judgment on the part of the Magsat Fine Attitude Task. I continued to tinker lovingly with the QUEST subroutines until the very last minute, when finally I was told gently but firmly that the final QUEST code was needed for end-to-end acceptance testing in two days.

In my effort to make QUEST more efficient, I had made the QUEST code somewhat murky by adding three parameters, which were thresholds for when one would invoke the method of sequential rotations, the maximum acceptable attitude error level, and the computational accuracy one wished for λ_{max}. To these parameters I gave the endearing names QUIBBL, FIBBL, and QUACC. Computing these for a given mission has often been the bane of QUEST users.

An additional level of opacity was dictated by the limitations of the computer resources of the day.[23] For even twenty-arcsec sensors (those on Magsat were more accurate), the value λ_{max}, the fundamental internal quantity in all of the computations, would differ from λ_o by only about one part in 10^8. Since this difference was crucial to the TASTE test, it was necessary that all of QUEST's computation be carried out in double precision. However, double precision was an impossible luxury for the rest of the Magsat Fine Attitude Determination software due to time constraints. Thus, the inputs and outputs of QUEST were in single precision while the internal computations were in double precision, necessitating two parallel sets of input/output and internal parameters.

The MAGSAT Launch and Mission Support

The Magsat spacecraft was launched on October 30, 1979. We would now see how QUEST would behave with real data. As I have said earlier, it was anticipated that fine attitude determination would require one year of data processing for six months of data, with definitive data processing not starting until six months after launch. The coarse attitude system would provide much less accurate attitude results on a daily basis as soon as orbit tapes and sufficient telemetry data became available.

Roger Werking, cautious as ever, had insisted that there be a back-up attitude determination algorithm in case QUEST didn't work. CSC's back-up algorithm (not by me) was simple. If there were only two observation vectors, TRIAD would be used. If all three observation vectors were present, TRIAD would be used for each of the three pairs, the three attitude matrices would be converted to Euler angles, and the results averaged. This *ad hoc* method would be clumsy and slow (ugly too), but in the frequently very crude and unsystematic way that attitude had been calculated up to then it would get the job done. This was the algorithm that a "real" engineer might have come up with back when I began this work. I, however, was a theoretical nuclear physicist and not an engineer, and the idea of proposing an *ad hoc* algorithm that was not derived mathematically from basic principles was totally alien to me.[24] Fortunately, this alternative method was never needed, nor,

[23]Limited though they seem in retrospect, they were the absolute state of the art at the time of Magsat.

[24]I now refer to this "evil" back-up algorithm as "The Anti-QUEST" [23]. For the Magsat fine attitude sensor configuration, it seems to work rather well, although it is slow and clumsy and provides none of the useful extras of QUEST.

I think, even exercised with real data, and the Magsat mission was able to benefit from having the extra tools provided by a covariance matrix and TASTE.

Routine coarse attitude data processing began four days after launch. During the first days after launch I would frequently check the performance of the near-real-time attitude determination system by calculating the Magsat spin-axis attitude on a Texas Instruments TI-59 programmable calculator, the latest thing at the time in personal computing. Roger, I'm sure, was pleased.

As it turned out, the fine-tuning and shakedown of the fine attitude system required five of the anticipated six months, due mostly to problems with the star-identification subroutines.[25] When the fine attitude determination system began routine processing in the spring of 1980, the TASTE test was implemented for data validation in the Fine Attitude System. However, an analyst still did data checking with the Tschebyscheff polynomial fit technique, just to be safe. To everyone's delight, the TASTE test worked very well at eliminating outliers before they could turn up on a graphics display terminal. After two weeks of fine attitude data processing without a single outlier in sight, the fits were subjected only to the most cursory inspection. It was also evident by this time that because of QUEST the Fine Attitude Determination System was operating much faster than the Coarse Attitude Determination System, something which had not been anticipated at all, given the factor of 240 in attitude computation frequency, not to mention the more complicated data reduction algorithms for the fine attitude sensors. As a result, after two weeks of Fine Attitude Determination System operation, the Coarse Attitude Determination System was shut down entirely and only the Fine Attitude Determination System was exercised for the remaining three months of the mission to provide data for the scientists. Thus, I caused the abandonment of two years of my own strenuous effort. The near-real-time system, of course, for which I had also been responsible, remained in operation throughout the lifetime of the Magsat spacecraft for mission operations.

Typically, the Fine Attitude System required about four hours (clock time) to process one day of fine attitude data. This time interval was smaller by a factor of 12 than the anticipated two days for processing one day of fine attitude data. Thus, the five-month backlog of unprocessed data owing to the shakedown was quickly eliminated. It was, in fact, the TASTE test which was the real time-saver for NASA, not the lightning speed of the QUEST attitude computations, a fact that is not generally known.[26]

I was very pleased.

