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A Tutorial on Attitude Kinematics

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For John L. Junkins, noble friend and the alpha
and omega cross of Attitude Dynamics.

Plus on apprend, moins on comprend. [1]

Introduction

This article continues the program begun in "A Tutorial on Vectors and Attitude" [2] by developing the equations of attitude kinematics or, equivalently, of rotational kinematics, to use an expression more current outside the aerospace community. Reference [2] was an improvement of part of an earlier survey of the attitude representations [3]. The present work, which continues [2], is mostly new.

We focus mostly on the basics, in particular, on the temporal derivative of vectors. The content of this article does not go much beyond the study of the transport equation, which describes the temporal development of a vector from the perspective of a rotating reference frame. We leave the detailed exposition of the attitude dynamics of particular systems to the many excellent texts on that subject, for example, [4] through [6].

The presentation of attitude kinematics in this work is very similar to the presentation of static attitude in [2], which concentrated on the very basic description of attitude. At its most primitive, static attitude, the subject of [2], is described by the axis and angle of rotation, or, more compactly, by the rotation vector [2, 3]. In the development of the attitude representations, the first step is usually to express an attitude representation as a function of the axis and angle of rotation [3]. In attitude kinematics, the corresponding quantity is the angular-velocity vector, which, in some cases, is the temporal derivative of the rotation vector. The most important result in the kinematics of each attitude representation is the expression of the rate of change of that representation in terms of the angular velocity [3]. As in [2], the attitude representation to which we give almost all our attention is the attitude matrix. There are, of course, other important representations (for example, the quaternion and the Euler angles), which are treated in detail in [3]. As in [2], our purpose here is to concentrate on principles rather than on a variety of forms.

The treatment of the present article, like that of [2], is distinguished by its emphasis on the differences and the connections between physical vectors and their column-vector representations with respect to a given basis. Textbooks treating rotational mechanics generally blur the distinction between physical vectors and column vectors or even equate the two. We examine here in great detail material usually passed over lightly in the textbooks. This work is not a review, nor is it an attempt to simplify the treatment of this topic. Clarity, we remind the reader, is purchased often only at the cost of greater intellectual effort. This work is a tutorial, but it is not "Attitude Kinematics Made Easy."

Attitude estimation, as remarked in [2], is dominated by the study of column-vector representations, because the attitude is estimated most often by comparing column-vector representations of the same physical vectors with respect to two different bases. The usual presentation of attitude kinematics (and dynamics), however, is dominated by the use of physical vectors. The column-vector representations, if they appear at all in the study of attitude kinematics and dynamics, are at best a poor cousin. A goal of this article and [2] is to bring uniformity to the treatment of static and dynamic attitude.

Like [2], this work depends on a hierarchy of vectors, namely, physical vectors, their representation as column-vector variables, and numerical column vectors. Of the last, the only examples in this work are the autorepresentations of physical bases (see [2] and below). We write simply of column vectors. The (coordinate-free) physical vector, we remind our readers, is an abstraction, more abstract than the column vector, in the sense that it is one step further removed from measurements. We cannot assign a numerical value to a physical vector, only to its column-vector representations. Nonetheless, when we derive equations of motion, we usually do so without drawing coordinate axes, although we must have an origin, at least implicitly. Without an origin, we cannot speak of a position vector but only of a position point. The coordinate-free derivation of equations of motion is the practice, even in the earliest university courses. The concept of a physical vector, the vector of drawings [2], is implicit in our thinking. We saw in [2] that the most basic result of the representation of attitude, Euler's formula, began with physical vectors. When we perform simulations or develop algorithms for project or mission support, however, we must use column vectors. Thus, a large part of our effort must be to make the transition from basic equations of motion in terms of physical vectors to practical algorithms and software in terms of column vectors. It is unavoidably but regretably characteristic of early university studies in Engineering and in Physics that physical vectors are identified with column vectors. We do not subscribe to that identification in attitude kinematics and dynamics, which we show, as in the study of static attitude [2], can lead to error.

We develop two approaches to attitude kinematics in this article. The first is a new approach characterized by making the transition from physical vectors to column-vector representations as early as possible in attitude kinematics studies. The second approach delays this transition until the very end of the development. This second approach is the traditional approach presented in textbooks treating rotational mechanics [4–8]. That approach is examined later in this article and made more rigorous. The two approaches are equivalent in that they lead to the same equations of motion for column vectors, as indeed they must. They are both correct. The first approach, however, seems to us better suited to the computer age.

An unfortunate characteristic of the traditional presentation of attitude kinematics is that it seems to have forgotten the basic definition of the temporal derivative and relies completely on a secondary construct, the derivative with respect to a frame. As a result, most engineers believe falsely that a vector can be differentiated only with respect to a frame. The temporal derivative with respect to a frame is examined rigorously later in this article. We assume only the basic concept of a temporal derivative, defined nearly 300 years ago.

