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A Tutorial on Points and Vectors

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And point by point, the treasure ... he shall again relate.
William Shakespeare (1564–1616)
Henry VIII, Act I, scene ii

I am resolved on two points.
William Shakespeare (1564–1616)
A Midsummer Night's Dream, Act I, scene v

Abstract

The properties of points and vectors are examined within the context of an affine space. The temporal derivative of a point is shown to be a vector and inertial frames are introduced in a natural way.

Introduction

In two previous works we presented a tutorial on vectors and the properties of static attitude [1] and a tutorial on attitude kinematics [2]. Reference [2] was the direct sequel to reference [1]. In the present work we present a prequel to reference [1] in which we present the properties of points and how they lead to vectors. Since points don't have a great deal of structure, this tutorial will be very short compared to the others, and ifs development will be a bit unusual to engineers and physicists, who seldom think about points.

A hallmark of the previous two tutorials was the emphasis on three different levels of abstraction. At the bottom level is the numerical column vector, whose components are simply numbers. Above that is the column-vector variable, in which the numerical components are replaced by variables. Of prime importance is the column-vector representation, which is the column vector of the components of a physical vector with respect to a physical basis. The physical vector is a coordinate-free object specified by a physical description

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only without reference to a specific basis. Implicit in the description of physical vectors is the existence of an origin, without which a physical vector would be only a point, the principal subject of this article. In the present work, we carry the abstraction one step further to a space in which there is not even a defined origin. Physical vectors also exist in this space, called an affine space. We call the process of identifying the origin *origination*. Thus, the extended hierarchy of a point U, a physical vector $\boldsymbol{u}$, a column-vector representation (variable) ${}^{\mathcal{E}}\mathbf{u}$, and a numerical vector $[1.2, 2.7, 0, 4]^{\mathrm{T}}$ (the superscript "T" denotes the matrix transpose) may be displayed as

$$\mathsf{U} \xrightarrow{\text{origination}} \boldsymbol{u} \xrightarrow{\text{representation}} {}^{\mathcal{E}}\mathbf{u} \xrightarrow{\text{numerical substitution}} \begin{bmatrix} 1.2 \\ 2.7 \\ 0.4 \end{bmatrix} .$$

Points and Affine Spaces

The geometric description of nature begins with points. Points A, B and C are not vectors. The sum of two points is not defined. Points simply cannot be added. This is related, of course, to the physical fact that there is no absolute origin of a reference frame for the description of Mechanics. There is no meaningful expression for points of the form

$$\mathsf{A} + \mathsf{B} = \mathsf{C} . \tag{1}$$

On the other hand, the displacements from one point to another, such as $\overrightarrow{\mathsf{AB}}$, the displacement from A to B, and $\overrightarrow{\mathsf{BC}}$, the displacement from B to C, satisfy

$$\overrightarrow{\mathsf{AB}} + \overrightarrow{\mathsf{BC}} = \overrightarrow{\mathsf{AC}} , \tag{2}$$

with $\overrightarrow{\mathsf{AC}}$ the displacement from point A to point C. The set of displacements are elements of a vector space [1, 3]. Points and displacements together are elements of an affine space, which provides the basis for making Geometry algebraic.

An *affine space* [4] consists of a vector space $\mathcal{V}$ [1, 2], whose elements are called displacements, a set $\mathcal{P}$, whose elements are called points, and a binary operation $\boxplus$ that combines a point and a displacement. The affine space has the following properties

(1) For every $\mathsf{P} \in \mathcal{P}$ and for every $\boldsymbol{v} \in \mathcal{V}$,

$$\mathsf{P} \boxplus \boldsymbol{v} \in \mathcal{P} . \tag{3}$$

(2) For $\boldsymbol{0}$ the zero element of $\mathcal{V}$ and for every $\mathsf{P} \in \mathcal{P}$,

$$\mathsf{P} \boxplus \boldsymbol{0} = \mathsf{P} . \tag{4}$$

(3) For any two points $\mathsf{P}, \mathsf{Q} \in \mathcal{P}$ there exists a unique vector $\boldsymbol{v} \in \mathcal{V}$ such that

$$\mathsf{Q} = \mathsf{P} \boxplus \boldsymbol{v} . \tag{5}$$

As a short hand we may write equation (6) as

$$\boldsymbol{v} = \mathsf{Q} \boxminus \mathsf{P} \equiv \overrightarrow{\mathsf{PQ}} . \tag{6}$$

We may say informally that points may be “subtracted” from one another, though not added.