Roger and QUEST

Was Roger Werking, iron chancelor of Attitude Operations at NASA/GSFC, won over now to quaternions? Well, maybe just a little bit. QUEST had saved NASA/GSFC (and Roger's branch) $300,000 possibly in operational support expenditures for Magsat alone, not to mention making certain that the project scientists would see fine definitive attitude estimates for Magsat before they had grown long white beards. Nonetheless, Roger's fundamental animosity towards quaternions probably never abated, although it softened slightly, and he even allowed the Magsat output data tapes to use quaternions instead of the usual Euler angles,

[25]"Subroutines," because everything was coded in FORTRAN IV in those days.
[26]Nonetheless, analysts proposing alternative algorithms to QUEST generally compare only flop counts for only the most basic part of the attitude computation algorithm.

because they would take up less space. Roger certainly developed respect and appreciation for what QUEST could do and played a key role in its expanded use at NASA/GSFC. He also knew what his contractors could do if left to their own devices, so he decreed that for NASA/GSFC attitude support, the QUEST code would never be modified from the version in the Magsat software, which I had last modified only one day before acceptance testing (see below). Those of us who have watched Roger nervously break pencils in two as he approached a major mission deadline appreciate the wisdom of his action. And so, the QUEST code as used by NASA/GSFC and its contractors remained frozen until after Roger retired from NASA, enshrining QUIBBL, FIBBL, QUACC, and the REAL*4/REAL*8 interfaces for nearly a decade. With exceptional wisdom and restraint on my part, I never told Roger that by accident I had inserted an error into QUEST when prettying up the code just two days before acceptance testing (a "+" had been replaced by an "$*$"), whose correction was my final modification to QUEST the following day. The next modification to NASA's QUEST software (a change in the order of certain computations to improve numerical significance), was made by Markley in 1987. I am certain that Roger, had he remained at NASA longer, would himself have caused the NASA code to be modified, certainly to meet the needs of the rapidly changing computer environment.

The Diffusion of QUEST

The diffusion of QUEST began very soon after it had proved itself in the Magsat mission. It began, naturally, at NASA/GSFC when QUEST became part of the attitude ground support system software for the Solar Maximum Mission. QUEST soon became a standard at NASA/GSFC, often replacing the TRIAD algorithm even when the latter algorithm was more than adequate.

One reason, certainly, that QUEST was adopted so quickly by NASA/GSFC was that I had daily contact with the attitude task leaders at CSC for all the other NASA/GSFC spacecraft (more than a half-dozen in preparation at any one time back in those days). I also had more than two years between my CSC seminar and the start of Magsat fine data processing in which to publicize QUEST. It was rare during the first of those two years to be within 30 feet of me and not hear about QUEST and its growing bag of tricks. All the same, immodest though I may have been back then (and since) about QUEST, I had not the faintest notion then that it would achieve the widespread fame it has today. The best thing I could say about QUEST's importance while I was at CSC was that it was probably good enough to be published in a journal.

QUEST appeared in the *Journal of Guidance and Control* in January 1981 [3], my very first journal article in Engineering. For an algorithm that would become so important, I had a hard time getting it accepted for publication. I had also submitted a second article, with S. D. Oh, which was devoted to a covariance analysis of the TRIAD algorithm using the QUEST measurement model. The associate editor, on the recommendation of a reviewer, insisted that the two articles be combined, which meant that Oh's name would be associated with the QUEST algorithm, in which she had played no part. I protested to the journal, but I had little say in the matter.[27]

[27]At least, I get to say that QUEST was my first journal publication in Engineering. Had the two submissions to the *Journal of Guidance and Control* been published separately as I had wished, then QUEST would have become my second journal publication in Engineering.

The most significant event in QUEST's diffusion came around 1987, when the NASA Jet Propulsion Laboratory adopted QUEST for its deep-space missions.[28] If anything conferred stature on QUEST, this was surely it. JPL had also suffered from the slings and arrows of inadequate computer resources, even worse than the situation at NASA/GSFC. Deep-space missions needed to be autonomous for long time intervals. Instantaneous direct control of the spacecraft was impossible from the Earth because of the finiteness of the speed of light and the very large distances to the planets. To make matters worse, the onboard software before the late 1980s did not reside on anything like an IBM-360 mainframe (try squeezing one of those into an unmanned spacecraft!) but on an Intel 8050 chip, which was less capable than the microprocessors in many microwave ovens today.[29]

Thus, there were enormous disincentives at JPL against using anything but the most primitive and most reliable algorithms. The success of the Voyager, Pioneer, and Mariner missions attests to the soundness of JPL's judgment. By 1987, however, the microprocessor revolution was in full swing and it became possible to use a more sophisticated algorithm like QUEST with all its special features. Thus, QUEST went to Jupiter on the Galileo mission, to Saturn on the Cassini mission, to Mars on the Explorer missions, to Venus on the Magellan mission, to the Eros asteroid on the NEAR mission, and it or a close relation may now be on its way to Mercury and Pluto. Whatever the failings of the QUEST algorithm, it has certainly gone far.