Prolegomena

Physical Vectors and Their Column-Vector Representations

As in [2], we can write a *physical vector* $\boldsymbol{u}$ in terms of a physical basis $\mathcal{E} = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ as

$$\boldsymbol{u} = {}^{\mathcal{E}}u_1\,\hat{\boldsymbol{e}}_1 + {}^{\mathcal{E}}u_2\,\hat{\boldsymbol{e}}_2 + {}^{\mathcal{E}}u_2\,\hat{\boldsymbol{e}}_3\,. \tag{1}$$

The ${}^{\mathcal{E}}u_k$, $k = 1,2,3$, are the *components* of $\boldsymbol{u}$. We assume throughout this work that bases are right-handed orthonormal [2, 3]. Physical vectors are coordinate-free, they cannot be written as 3×1 arrays of components, which is the rôle of the column-vector representation of a physical vector. The three physical basis vectors in (1) are likewise coordinate-free.

We write the *column-matrix representation* of $\boldsymbol{u}$ with respect to the basis $\mathcal{E}$ as

$${}^{\mathcal{E}}\mathbf{u} = \begin{bmatrix} {}^{\mathcal{E}}u_1 \\ {}^{\mathcal{E}}u_2 \\ {}^{\mathcal{E}}u_3 \end{bmatrix}. \tag{2}$$

It follows (for a right-handed orthonormal basis) that

$${}^{\mathcal{E}}u_k \equiv \hat{\boldsymbol{e}}_k \cdot \boldsymbol{u}\,, \qquad k = 1,2,3\,. \tag{3}$$

The difference between physical vectors and their column-vector representations can be made to appear more subtle by writing the autorepresentation of the basis $\mathcal{E}$, that is, the representation of the basis with respect to itself, as [2, 3]

$${}^{\mathcal{E}}\hat{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \hat{\mathbf{1}}\,, \qquad {}^{\mathcal{E}}\hat{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \hat{\mathbf{2}}\,, \qquad {}^{\mathcal{E}}\hat{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \hat{\mathbf{3}}\,, \tag{4}$$

in which case, (2) becomes equivalently

$$\begin{aligned} {}^{\mathcal{E}}\mathbf{u} &= {}^{\mathcal{E}}u_1\,{}^{\mathcal{E}}\hat{\mathbf{e}}_1 + {}^{\mathcal{E}}u_2\,{}^{\mathcal{E}}\hat{\mathbf{e}}_2 + {}^{\mathcal{E}}u_3\,{}^{\mathcal{E}}\hat{\mathbf{e}}_3 \\ &= {}^{\mathcal{E}}u_1\,\hat{\mathbf{1}} + {}^{\mathcal{E}}u_2\,\hat{\mathbf{2}} + {}^{\mathcal{E}}u_3\,\hat{\mathbf{3}}\,. \end{aligned} \tag{5}$$

The vectors $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ are *numerical column vectors*, defined completely by their numerical value. Note that $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$, despite their appearance, constitute a right-handed orthonormal basis if and only if $\mathcal{E}$ is right-handed orthonormal (see [2]).

The difference between physical vectors and their column-vector representations now becomes the difference between the physical basis vectors $\hat{\boldsymbol{e}}_k$, $k = 1,2,3$, and the column-vector basis vectors ${}^{\mathcal{E}}\hat{\mathbf{e}}_k$, $k = 1,2,3$. These differences, though minor in appearance, are stark, as shown in [2] and [3]. In particular, the physical basis vectors $\hat{\boldsymbol{e}}(t)$, $k = 1,2,3$, may be time-varying, while the column vectors ${}^{\mathcal{E}}\hat{\mathbf{e}}_k$, $k = 1,2,3$, (or $\{\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}\}$) are constant in time. Thus, the temporal derivative of ${}^{\mathcal{E}}\mathbf{u}(t)$ differs from the representation with respect to $\mathcal{E}$ of $d\boldsymbol{u}(t)/dt$. The operations of "representation" and temporal differentiation do not commute. There is an advantage in studying the temporal derivative of physical vectors before that of their column-vector representations, which lose some information through the process of "representation." We illustrate the hierarchy of vectors below.

$$\boldsymbol{u} \xrightarrow{\text{representation}} {}^{\mathcal{E}}\mathbf{u} \xrightarrow{\text{numerical substitution}} \begin{bmatrix} 1.2 \\ 2.7 \\ 0.4 \end{bmatrix} .$$

Let us add that one cannot "see" physical vectors, even though one can draw them abstractly in diagrams. "Seeing" means measurement, and measurements of physical vectors, because they can assume numerical values, are column vectors. The physical vector is a pure abstraction, farther removed from concrete description than column-vector variables. Directly or indirectly, the properties of physical vectors are inferred from those of column vectors, but they are not the same. Carrying over intuition gained from the study of column vectors can lead to misunderstanding and sometimes error, especially in the matter of temporal derivation. A physical vector may have components, but it does not consist of components. That honor belongs to the column-vector representation.

How does one determine the properties of physical vectors? Often, the properties of physical vectors have been inferred from those of their column-vector representations. In general, if an equation in terms of column vectors, with all column vectors representations with respect to the same common right-handed orthonormal basis, and without any explicit transformation of representations, is satisfied for every such basis, then the same equation is satisfied by the physical vector. Thus, for example, if $d\,{}^{\mathcal{E}}\mathbf{r}(t)/dt = {}^{\mathcal{E}}\mathbf{v}(t)$ for every right-handed orthonormal basis $\mathcal{E}$, then $d\boldsymbol{r}(t)/dt = \boldsymbol{v}(t)$. In some cases, for example, the usual derivation of (7) (see below), the relationship is derived generally directly in terms of physical vectors.