(4) The binary operator $\boxplus$ is associative in the sense that for every $\mathsf{P} \in \mathcal{P}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$.

$$(\mathsf{P} \boxplus \boldsymbol{u}) + \boldsymbol{v} = \mathsf{P} \boxplus (\boldsymbol{u} + \boldsymbol{v}) . \tag{7}$$

Henceforth, we write simply “+” for “$\boxplus$” and “−” for “$\boxminus$.” Thus, there are three different binary addition operators, one for the scalars in the vector space, one for the vectors in the vector space, and one for adding a vector to a point, all represented by the same symbol. The nature of each addition operator is obvious from the context.

It follows that if

$$\mathsf{A} + \boldsymbol{u} = \mathsf{B} + \boldsymbol{v} , \tag{8}$$

then

$$\overrightarrow{\mathsf{AB}} = \boldsymbol{u} - \boldsymbol{v} . \tag{9}$$

Also, if

$$\mathsf{A} = \mathsf{B} + \boldsymbol{v} \qquad \text{and} \qquad \mathsf{C} = \mathsf{D} + \boldsymbol{v} , \tag{10}$$

for identical $\boldsymbol{v}$, then

$$\overrightarrow{\mathsf{AC}} = \overrightarrow{\mathsf{BD}} . \tag{11}$$

One can show that

$$\overrightarrow{\mathsf{AB}} = -\overrightarrow{\mathsf{BA}} \tag{12}$$

for every $\mathsf{A}, \mathsf{B} \in \mathcal{P}$, which is equivalent to

$$\overrightarrow{\mathsf{PP}} = \boldsymbol{0} \tag{13}$$

for every $\mathsf{P} \in \mathcal{P}$. We include also in this list of properties equation (2) above.

We note because of the *translation theorem* for points, equations (10) and (11), that the set of elements of the vector space $\mathcal{V}$ is isomorphic to the set of equivalent classes of the Cartesian product $\mathcal{P} \times \mathcal{P}$, namely, the set

$$\mathcal{P}(\boldsymbol{v}) \equiv \left\{ \overrightarrow{\mathsf{AB}} \mid \mathsf{A} \in \mathcal{P}, \mathsf{B} \in \mathcal{P}, \mathsf{B} - \mathsf{A} = \boldsymbol{v} \right\} . \tag{14}$$

The line segment $\overline{\mathsf{AB}}$ may be represented as the set

$$\overline{\mathsf{AB}} \equiv \left\{ \mathsf{A} + \beta \overrightarrow{\mathsf{AB}} \mid 0 \leq \beta \leq 1 \right\} , \tag{15}$$

and $L(\mathsf{A}, \mathsf{B})$, the infinite line passing through A and B, as

$$L(\mathsf{A}, \mathsf{B}) \equiv \left\{ \mathsf{A} + \beta \overrightarrow{\mathsf{AB}} \mid -\infty \leq \beta \leq \infty \right\} . \tag{16}$$

If one point $\mathcal{O}$ of the affine space is singled out as the *origin*, then points can be written in general as

$$\mathcal{P} = \mathcal{O} + \overrightarrow{\mathcal{O}\mathcal{P}} \equiv \mathcal{O} + \mathbf{p} \equiv \mathcal{O} + \boldsymbol{r}_{\mathcal{P}} \,. \tag{17}$$

with $\boldsymbol{r}_{\mathcal{P}}$ a vector, an element of the vector space,. Equation (17) defines an isomorphism $T_{\mathcal{O}}$ between the set of points and the set of vectors

$$T_{\mathcal{O}} : \ \mathcal{P} \longrightarrow \boldsymbol{r}_{\mathcal{P}} \,. \tag{18}$$