In 1989 I discovered that the Instituto Nacional de Pesquisas Espaciais (INPE) in Brazil had carried out some interesting QUEST studies. The level of Astronautics work at INPE was very high, and engineers there had even anticipated [24] my use of QUEST as a preprocessor in the Kalman filter [25, 26] by several years. (References [25] and [26] both cite the Brazilian work.) This led to a stream of letters and even a telephone call from INPE engineers complaining good-naturedly about all the headaches they had suffered in trying to understand QUEST.[30] Eventually these communications would lead to a fruitful collaboration with INPE engineers, which continues to this day. I was by no means insensitive to the pain and suffering which I had had caused the Brazilians. As a result, for more than a decade there has hung in the secretary's office of the Department of Control and Mechanics at INPE a wooden plaque bearing a bottle of Bayer aspirin and the inscription: *From Malcolm Shuster to his colleagues at INPE.*[31] For a while it was customary at INPE to remove an aspirin from the bottle to alleviate QUEST-aches, so much so that it has been necessary to ship refills periodically from the U.S.A.

There was a problem with the publication of QUEST, namely, the q-method had not yet received journal publication. From 1978, when I had seen that QUEST would be worth publishing, I began nagging Paul Davenport to publish the q-method. When I was about to submit my QUEST article to the *Journal of Guidance, Control and Dynamics,* I discovered that the journal would not permit the citing of private communications, which was how Paul wished to be cited, because he disliked the CSC company report. I suggested to Paul that he let me be the ghost-writer of an article by him on his q-method, which I could then cite, but he refused. Had Paul accepted my offer, the resulting paper would have become one of the great classics of Attitude Estimation and would have been cited much more frequently than reference [3].

[28] I am grateful to Dr. Fred Hadaegh of JPL (finally, an engineer!) for providing me with information about JPL's early QUEST experience.

[29] The Intel 8050 chip, to no one's surprise, has found far more extensive applications in microwave ovens than in spacecraft. Magsat also incorporated an Intel 8050 chip onboard in the attitude control system.

[30] Giorgio Giacaglia, professor emeritus of Engineering at the University of São Paulo and first head of the Brazilian Space Agency, has even commented in a course on Astrodynamics that he knew no better hazing for new graduate students than to make them rederive the QUEST algorithm!

[31] Da parte de Malcolm Shuster para seus colegas no INPE.

QUEST has now gained a firm foothold throughout the Solar System. It is difficult to imagine that the enthusiasm for QUEST would have been as great had its unpromising origin been more widely known.

QUEST's Great Shining Moment

I can take no credit for what must certainly be the greatest achievement of QUEST. In early 1982 I received a telephone call from Dr. Hermann Woltring, a research fellow then at the Free University of Amsterdam, who wished to know if I had done any further work on QUEST beyond that published the year before in the *Journal of Guidance and Control.* Dr. Woltring, who died in an automobile accident in 1992, was a biomedical engineer who had applied QUEST to the determination of limb orientation in studies of the human gait. His goal: to design better human prostheses. How much brighter must QUEST shine than all the stars and planets if it has helped a disabled child to walk.

Life after QUEST

What did I do after the QUEST article had been published in the *Journal of Guidance and Control?* A few days before the article appeared I left CSC and began work on submarine-launched ballistic missile systems, never expecting to work on problems of Spacecraft Attitude Determination again.

My career in spacecraft attitude determination, however, did not end abruptly at this point. My second job in the aerospace industry was at BTS, Inc., the company founded by Andrew Jazwinski, who had written a famous book on Estimation Theory and Kalman Filtering [27]. Clearly, I was to continue learning more about Estimation Theory. At the same time, I maintained close contact with my former colleagues at the Computer Sciences Corporation, many of whom have remained among my closest friends, so that continued stimulation to work on problems of attitude determination, if only as a hobby, was inescapable.

Much of my work *post QUEST* was to extend the utility of QUEST or any solution of the Wahba problem. In order to determine the real σ_k for focal-plane sensors, specifically the Magsat star trackers and fine Sun sensor, I developed a method [28] for inferring these error levels from QUEST computations using real data. This same paper also developed a somewhat lame method for determining spacecraft attitude sensor alignments also using the QUEST measurement model.[32] That alignment estimation work has been totally superceded by references [29] and [30], which assume no specific sensor error models but use the QUEST measurement model in the examples.

About eight years after the publication of QUEST, when I had just joined the Space Department of the Johns Hopkins University Applied Physics Laboratory, I showed formally that if one started with the QUEST measurement model and applied the principles of maximum-likelihood estimation, one was led directly to the Wahba problem [31]. Thus, the Wahba problem was no longer an *ad hoc* optimization problem but belonged to the mainstream of Estimation Theory. With this knowledge, the simple expression for the QUEST attitude-error covariance matrix

[32]Despite its lameness, the alignment algorithm of reference [28] was the first to handle redundancy properly.

now fell out immediately as the inverse of the Fisher information matrix. I had known this fact in an heuristic manner for some time and had used it to motivate the Wahba problem as maximum-likelihood estimation of attitude in my attitude determination courses since 1983. I am not quick to publish.