Notation

As in [2], we use the Palatino bold italic font or Times bold italic font to denote a physical vector ($\boldsymbol{r}$, $\boldsymbol{u}$, $\boldsymbol{v}$, $\hat{\boldsymbol{e}}_k$, $\hat{\boldsymbol{n}}$, $\boldsymbol{\omega}$, ...) and the Helvetica bold font to denote column vectors ($\mathbf{r}$, $\mathbf{u}$, $\mathbf{v}$, $\hat{\mathbf{e}}_k$, $\boldsymbol{\omega}$,...). The entries of column vectors are denoted by Times italic letters (r, u, n, ω,...). For the most part, matrices are denoted by upper-case Helvetica letters (A, C, Ω, ...), and their entries by the corresponding upper-case Times italic letters (A_{ij}, C_{ij}, Ω_{ij}, ...). In handwriting, we usually write a physical vector $\boldsymbol{u}$ as $\vec{u}$ and a column vector u as $\underline{u}$. The identification of a particular bold symbol as either a column vector or a physical vector is often made easier by the absence or presence, respectively, of a presuperscript denoting the basis of representation. Our general philosophy of notation is that any matrix, whether 3×1, 3×3, or of any dimension, is represented by an unslanted Helvetica symbol (bold for column vectors and nonbold otherwise), and almost any other quantity is written using Times or Palatino italic (bold for physical vectors, nonbold otherwise). Bases are denoted by the Zapf Calligraphic font $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{E}$). Dyadics are denoted by the Zapf Chancery Italic font ($\mathscr{A}$, $\mathscr{I}$). In handwriting, we might indicate dyadics by bidirectional arrows ($\overleftrightarrow{A}$, $\overleftrightarrow{I}$). Presuperscripts are reserved for indicating the basis of representation, postsuperscripts for indicating the frame or frames characterizing the quantity, and postsubscripts for entry indices.

Equations of Motion

To illustrate explicitly what we mean by kinematic equations and dynamical equations, we consider the archetypical equations of motion. The kinematic equation for the translational motion of the physical position of a point particle is simply

$$\frac{d}{dt}\boldsymbol{r} = \boldsymbol{v} \,. \tag{6}$$

Essentially, (6) is the definition of the physical linear velocity, and *ipso facto* correct.

The kinematic equation for the rotational motion of a physical direction $\hat{\boldsymbol{b}}$ fixed and constant with respect to a rotating frame $\mathcal{E}$ is simply

$$\frac{d}{dt}\hat{\boldsymbol{b}} = \boldsymbol{\omega}^{\mathcal{E}} \times \hat{\boldsymbol{b}} \,, \tag{7}$$

with $\boldsymbol{\omega}^{\mathcal{E}}$ the physical angular velocity vector of the frame. These equations should require no explanation. This does not mean, of course, that these equations do not require derivation, at least (7), which is a fundamental relation in attitude kinematics and not a definition.

A Word of Caution

Some readers may be confused about the meaning of a physical vector due to the often ambiguous treatment of physical vectors and column vectors in traditional presentations of attitude kinematics. These traditional treatments are generally not clear on the differences between physical vectors and column vectors and tend to endow physical vectors with the properties of column vectors. As an example, we consider the direction of a physical vector.

If $\boldsymbol{u}$ is a physical vector then $\hat{\boldsymbol{u}} \equiv \boldsymbol{u}/\|\boldsymbol{u}\|$ is a unit vector, its direction. The quantity $\|\boldsymbol{u}\|$ is the norm of $\boldsymbol{u}$ explained below. The physical vector $\hat{\boldsymbol{u}}$ is component-free and, therefore, has no numerical direction. For the direction of $\hat{\boldsymbol{u}}$ relative to a frame we must examine the column vector representation of $\hat{\boldsymbol{u}}$ with respect to that frame. The physical direction $\hat{\boldsymbol{u}}$ has a direction with respect to a frame, but it is not a direction with respect to a frame.

The temporal derivative of a physical vector is another physical vector, whose direction with respect to a frame must likewise be determined by the column-vector representation of the physical derivative vector with respect to that frame. The direction of a physical vector, like the physical derivative vector, is "absolute." Only the column-vector representation with respect to a frame is "with respect to a frame." The physical angular velocity vector of (7) is neither with respect to a frame nor relative to a frame.

Since most readers are unfamiliar with physical vectors, the properties of physical vectors may seem "unphysical." They are not. When we examine the column-vector representations of physical vectors, we recover the more familiar relationships of column vectors. Unsupported dogmatic statements like "one cannot differentiate a vector except with respect to a frame" have no place in this work. Such statements focus on the components of a vector, but a physical vector, as has been said repeatedly does not consist of components.