Choosing a different origin $\mathcal{O}'$ leads to the different physical vector $\boldsymbol{r}'_{\mathcal{P}}$ given by

$$\boldsymbol{r}'_{\mathcal{P}} = \boldsymbol{r}_{\mathcal{P}} + \overrightarrow{\mathcal{O}\mathcal{O}'} \,. \tag{19}$$

The Temporal Derivation of Vectors and Points

The Temporal Derivation of Physical Vectors

If $\boldsymbol{u}(t)$ is a time-dependent physical vector defined on an open interval (t_i, t_f), then the (ordinary) temporal derivative of $\boldsymbol{u}(t)$ on (t_i, t_f), if it exists, is simply

$$\frac{d}{dt}\boldsymbol{u}(t) \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)\right] , \qquad t_i < t < t_f \,. \tag{20}$$

The argument of the limit in the right member of equation (20) is in $\mathcal{V}$ for every finite value of Δt. In order to be able to show that the derivative exists, the vector space must possess a norm. A *norm* is a scalar function on a vector space that satisfies the three relations

$$\|\boldsymbol{u}\| \geq 0 \,, \tag{21}$$

$$\|\boldsymbol{u}\| = 0 \quad \text{if and only if} \quad \boldsymbol{u} = \boldsymbol{0} \,, \tag{22}$$

and

$$\|a\,\boldsymbol{u}\| = |a|\,\|\boldsymbol{u}\| \,. \tag{23}$$

We often write $|\boldsymbol{u}|$ for $\|\boldsymbol{u}\|$. A norm $\|\cdot\|$ always exists in an inner-product space, namely,

$$\|\boldsymbol{u}\| \equiv (\boldsymbol{u}\cdot\boldsymbol{u})^{1/2} \,. \tag{24}$$

We have not, however, specified that $\mathcal{V}$ is an inner-product space. We specify now that $\mathcal{V}$ is a normed vector space [5].

Given a norm, for the limit of equation (20) to exist, there must exist a vector-valued function $d\boldsymbol{u}(t)/dt$ such that

$$\lim_{\Delta t \to 0}\left\| \frac{d}{dt}\boldsymbol{u}(t) - \frac{1}{\Delta t}\left[\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)\right]\right\| = 0 \,, \qquad t_i < t < t_f \,. \tag{25}$$

If equation (25) is true, we say that $\boldsymbol{u}(t)$ is *differentiable* on (t_i, t_f). Note that the statement of the existence of the derivative of a physical vector does not require the use of representations nor of a basis. Thus, for the *physical* vector $\boldsymbol{x}(t) = \boldsymbol{x}_o + \boldsymbol{v}_o t + (1/2)\boldsymbol{a}_o t^2$, with $\boldsymbol{x}_o$, $\boldsymbol{v}_o$, and $\boldsymbol{a}_o$ constant physical vectors, one has immediately that $d\boldsymbol{x}(t)/dt = \boldsymbol{v}_o + \boldsymbol{a}_o t$.

Note that without an inner product, there is even no mechanism in the vector space $\mathcal{V}$ for determining the components of a vector., but that no components appear in equations (20) or (25), The determination of the temporal derivative of a physical vector does not require the existence of components, as shown by the example above.

The Temporal Derivation of Points

The temporal derivation of points follows trivially from that of physical vectors. If $\mathsf{P}(t)$ is a time-varying point, then

$$\mathsf{P}(t+\Delta t) - \mathsf{P}(t) = \overrightarrow{\mathsf{P}(t+\Delta t)\mathsf{P}(t)}\,, \tag{26}$$

which is a vector. Thus,

$$\frac{d}{dt}\,\mathsf{P}(t) \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\mathsf{P}(t+\Delta t) - \mathsf{P}(t)\right] \equiv \boldsymbol{v}_{\mathsf{P}}(t)\,, \qquad t_i < t < t_f \tag{27}$$

is well defined. The right member of equation (27) is the physical velocity vector of the point P.