Next I showed how to make the QUEST algorithm itself into a Kalman filter and Kalman smoother[33] and developed an approximate means for simulating the effects of process noise using fading memory [32]. This suboptimal algorithm was seriously considered for the MSX mission, but in simulations I found that it missed the accuracy requirement for that mission by a factor of two. At least the algorithms provided valuable insights.

Since QUEST was a maximum likelihood estimator, it could be used as a measurement preprocessor within the Kalman filter [25, 26]. In more rigorous terms, the QUEST attitude solution is a sufficient statistic for the attitude, assuming the QUEST measurement model. Thus, given a star camera which measures typically the directions of ten stars simultaneously, rather than process these ten star directions individually in a Kalman filter, one could compute the sensor attitude from these ten star measurements using QUEST and then use the QUEST attitude as an effective measurement in the filter.[34] It is this form of the Kalman filter that I finally selected for the MSX mission. I learned six years later that the Jet Propulsion Laboratory implements QUEST and the Kalman filter in this same way in its deep-space missions. The Brazilians at INPE, as I have mentioned above, had (unbeknownst to me) beaten everyone to the punch here [24].

In this same paper [26], using a mathematical trick common in Quantum Scattering Theory (old habits die hard), I also showed that one could ignore the unit-norm constraint on vector measurements in certain cases and replace the QUEST measurement error model covariance in a Kalman filter with

$$E\{\Delta\hat{\mathbf{W}}_k \Delta\hat{\mathbf{W}}_k^{\mathrm{T}}\} = \sigma_k^2 \, I_{3\times 3} \tag{28}$$

This works, because the measurement sensitivity matrix cancels any contribution from the extra term in the covariance matrix (because of the constraint on the attitude matrix that it be proper orthogonal).[35] The advantage of such a substitution is that the measurement covariance matrix for a direction measurement is now invertible. The implementation of this purely mathematical trick has been called the *unit-vector filter* by Joseph Sedlak and Donald Chu [33], who demonstrated that it works quite well.

It could be said that nearly half of my publications use results from the QUEST work in some way. For the most part, this is because they use the QUEST measurement model either as a version of the truth or for simulation purposes. Only about a half-dozen of my journal articles (in 2006), however, are concerned directly with the QUEST attitude computation algorithm or the Wahba problem.

[33]As reference [32] is presented, the filter and smoother implementations apply to any solution algorithm for the Wahba Problem. However, at the time QUEST was almost the only game in town.

[34]Many commercial star trackers, in fact, now output not only the star positions but also an attitude quaternion calculated using QUEST.

[35]It is for this reason, in fact, that maximum-likelihood estimation applied to the QUEST measurement model with its non-invertible covariance matrix leads to Wahba's cost function with scalar weights rather than to a cost function with weight matrices. This is the essence of reference [31].

Alternatives to QUEST

Mirror, Mirror on the wall, who is the fairest of them all?[36]

Over the past two decades many alternative algorithms have been presented as solutions of the Wahba problem. Most remarkable of these newer algorithms are Landis Markley's SVD algorithm [34], which relies on the singular value decomposition method,[37] and his FOAM algorithm [35], which uses a very novel form for the attitude matrix. These two algorithms work entirely in terms of the direction-cosine matrix rather than the quaternion. I have enormous admiration for both. Daniele Mortari has presented numerous quaternion-based solutions[38] to the Wahba problem, all of considerable interest, the most prominent of which is the ESOQ2 algorithm [36], which he reports as being ten percent faster than QUEST.[39] With sometimes a few exceptions for each algorithm, many parts of QUEST, namely, the Newton-Raphson calculation of the overlap eigenvalue, the initial value for that iteration, the optimal weights, the expression for the QUEST attitude-error covariance matrix, the TASTE test, and the method of sequential rotations, are carried over into these new algorithms, so that a very large part of many of these algorithms (especially ESOQ1) is, in fact, QUEST. Further information on these alternative algorithms and references can be found in Markley's and Mortari's review [12].

It is difficult to read reference [12] without coming away with the misimpression on several fronts that QUEST performs less well than other algorithms. QUEST does, in fact, perform more poorly in reference [12] but only for an extreme case (Scenario 2 of reference [12]) which requires an unreasonable design for the attitude determination system and the retention of data which mission analysts generally discard as unreliable. It requires also that the iteration of the overlap eigenvalue be performed, which is unnecessary for the attitude computation.[40]

The ranking of the relative speeds of QUEST and the ESOQ algorithms is also more complicated than reported by reference [12]. For the case that there are no

[36]The word *fair* entered English from Old Norse *fagr* (proto-Germanic **fagraz,* proto-Indo-European **fag-*) probably soon after the Norse invasion of Britain. Its original meaning was as in "fair weather," that is, the opposite of *foul* (*c.f. Macbeth,* Act I, scene iii: "So foul and fair a day I have not seen."). The meanings "light-complexioned," "beautiful," and "morally pure," which characterize Snow White, are first attested toward the end of the 12th century (when, by the way, the proper name "Malcolm" is also attested in English for the first time). The original German text (1812) of the Brothers Grimm—"Spieglein, Spieglein an der Wand, wer ist die Schönste im ganzen Land?"—is more ambiguous. German *schön* can be applied both to weather and to people. The meaning "free from bias" is first attested only in the 14th century.