The Temporal Derivation of Physical Vectors

If $\boldsymbol{u}(t)$ is a time-dependent physical vector defined on an open interval $(t_i,\, t_f)$, then the (ordinary) temporal derivative of $\boldsymbol{u}(t)$ on $(t_i,\, t_f)$, if it exists, is simply

$$\frac{d}{dt}\,\boldsymbol{u}(t) \equiv \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left[\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)\right], \qquad t_i < t < t_f\,. \tag{8}$$

The argument of the limit in the right-hand side of (8) is in $\mathcal{V}$ for every finite value of Δt. As a necessary condition to determine whether the limit exists, the vector space must possess a norm. A *norm* on a vector space is a scalar-valued function that satisfies the three relations

$$\|\boldsymbol{u}\| \geq 0\,, \tag{9}$$

$$\|\boldsymbol{u}\| = 0 \quad \text{if and only if} \quad \boldsymbol{u} = \boldsymbol{0}\,, \tag{10}$$

and

$$\|a\,\boldsymbol{u}\| = |a|\,\|\boldsymbol{u}\|\,. \tag{11}$$

We often write $|\boldsymbol{u}|$ for $\|\boldsymbol{u}\|$. A norm $\|\cdot\|$ always exists in an inner-product space, namely,

$$\|\boldsymbol{u}\| \equiv (\boldsymbol{u}\cdot\boldsymbol{u})^{1/2}\,. \tag{12}$$

However, the temporal derivative exists even in vector spaces that do not possess an inner product, in which case there does not exist a right-handed orthonormal basis or even a mechanism for determining the components of vectors. The existence of a norm on $\mathcal{V}$ does not imply that $\mathcal{V}$ possesses an inner product. For more on normed spaces, see [9].

Given a norm, for the limit of (8) to exist, there must exist a physical-vector-valued function $d\boldsymbol{u}(t)/dt$ of finite norm such that

$$\lim_{\Delta t\to 0} \left\| \frac{d}{dt}\,\boldsymbol{u}(t) - \frac{1}{\Delta t}\left[\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)\right] \right\| = 0\,, \qquad t_i < t < t_f\,. \tag{13}$$

If (13) is true, we say that $\boldsymbol{u}(t)$ is *differentiable* on $(t_i,\, t_f)$. Note that the statement of the existence of the derivative of a physical vector does not require the use of representations nor of a basis. Thus, for example, for the *physical* vector $\boldsymbol{x}(t) = \boldsymbol{x}_o + \boldsymbol{v}_o t + (1/2)\boldsymbol{a}_o t^2$, with $\boldsymbol{x}_o$, $\boldsymbol{v}_o$, and $\boldsymbol{a}_o$ constant physical vectors, one has immediately that $d\boldsymbol{x}(t)/dt = \boldsymbol{v}_o + \boldsymbol{a}_o t$.

Equations (8) and (13) are not new. Equation (8) has been with us more and less since the time of Newton and Leibniz and in the form of (8) and (13) for about 150 years, since Bolzano and Weierstrass began making the Calculus more rigorous. In the form above, for vectors in a normed vector space, they have been in existence for a century. They do not depend on a frame. Equation (8) is also the starting point for the development of rigid-body mechanics in the recent book by Hurtado [10]. The approach of [10] to the distinction between physical vectors and their column-vector representations is identical to that of [2] and [3], although the notation is somewhat different.

The Attitude Kinematics of Physical Reference Frames

Equation (7) is most often derived diagrammatically. In the present section we present a more formal rigorous derivation.

For any differentiable time-dependent right-handed orthonormal basis $\mathcal{E}(t) = \{\hat{\boldsymbol{e}}_1(t), \hat{\boldsymbol{e}}_2(t), \hat{\boldsymbol{e}}_3(t)\}$, we must have, for $t_i < t < t_f$,

$$\frac{d}{dt}\hat{\boldsymbol{e}}_i(t) = \sum_{j=1}^{3} \Omega^{\mathcal{E}}_{ij}(t)\,\hat{\boldsymbol{e}}_j(t), \qquad i = 1, 2, 3\,, \tag{14}$$

for some real 3×3 matrix $\Omega^{\mathcal{E}}(t)$. We do not generally write the temporal dependence of $\mathcal{E}(t)$ explicitly in superscripts except in cases where there are multiple times in an equation or for emphasis. For each value of i, $i = 1, 2, 3$, the matrix entries $\Omega^{\mathcal{E}}_{ij}(t)$, $j = 1, 2, 3$, are just the three components of $d\hat{\boldsymbol{e}}_i(t)/dt$ with respect to $\mathcal{E}(t)$. Differentiating $\hat{\boldsymbol{e}}_i(t) \cdot \hat{\boldsymbol{e}}_j(t) = \delta_{ij}$, $i, j = 1, 2, 3$, with respect to time, leads to the condition

$$\Omega^{\mathcal{E}}_{ij}(t) = -\Omega^{\mathcal{E}}_{ji}(t)\,, \qquad i, j = 1, 2, 3\,, \tag{15}$$

that is, that $\Omega^{\mathcal{E}}(t)$ be antisymmetric. We define $\boldsymbol{\omega}^{\mathcal{E}}(t)$, the *physical angular velocity vector* of the reference frame $\mathcal{E}(t)$ at time t, as

$$\boldsymbol{\omega}^{\mathcal{E}}(t) \equiv \frac{1}{2} \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\, \Omega^{\mathcal{E}}_{ij}(t)\,\hat{\boldsymbol{e}}_k(t)\,. \tag{16}$$