Let $\mathsf{O}(t)$ be a time‘dependent point. We say that $\mathsf{O}(t)$ is *non-accelerating* over an open interval (t_i, t_f) if

$$\frac{d^2}{dt^2}\,\mathsf{O}(t) = \boldsymbol{0}\,, \qquad t_i < t < t_f\,. \tag{28}$$

If $\mathsf{O}(t)$ is an non-accelerating origin over the time interval $(t_i, t_f$ and $\boldsymbol{r}(t)$, defined as $\overrightarrow{\mathsf{P}(t)\mathsf{O}(t)}$, is the physical position vector of a point particle at point P with mass m, then $\boldsymbol{r}(t)$ will satisfy Newton's second law

$$m\,\frac{d^2}{dt^2}\,\boldsymbol{r}(t) = \boldsymbol{F}(t)\,, \tag{29}$$

where $\boldsymbol{F}(t)$ is the applied force. If the origin $\mathsf{O}(t)$ is accelerating, however, then equation (29) becomes

$$m\,\frac{d^2}{dt^2}\,\boldsymbol{r}(t) = \boldsymbol{F}(t) + m\,\boldsymbol{a}_{\mathsf{O}}(t)\,, \tag{30}$$

where

$$\boldsymbol{a}_{\mathsf{O}}(t) \equiv \frac{d^2}{dt^2}\,\mathsf{O}(t)\,. \tag{31}$$

We restrict our attention now to affine spaces in which the vector space is an inner-product space, in which case it possesses a right-handed orthonormal basis. If $\mathcal{V}$ is an inner-product space, then we may define a reference frame $\{\mathsf{O}(t), \mathcal{E}(t)\}$ as consisting of an origin $\mathsf{O}(t)$ and a physical right-handed orthonormal basis

$\mathcal{E}(t) = \{\hat{\boldsymbol{e}}_1(t), \hat{\boldsymbol{e}}_3(t), \hat{\boldsymbol{e}}_3(t)\}$. If the origin is non-accelerating, and the right-handed orthonormal basis is irrotational, then the reference frame is said to be *inertial*.

We have reached the starting point of reference [1].

If we wish to represent an affine space with respect to a particular reference frame, then we must first transform the set of points into a set of vectors according to equation (18). The result, of course is a vector space which we may write as $\mathcal{V}_O$. We must, however, recall the defined origin in the symbol for the the column-vector representations, just as we recalled the right-handed orthonormal basis in the symbol for column-vector representations in references [1] and [2]. The column-vector space would then be denoted by ${}^{\mathcal{E}}_{O}\mathcal{V}$ and its elements by ${}^{\mathcal{E}}_{O}\mathbf{u}$. This leads to the possibility in the treatment of attitude kinematics to symbols for the kth component of the angular velocity of the form

$$ {}^{\mathcal{E}}_{O}\omega_k^{\mathcal{E}/\mathcal{N}} \, . $$

Discussion

There is not, as we have seen, a great deal that can be said about points. What value there is in this work is the knowledge that the temporal derivative of a point is a vector and to show formally how the acceleration of a non-inertial frame enters into the equations of motion.

The displacements in this work are free vectors because of equations (10) and (11), as apposed to bound vectors. In normal parlance, $\overrightarrow{\mathsf{AB}}$ is often considered to be bound to the point A. We regain the possibility of having a vector bound to a point by considering a vector field $\boldsymbol{v}(\boldsymbol{x})$, where $\boldsymbol{x}$ is confined to some region of space. $\boldsymbol{v}(\boldsymbol{x})$ is then "bound" to the position $\boldsymbol{x}$.

With good reason, the study of points has not received much attention in Engineering or Physics. Needless to say, we will not continue this work with a further pre-prequel on spaces without points or vectors. That would be pointless.

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