[37]Landis has stated privately that the idea for the SVD algorithm came from my SVD treatment of spacecraft sensor alignment estimation [29, 30], and that for a time he was afraid that I would discover the algorithm before him, a high compliment indeed, but unfounded. I wish I had.

[38]Recently, Bruccoleri, Lee, and Mortari have unveiled the MRAD algorithm [37], which employs the modified Rodrigues parameters [6] for the attitude solution.

[39]We note with amusement that 28 years of research (in 2006) since the development of the QUEST algorithm have managed to produce an algorithm which *allegedly* is only barely ten percent faster than QUEST [12]. The allegation is true, in fact, only in a limited domain (see below).

[40]Furthermore, if one makes a simple rearrangement of terms in the QUEST characteristic polynomial, then the alleged problem disappears entirely [38], even for the extreme and unreasonable Scenario 2 of reference [12].

The discoverer of the cause of the problem reported by reference [12] and of the simple rearrangement of terms which makes QUEST as robust (and, unquestionably, as accurate, even according to the point of view of reference [12]) as the other fast algorithms is a brilliant young engineer, Dr. Yang Cheng. The poorer numerics of the QUEST characteristic polynomial has been a topic of great interest to this writer for more than a decade. As often happens, however, it is the imaginative newcomer, not the dull expert, who finds the solution. References [38] and [39] are mostly the work of Dr. Cheng.

iterations in the computation of the overlap eigenvalue (i.e., $\lambda_{\max} = \lambda_o$), the only case considered in reference [12], ESOQ2 does indeed require fewer Matlab floating-point operations. However, more extensive simulations [38] have shown that the opposite is true in all the other cases.[41] In far more significant comparisons in terms of the execution times of compiled stand-alone implementations using the C language, a much closer approximation to actual mission software, the differences in the execution times are smaller still and do not favor either QUEST or one of the ESOQ algorithms generally.

Neither QUEST nor one of the ESOQ algorithms is fastest generally. It would seem best to say simply that QUEST, ESOQ1, and ESOQ2 are in the same speed class. In particular, when one considers that the attitude computation from unit vectors, however central, constitutes only a minute fraction of the attitude determination software, less than one percent, the difference hardly matters. In addition, the speed tests of references [12] and [39] did not examine complete implementations of these algorithms as might be used in actual mission software, but minimal implementations of only the most basic attitude computation part and that far from singularity. Given the speed of current computers and the minuteness of the computational burden of the (minimal) attitude computation algorithm, it is obvious that algorithm speed in attitude estimation lost its importance long ago.

Thus, QUEST has not yet been surpassed by any other fast algorithm, although it does have equals [38, 39]. These last lack only QUEST's quarter-century record of reliability, though that should not exclude them as mission algorithms. For now, the Magic Mirror of the Wahba problem answers all inquiries with a satisfied "not you."

A Mistake in QUEST

As soon as QUEST was published, I began receiving correspondence regarding errors in QUEST. With one exception, all of these were false alarms. For the most part, the writers had tried to apply QUEST to a problem for which it was not appropriate. One writer, however, Gregory Natanson of CSC, writing to me ten years after QUEST's publication, pointed out a real mistake.

The mistake was not a mistake in the computation, but in an apparently poorly considered statement I had made, namely, that by the method of sequential rotations, the angle of rotation could always be made less than 90 deg. What Greg showed quite beautifully was that the angle of rotation could only be made less than 120 deg with certainty. This, of course, is more than adequately less than 180 deg, the angle of rotation which the method had sought to avoid. Greg never published his result. Two years later I found more general applications for the method of sequential rotations, and he and I published our results together [40].

The Achievements and Future of QUEST

While challengers hoping to unseat QUEST from its privileged position try to do so on the basis of relative computational speed, speed is now the least important of QUEST's achievements, at least today (2006) when even modest notebook computers are faster by three orders of magnitude than the mainframes of three decades

[41] In terms of Matlab execution times QUEST is fastest in *all* cases.

ago. Speed was not even the most important of QUEST's achievements twenty years ago, although it was certainly important then. It was the TASTE test, even in the beginning, which was the real mission time saver. The principal achievements of the QUEST work (thus far) are: (1) the QUEST attitude computation method itself, (2) the QUEST measurement model, (3) the demonstration that the Wahba problem is the maximum-likelihood estimation problem for this measurement model, (4) the QUEST formula for the attitude-error covariance matrix, (4) the TASTE test, (5) the method of sequential rotations, (6) the use of QUEST as a sensor accuracy estimator, (7) the use of QUEST as a preprocessor in the Kalman filter, and, perhaps, (8) the unit-vector filter idea. Four of these eight innovations were presented in the initial journal publication of QUEST.[42] Even if the QUEST attitude computation method were to be replaced in common usage, these other results of the QUEST work would certainly remain in place as integral components of the newer methods. In this larger sense, QUEST is very unlikely ever to disappear from the scene.