Equation (16) is equivalent to

$${}^{\mathcal{E}}\omega^{\mathcal{E}}_1(t) = \Omega^{\mathcal{E}}_{23}(t)\,, \qquad {}^{\mathcal{E}}\omega^{\mathcal{E}}_2(t) = \Omega^{\mathcal{E}}_{31}(t)\,, \qquad {}^{\mathcal{E}}\omega^{\mathcal{E}}_3(t) = \Omega^{\mathcal{E}}_{12}(t)\,, \tag{17}$$

for the three components of $\boldsymbol{\omega}^{\mathcal{E}}$ with respect to $\mathcal{E}$. It follows that

$$\frac{d}{dt}\hat{\boldsymbol{e}}_k(t) = \boldsymbol{\omega}^{\mathcal{E}}(t) \times \hat{\boldsymbol{e}}_k(t), \qquad k = 1, 2, 3\,. \tag{18}$$

Equation (18) is the *kinematic equation for physical basis vectors.* If $\boldsymbol{b}(t)$ is fixed with respect to the basis $\mathcal{E}(t)$, namely,

$$\boldsymbol{b}(t) = b_1\,\hat{\boldsymbol{e}}_1(t) + b_2\,\hat{\boldsymbol{e}}_2(t) + b_3\,\hat{\boldsymbol{e}}_3(t)\,, \tag{19}$$

with b_1, b_2, b_3 constant (equivalently, ${}^{\mathcal{E}(t)}\mathbf{b}(t)$ is independent of time), then we obtain straightforwardly (7).

By *reference frame*, we mean generally an origin together with a set of basis vectors. Since all reference frames in this work have the same origin, *reference frame* becomes synonymous with *basis*, and we use the two terms almost interchangeably. Reference frame, or frame, is the more frequent expression in the discussion of rotational mechanics. The postsuperscripts in (17) indicate that $\boldsymbol{\omega}^{\mathcal{E}}$ is the angular velocity of the frame $\mathcal{E}$. The presuperscripts indicate that the components are also with respect to $\mathcal{E}$. Note that both $\boldsymbol{\omega}^{\mathcal{E}}$ and ${}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}$ are both absolute angular-velocity vectors. They are not angular velocities relative to another frame.

From (18), we obtain readily

$$\begin{aligned}\left(\frac{d}{dt}\hat{\boldsymbol{e}}_k(t)\right)\times\hat{\boldsymbol{e}}_k(t) &= \left(\boldsymbol{\omega}^{\mathcal{E}}(t)\times\hat{\boldsymbol{e}}_k(t)\right)\times\hat{\boldsymbol{e}}_k(t)\\ &= \boldsymbol{\omega}^{\mathcal{E}}(t) - \left(\hat{\boldsymbol{e}}_k(t)\cdot\boldsymbol{\omega}^{\mathcal{E}}(t)\right)\hat{\boldsymbol{e}}_k(t)\\ &= \left(\mathcal{I}_{(3)} - \hat{\boldsymbol{e}}_k(t)\,\hat{\boldsymbol{e}}_k^{\dagger}(t)\right)\boldsymbol{\omega}^{\mathcal{E}}(t)\,, \qquad k = 1,2,3\,,\end{aligned} \tag{20}$$

where $\hat{\boldsymbol{e}}_k^{\dagger}(t)$ is the dual vector to $\hat{\boldsymbol{e}}_k(t)$, and $\mathcal{I}_{(3)}$ is the identity dyadic on a three-dimensional vector space [2],

$$\mathcal{I}_{(3)} = \hat{\boldsymbol{e}}_1(t)\,\hat{\boldsymbol{e}}_1^{\dagger}(t) + \hat{\boldsymbol{e}}_2(t)\,\hat{\boldsymbol{e}}_2^{\dagger}(t) + \hat{\boldsymbol{e}}_3(t)\,\hat{\boldsymbol{e}}_3^{\dagger}(t)\,. \tag{21}$$

Summation of (20) over k leads to

$$\boldsymbol{\omega}^{\mathcal{E}}(t) = \frac{1}{2}\sum_{k=1}^{3}\left(\frac{d}{dt}\hat{\boldsymbol{e}}_k(t)\right)\times\hat{\boldsymbol{e}}_k(t)\,. \tag{22}$$

Using dyadics [2], we can write (18) and its conjugate equation equivalently as

$$\frac{d}{dt}\hat{\boldsymbol{e}}_k = \{\boldsymbol{\omega}^{\mathcal{E}}\times\}\,\hat{\boldsymbol{e}}_k\,, \qquad k = 1,2,3\,, \tag{23}$$

$$\frac{d}{dt}\hat{\boldsymbol{e}}_k^{\dagger} = \hat{\boldsymbol{e}}_k^{\dagger}\{\boldsymbol{\omega}^{\mathcal{E}}\times\}^{\dagger} = -\hat{\boldsymbol{e}}_k^{\dagger}\{\boldsymbol{\omega}^{\mathcal{E}}\times\}\,, \qquad k = 1,2,3\,, \tag{24}$$

where, for any physical vector $\boldsymbol{u}$ [2],

$$\{\boldsymbol{u}\times\} \equiv -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,u_i\,\hat{\boldsymbol{e}}_j\,\hat{\boldsymbol{e}}_k^{\dagger}\,. \tag{25}$$