It is well to ask at this point: What constitutes QUEST? Even for me the answer isn't very clear anymore. For almost a decade, QUEST (officially) was simply the Magsat algorithm, frozen in the MAGFINE code (called MSAD-MAGSAT in official NASA/GSFC documents). Certainly, the core of QUEST is the computation of the attitude quaternion and $\lambda_{\max}$ from the Davenport matrix K via the Cayley-Hamilton Theorem and the characteristic polynomial as well as the method of sequential rotations. I would argue forcefully that the model attitude-error covariance matrix and the TASTE test should be inseparable parts of QUEST as well. QUIBBL, FIBBL, and QUACC, however, and the REAL*4/REAL*8 interfaces are certainly not an integral part of QUEST nor are all of the input- and intermediate-variable checking that takes place in the Magsat QUEST code to make sure that QUEST returns an error code for really bad data rather than crashing, a necessary precaution in flight code[43] And is the algorithm no longer QUEST if Markley's expression[44] for the characteristic polynomial is substituted for the one currently in use?[45] QUEST will certainly still be QUEST if the rearrangement of terms in the characteristic polynomial [38] is implemented. At the other extreme some workers even use "QUEST" to label any executable file for solving the Wahba problem.[46]

One disadvantage of QUEST's competitors is that they came at least fifteen years after QUEST and, therefore, cannot compete with QUEST's twenty-six-year record (in 2006) of proven reliability in actual mission support. QUEST has been executed, perhaps, more than a trillion times with real data in more than a hundred very different missions. No amount of simulated testing can match that. Project managers, therefore, if they are sufficiently knowledgable, will almost always choose QUEST over competing algorithms, and with some missions costing nearly one gigadollar (USD) one can hardly blame them.

[42]Strangely, the TASTE test wasn't mentioned in either the 1978 conference report [2] or the 1981 journal article on QUEST [3], although it had been part of the QUEST computer code since 1977. The TASTE test was not published in the open literature until 2005 [20].

[43]This is the part of the QUEST code with which I tinkered up to the last minute in the development of the MAGFINE software, trying always to anticipate one more thing that could go wrong.

[44]As we know now, the QUEST expression is as good as the FOAM expression if we make a slight rearrangement of terms [38].

[45]This has already occurred once in actual mission software designed by this writer [38]!

[46]This writer has even committed that sin in the earliest stages of prototyping flight software in Matlab, because far fewer lines of Matlab code are required to code Markley's SVD algorithm than the QUEST algorithm.

In the long run, this writer believes, as on-board computers become faster and more capable, it will be either Davenport's original implementation of the q-algorithm using Householder's method or, perhaps, Markley's SVD algorithm which will become standard practice. There are even indications that we might already make this transition [39]. The QUEST measurement model, with its many theoretical and practical consequences, including even the derivation of the Wahba problem as maximum likelihood estimation, and other achievements of the QUEST work will prove of more lasting value than the QUEST computation procedure itself. But without QUEST, the QUEST measurement model might never have been proposed.[47] In any event, an obsequy is premature; QUEST has a lot of life yet.

Epilogue

If any lessons are to be learned from my history of the QUEST algorithm, which unavoidably (and for me very happily) has also been my history, they are that there is always something new to be done, and that these new things will sometimes be done by people who know least what has been done before. Expertise and experience can even be a disadvantage, since they cause us to follow well-worn paths. If I look back on the *annus mirabilis* during which I invented QUEST, when I knew nothing about spacecraft attitude, and every step was a fearful leap into the unknown, what I do now seems to be far less exciting,[48] even if technically of higher quality. The lessons which I myself learned while developing QUEST were enormous. I no longer stumble through attitude determination problems, perhaps, because I managed to make almost every conceivable mistake during that first year. In some sense my career in spacecraft attitude did indeed peak early. Perhaps, that is as it should be.

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Jerry Lerner receives special thanks for having brought me into this business in 1977. Paul Davenport deserves special mention for inventing the q-method, without which modern attitude estimation might have been very dull and without which this speaker would have had no story to tell. Roger Werking was very important to the development of QUEST. He will be sorely missed, as is Jerry Bierman, an excellent estimator and friend. It would be hard to imagine working in attitude determination without the presence of Landis Markley, friend and gentle nemesis (sometimes not so gentle) for more than three decades, who greatly influenced QUEST and much of my other work. To John Junkins, astrodynamicist nonpareil, I must acknowledge debts more numerous than I can ever hope to repay.

[47]The author believes that it is the QUEST measurement model, contrived within an instant late one night in September 1978, which is his most important contribution to Spacecraft Attitude Estimation.