Of particular interest is an *irrotational basis* $\mathcal{N}(t) = \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$, by which we mean simply a basis whose elements satisfy

$$\frac{d}{dt}\hat{\boldsymbol{n}}_k(t) = \boldsymbol{0}\,, \qquad k = 1,2,3\,. \tag{26}$$

Hence, from (14) and (16),

$$\boldsymbol{\omega}^{\mathcal{N}} = \boldsymbol{0}\,. \tag{27}$$

Although every inertial frame is irrotational, not every irrotational frame is inertial. For a frame to be inertial, its origin must be non-accelerating, which can hardly be true for a spacecraft in orbit. An obvious irrotational basis is $\mathcal{E}(t_o)$ for some fixed t_o, $t_i < t_o < t_f$.

Note that (27) is the definition of an irrotational frame. Hence, it is axiomati. The physical vector space is purely geometrical and not endowed *ab initio* with any inertial properties. This work is one of geometry, not of physics.

If $\mathcal{E}'(t)$ is any other right-handed orthonormal basis related to $\mathcal{E}(t)$ by a constant proper orthogonal transformation, then $\boldsymbol{\omega}^{\mathcal{E}'} = \boldsymbol{\omega}^{\mathcal{E}}$. Thus, if $\mathcal{E}(t)$ is fixed in a rigid body, it might be more appropriate to speak rather of a *body* physical angular-velocity vector, without reference to a particular set of body-fixed coordinate axes. Our focus in this work, however, is not on physical bodies but on reference frames, and we speak of a body angular velocity only with respect to a particular body-fixed reference frame..

We emphasize again that the physical angular-velocity vector defined by (16) is not intrinsically the angular velocity of one frame relative to another. No frame, other than the one being differentiated, occurs in (16), (17) or (18). The absolute angular velocity may seem "unphysical" to some readers, but we show in a later section (see (82) below) that the absolute angular velocity is, in fact, equal to the angular velocity relative to a non-rotating frame.

The Column-Vector Representation of Attitude Kinematics

For column-vector representations, we have, analogously to (16),

$${}^{\mathcal{E}}\mathbf{\omega}^{\mathcal{E}}(t) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,\Omega_{ij}^{\mathcal{E}}(t)\;{}^{\mathcal{E}}\hat{\mathbf{e}}_k(t)\,, \tag{28}$$

and, from (14), we have that

$$\Omega_{ij}^{\mathcal{E}}(t) = \left(\frac{d}{dt}\hat{\boldsymbol{e}}_i(t)\right)\cdot\hat{\boldsymbol{e}}_j(t) = {}^{\mathcal{E}}\!\left(\frac{d}{dt}\hat{\mathbf{e}}_i(t)\right)_j, \tag{29}$$

from which,

$${}^{\mathcal{E}}\mathbf{\omega}^{\mathcal{E}}(t) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\,{}^{\mathcal{E}}\!\left(\frac{d\hat{\mathbf{e}}_i(t)}{dt}\right)_j{}^{\mathcal{E}}\hat{\mathbf{e}}_k(t)\,, \tag{30}$$

or

$${}^{\mathcal{E}}\mathbf{\omega}^{\mathcal{E}}(t) = \frac{1}{2}\sum_{k=1}^{3}{}^{\mathcal{E}}\!\left(\frac{d}{dt}\hat{\mathbf{e}}_k(t)\right)\times{}^{\mathcal{E}}\hat{\mathbf{e}}_k(t)\,, \tag{31}$$

in analogy with (22). Note that

$$
{}^{\mathcal{E}}\!\left(\frac{d\hat{\mathbf{e}}_i(t)}{dt}\right) \equiv \sum_{k=1}^{3}\left(\hat{\boldsymbol{e}}_k(t)\cdot\frac{d\hat{\boldsymbol{e}}_i(t)}{dt}\right){}^{\mathcal{E}}\hat{\mathbf{e}}(t)
$$

$$
= {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times{}^{\mathcal{E}}\hat{\mathbf{e}}_i(t)\,, \qquad i=1,2,3\,, \tag{32}
$$

$$
\frac{d}{dt}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_i(t) = \mathbf{0}\,, \qquad i=1,2,3\,, \tag{33}
$$

displaying the difference between the representation of a temporal derivative and the temporal derivative of a representation. In (29) through (32), note that it is not ${}^{\mathcal{E}}\hat{\mathbf{e}}_k(t)$ but $\hat{\boldsymbol{e}}_k(t)$ that is being differentiated. Finally,

$$
\Omega^{\mathcal{E}}(t) = -[{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times]\,, \tag{34}
$$

with [2, 3]

$$
[\mathbf{u}\times] \equiv \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{35}
$$

for any column vector $\mathbf{u}$. If $\mathcal{E}(t)$ and $\mathcal{E}'(t)$ are two reference frames related by a constant proper-orthogonal transformation, then

$$
\boldsymbol{\omega}^{\mathcal{E}}(t) = \boldsymbol{\omega}^{\mathcal{E}'}(t)\,, \tag{36}
$$

but

$$
{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t) \neq {}^{\mathcal{E}'}\boldsymbol{\omega}^{\mathcal{E}'}(t)\,. \tag{37}
$$

The Transport Equation for Column-Vector Representations

Let $\mathcal{N} = \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$ be an irrotational right-handed orthonormal basis, and let $\mathcal{E} = \{\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\}$ be a right-handed orthonormal basis rotating with angular velocity $\boldsymbol{\omega}^{\mathcal{E}}(t)$. Then,

$$
\frac{d}{dt}\,\hat{\boldsymbol{n}}_k(t) = \boldsymbol{0}\,, \qquad k = 1,2,3\,, \tag{38}
$$

$$
\frac{d}{dt}\,\hat{\boldsymbol{e}}_k(t) = \boldsymbol{\omega}^{\mathcal{E}}(t)\times\hat{\boldsymbol{e}}_k(t)\,, \qquad i=1,2,3\,. \tag{39}
$$

Let $\boldsymbol{r}(t)$ be a time-dependent physical vector, and write the components of the representations of $\boldsymbol{r}(t)$ with respect to irrotational and rotating frames in the usual way as

$$
{}^{\mathcal{N}}r_k(t) \equiv \hat{\boldsymbol{n}}_k(t)\cdot\boldsymbol{r}(t)\,, \qquad {}^{\mathcal{E}}r_k(t) \equiv \hat{\boldsymbol{e}}_k(t)\cdot\boldsymbol{r}(t)\,, \qquad k = 1,2,3\,. \tag{40}
$$

It follows that

$$
\hat{\boldsymbol{n}}_k\cdot\frac{d\boldsymbol{r}(t)}{dt} = \frac{d}{dt}\,{}^{\mathcal{N}}r_k(t)\,, \tag{41}
$$

or

$$
{}^{\mathcal{N}}\!\left(\frac{d\mathbf{r}(t)}{dt}\right) = \frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{r}(t)\,. \tag{42}
$$

The reduction of ${}^{\mathcal{E}}(d\mathbf{r}(t)/dt)$ to an expression in terms of $d^{\mathcal{E}}\mathbf{r}(t)/dt$ is more difficult and is also the central result of this work. Again, note (32) and the comment that follows it.

We are now prepared to develop the transport equation for ${}^{\mathcal{E}}\mathbf{r}(t)$, the column-vector representation of the time-varying physical vector $\boldsymbol{r}(t)$ with respect to $\mathcal{E}(t)$. Expanding an arbitrary physical vector $\boldsymbol{r}(t)$ in terms of a basis $\mathcal{E}(t)$ as

$$\boldsymbol{r}(t) = \sum_{k=1}^{3} {}^{\mathcal{E}}r_k(t)\,\hat{\boldsymbol{e}}_k(t)\,. \tag{43}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\boldsymbol{r}(t) &= \sum_{k=1}^{3}\left[\left(\frac{d}{dt}{}^{\mathcal{E}}r_k(t)\right)\hat{\boldsymbol{e}}_k(t) + {}^{\mathcal{E}}r_k(t)\,\boldsymbol{\omega}^{\mathcal{E}}(t)\times\hat{\boldsymbol{e}}_k(t)\right]. \\ &= \sum_{k=1}^{3}\left(\frac{d}{dt}{}^{\mathcal{E}}r_k(t)\right)\hat{\boldsymbol{e}}_k(t) + \boldsymbol{\omega}^{\mathcal{E}}(t)\times\boldsymbol{r}(t))\,,\end{aligned} \tag{44}$$

which has the representation with respect to $\mathcal{E}(t)$,

$${}^{\mathcal{E}}\!\left(\frac{d}{dt}\mathbf{r}(t)\right) = \sum_{k=1}^{3}\left(\frac{d}{dt}{}^{\mathcal{E}}r_k(t)\right){}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) + {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times{}^{\mathcal{E}}\mathbf{r}(t)\,. \tag{45}$$

Recalling (4) and (5), we have

$$\begin{aligned}\sum_{k=1}^{3}\left(\frac{d}{dt}{}^{\mathcal{E}}r_k(t)\right){}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) &= \frac{d}{dt}{}^{\mathcal{E}}r_1(t)\,\hat{\mathbf{1}} + \frac{d}{dt}{}^{\mathcal{E}}r_2(t)\,\hat{\mathbf{2}} + \frac{d}{dt}{}^{\mathcal{E}}r_3(t)\,\hat{\mathbf{3}} \\ &= \begin{bmatrix} d\,{}^{\mathcal{E}}r_1(t)/dt \\ d\,{}^{\mathcal{E}}r_2(t)/dt \\ d\,{}^{\mathcal{E}}r_3(t)/dt \end{bmatrix} = \frac{d}{dt}{}^{\mathcal{E}}\mathbf{r}(t)\,.\end{aligned} \tag{46}$$

Thus,

$${}^{\mathcal{E}}\!\left(\frac{d}{dt}\mathbf{r}(t)\right) = \frac{d}{dt}{}^{\mathcal{E}}\mathbf{r}(t) + {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times{}^{\mathcal{E}}\mathbf{r}(t)\,. \tag{47}$$