[48]Exciting, perhaps, only in retrospect, though there were moments. The first part of the QUEST work consisted mostly of two years of frustration compressed into six months. Leading the Magsat Coarse Attitude Determination task, however, which contained two years of frustration of quite another kind, really was exciting as it was happening.

References

[1] SHUSTER, M. D. "In Quest of Better Attitudes" (Dirk Brouwer Lecture), presented as paper No. AAS-01-250 at the 11th AAS/AIAA Space Flight Mechanics Meeting, Santa Barbara, California, February 11–14; 2001; reprinted in *Advances in the Astronautical Sciences,* Vol. 108, December 2001, pp. 2089–2117.

[2] SHUSTER, M. D. "Approximate Algorithms for Fast Optimal Attitude Computation," Paper No. 78-1249, *Proceedings, AIAA Guidance and Control Conference,* Palo Alto, California, August 1978, pp. 88–95.

[3] SHUSTER, M. D. and OH, S. D. "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control,* Vol. 4, No. 1, January–February 1981, pp. 70–77.

[4] WERTZ, J. R. (ed.) *Spacecraft Attitude Determination and Control,* Springer, New York and Berlin, 1978.

[5] BLACK, H. D. "A Passive System for Determining the Attitude of a Satellite," *AIAA Journal,* Vol. 2, July 1964, pp. 1350–1351.

[6] SHUSTER, M. D. "A Survey of Attitude Representations," *The Journal of the Astronautical Sciences,* Vol. 41, No. 4, October–December 1993, pp. 439–517.

[7] SORENSON, H. *Parameter Estimation,* Marcel Dekker, New York, 1980.

[8] LERNER, G. M., HUANG, W. H., and SHUSTER, M. D. "Investigation of the AEM-A/HCMM Attitude Control System Performance," presented as paper No. AAS-77-116 at the AAS/AIAA Astrodynamics Conference, Grand Teton National Park, Wyoming, September 1977.

[9] POTEMRA, T. A., MOBLEY, F. F., and ECKARD, F. L. (eds.) *The Johns Hopkins University APL Technical Digest—Special Issue on Magsat,* Vol. 1, No. 3, 1980, pp. 162–248.

[10] SHUSTER, M. D. "Spacecraft Attitude Determination and Control," Chapter 5 in V. L. Pisacane and R. C. Moore, *Fundamentals of Space Systems,* Oxford University Press, New York and Oxford, 1994, pp. 245–336; second edition (with W. Dellinger): 2005, pp. 236–325.

[11] WAHBA, G. "Problem 65–1: A Least Squares Estimate of Spacecraft Attitude," *SIAM Review,* Vol. 7, No. 3, July 1965, p. 409.

[12] MARKLEY, F. L. and MORTARI, D. "Quaternion Attitude Estimation Using Vector Observations," *The Journal of the Astronautical Sciences,* Vol. 48, Nos. 2 and 3, April–September 2000, pp. 359–380.

[13] FALLON, L. III, HARROP, I. H., and STURCH, C. R. "Ground Attitude Determination and Gyro Calibration Procedures for the HEAO Missions," *Proceedings, AIAA 17th Aerospace Sciences Meeting,* New Orleans, Louisiana, January 1979.

[14] KEAT, J. "Analysis of Least Squares Attitude Determination Routine DOAOP," Computer Sciences Corporation, CSC/TM–77/6034, February 1977.

[15] DAVENPORT, P. R. "Attitude Determination and Sensor Alignment via Weighted Least Squares Affine Transformations," NASA X-514-71-312, November 1965.

[16] DAVENPORT, P. R. "A Vector Approach to the Algebra of Rotations with Applications," NASA TN-D-4696, August 1968.

[17] DIRAC, P.A.M. *Principles of Quantum Mechanics,* 4th ed., Oxford University Press, Oxford, 1958.

[18] SHUSTER, M. D. "QUEST, a Longa História de um Algoritmo Rápido," (in Portuguese) Plenary Talk, Primeiro Simpósio de Tecnologia Aeroespacial, São José dos Campos (SP), Brazil, August 1990; Instituto Nacional de Pesquisas Espaciais, São José dos Campos (SP), Brazil, NTI-002/91-DMC, December 1991.

[19] SHUSTER, M. D. "Algorithms for Determining Optimal Attitude Solutions," Computer Sciences Corporation, CSC/TM-78/6056, 1978.

[20] SHUSTER, M. D. and FREESLAND, D. C. "The Statistics of TASTE and the Inflight Estimation of Sensor Precision," Paper 56, *Proceedings, Flight Mechanics Symposium 2005,* NASA Goddard Space Flight Center, Greenbelt, Maryland, October 18–20, 2005.

[21] MURRELL, J. W. "Precision Attitude Determination for Multimission Spacecraft," Paper AIAA 78-1248, *Proceedings, AIAA Guidance and Control Conference,* Palo Alto, California, August 1978, pp. 70–87.

[22] LEFFERTS, E. J., MARKLEY, F. L., and SHUSTER, M. D. "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control and Dynamics,* Vol. 5, No. 5, September-October 1982, pp. 417–429.

[23] CHENG, Y. and SHUSTER, M. D. "QUEST and The Anti-QUEST: Good and Evil Attitude Estimation," *The Journal of the Astronautical Sciences,* Vol. 53, No. 3, July–September 2005, pp. 337–351.

[24] VAROTTO, S. E. C., ORLANDO, V., and LOPES, R. V. F. "Um procedimento para determinação da atitude de satélites artificiais utilizando técnicas de estimação ótima estática e dinâmica," *Atas, 6º. Congresso Brasileiro de Automática,* Belo Horizonte, 1986, pp. 946–951.

[25] FISHER, H. L., SHUSTER, M. D., and STRIKWERDA, T. E. "Attitude Determination for the Star-Tracker Mission," paper No. AAS-89-365, AAS/AIAA Astrodynamics Specialists Conference, Stowe, Vermont, August 7–10, 1989; *Advances in the Astronautical Sciences,* Vol. 71, 1990, pp. 138–150.

[26] SHUSTER, M. D. "Kalman Filtering of Spacecraft Attitude and the QUEST Model," *The Journal of the Astronautical Sciences,* Vol. 38, No. 3, July–September 1990, pp. 377–393; Erratum: Vol. 51, No. 3, July–September 2003, p. 359.

[27] JAZWINSKI, A. H. *Stochastic Processes and Filtering Theory,* Academic Press, New York and Orlando, 1970.

[28] SHUSTER, M. D., CHITRE, D. K., and NIEBUR, D. P. "Inflight Estimation of Spacecraft Attitude-Sensor Accuracy and Alignments," *Journal of Guidance, Control and Dynamics,* Vol. 5, No. 4, July–August 1982, pp. 339–343.

[29] SHUSTER, M. D., PITONE, D. S., and BIERMAN, G. J. "Batch Estimation of Spacecraft Sensor Alignments, I. Relative Alignment Estimation," *The Journal of the Astronautical Sciences,* Vol. 39, No. 4, October–December 1991, pp. 519–546.

[30] SHUSTER, M. D. and PITONE, D. S. "Batch Estimation of Spacecraft Sensor Alignments, II. Absolute Alignment Estimation," *The Journal of the Astronautical Sciences,* Vol. 39, No. 4, October–December 1991, pp. 547–571.

[31] SHUSTER, M. D. "Maximum Likelihood Estimation of Spacecraft Attitude," *The Journal of the Astronautical Sciences,* Vol. 37, No. 1, January–March 1989, pp. 79–88.

[32] SHUSTER, M. D. "A Simple Kalman Filter and Smoother for Spacecraft Attitude," *The Journal of the Astronautical Sciences,* Vol. 37, No. 1, January–March 1989, pp. 89–106.

[33] SEDLAK, J. and CHU, D. "Kalman Filter Estimation of Attitude and Gyro Bias with the QUEST Observation Model," Paper AAS-93-297, *AAS/GSFC International Symposium on Space Flight Dynamics,* NASA Goddard Space Flight Center, April 26–30, 1993; *Advances in the Astronautical Sciences,* Vol. 84, 1993, pp. 683–696.

[34] MARKLEY, F. L. "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *The Journal of the Astronautical Sciences,* Vol. 36, No. 3, July–September 1988, pp. 245–258.

[35] MARKLEY, F. L. "Attitude Determination Using Vector Observations: a Fast Optimal Matrix Algorithm," *The Journal of the Astronautical Sciences,* Vol. 41, No. 2, April–June 1993, pp. 261–280.

[36] MORTARI, D. "Second Estimator for the Optimal Quaternion," *Journal of Guidance, Control and Dynamics,* Vol. 23, No. 4, September–October 2000, pp. 885–888.

[37] BRUCCOLERI, C. and MORTARI, D. "MRAD: Modified Rodrigues Vector Attitude Determination," *The Journal of the Astronautical Sciences,* Vol. 54, Nos. 3 and 4, July–December 2006, pp. 383–390.

[38] CHENG, Y. and SHUSTER, M. D. "Robustness and Accuracy of the QUEST Algorithm," presented as paper AAS-07-102 at the AAS/AIAA 17th Space Flight Mechanics Meeting, Sedona, Arizona, January 28–February 2, 2007; *Advances in the Astronautical Sciences,* Vol. 127, 2007, pp. 41–61.

[39] CHENG, Y. and SHUSTER, M. D. "The Speed of Attitude Estimation," presented as paper AAS-07-105 at the AAS/AIAA 17th Space Flight Mechanics Meeting, Sedona, Arizona, January 28–February 2, 2007; *Advances in the Astronautical Sciences,* Vol. 127, 2007, pp. 101–116.

[40] SHUSTER, M. D. and NATANSON, G. A. "Quaternion Computation from a Geometric Point of View," *The Journal of the Astronautical Sciences,* Vol. 41, No. 4, October–December 1993, pp. 545–556.