If we wish to write

$${}^{\mathcal{E}}\!\left(\frac{d}{dt}\mathbf{r}(t)\right) = {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right){}^{\mathcal{E}}\mathbf{r}(t)\,, \tag{48}$$

then we must define

$${}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right) \equiv \frac{d}{dt} + [{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}\times]\,. \tag{49}$$

The "$\boldsymbol{\omega}\times$" term emerges naturally from the representation of the ordinary temporal derivative with respect to a rotating basis. We call ${}^{\mathcal{E}}\mathsf{d/dt}$ *the representation with respect to the frame $\mathcal{E}$* or simply the *representation of the temporal derivative*. It is different from the temporal derivative with respect to the frame $\mathcal{E}$ of traditional presentations of rotational mechanics [4–6]. The latter, as shown later in this work, operates not on column vectors but on physical vectors.

If we now note (42) and write

$$ {}^{\mathcal{E}}\!\left(\frac{d}{dt}\,\mathbf{r}(t)\right) = \mathsf{A}^{\mathcal{E}(t)/\mathcal{N}}\;{}^{\mathcal{N}}\!\left(\frac{d}{dt}\,\mathbf{r}(t)\right) = \mathsf{A}^{\mathcal{E}(t)/\mathcal{N}}\;\left(\frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{r}(t)\right)\,, \tag{50} $$

where $\mathsf{A}^{\mathcal{E}(t)/\mathcal{N}}$ denotes the attitude matrix of $\mathcal{E}(t)$ relative to $\mathcal{N}$, which is resolved along the axes of either $\mathcal{E}(t)$ or $\mathcal{N}$ [2, 3], then

$$ \frac{d}{dt}\,{}^{\mathcal{E}}\mathbf{r}(t) + {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times{}^{\mathcal{E}}\mathbf{r}(t) = \mathsf{A}^{\mathcal{E}(t)/\mathcal{N}}\;\left(\frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{r}(t)\right)\,. \tag{51} $$

This is the *transport equation for a column-vector representation* ${}^{\mathcal{E}}\mathbf{r}(t)$. Note that the attitude matrix is specified completely by the postsuperscripts for the prior and posterior bases [2].

Rigorously, ${}^{\mathcal{E}}(\mathrm{d/dt})$ is a matrix operator.

$$ {}^{\mathcal{E}}\!\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) = \mathsf{I}_{3\times3}\,\frac{d}{dt} + [{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}\times] = \begin{bmatrix} d/dt & -{}^{\mathcal{E}}\omega^{\mathcal{E}}_3 & {}^{\mathcal{E}}\omega^{\mathcal{E}}_2 \\ {}^{\mathcal{E}}\omega^{\mathcal{E}}_3 & d/dt & -{}^{\mathcal{E}}\omega^{\mathcal{E}}_1 \\ -{}^{\mathcal{E}}\omega^{\mathcal{E}}_2 & {}^{\mathcal{E}}\omega^{\mathcal{E}}_1 & d/dt \end{bmatrix}\,. \tag{52} $$

To be rigorous, one should always write the factor $\mathsf{I}_{3\times3}$ when one expands ${}^{\mathcal{E}}(\mathrm{d/dt})$. However, since the results of $(d/dt)\,\mathbf{u}(t)$ and $(\mathsf{I}_{3\times3}\,d/dt)\,\mathbf{u}(t)$ are identical, we tend to be lax in writing $\mathsf{I}_{3\times3}$. One has for any positive integer n,

$$ {}^{\mathcal{E}}\!\left[\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)^{n}\right] = \left[{}^{\mathcal{E}}\!\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\right]^{n}. \tag{53} $$

Thus, (38) and (39) for physical basis vectors lead to the following results for vector representations with respect to irrotational and to rotating frames.

$$ \frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{r}(t) = {}^{\mathcal{N}}\mathbf{v}(t)\,, \tag{54} $$

$$ \frac{d}{dt}\,{}^{\mathcal{E}}\mathbf{r}(t) + {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times{}^{\mathcal{E}}\mathbf{r}(t) = A^{\mathcal{E}/\mathcal{N}}(t)\,\frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{r}(t) = {}^{\mathcal{E}}\mathbf{v}(t)\,. \tag{55} $$

The left-hand side of (54) and (55) is the representation with respect to $\mathcal{N}$ and to $\mathcal{E}$ of the ordinary derivative of $\boldsymbol{r}(t)$. In general, ${}^{\mathcal{E}}(d\mathbf{r}(t)/dt) \neq d\,{}^{\mathcal{E}}\mathbf{r}(t)/dt$ unless $\mathcal{E}(t)$ is irrotational ((42)). Equations (54) and (55) are frame-dependent, because they give the temporal derivatives of the components of a physical vector with respect to a particular frame. However, nothing other than the ordinary temporal derivative, which is frame-independent, is used to derive the equations for physical vectors analogously to (54) and (55).

Generally, we have a particular interest in ${}^{\mathcal{B}}\mathbf{r}(t)$, the representation with respect to a body-fixed basis, because it is with respect to body axes that the inertia tensor (matrix) of a rigid body is constant in time. Note again that $\boldsymbol{r}(t)$ in the above development is an arbitrary physical vector, not necessarily the position.

A Useful Identity

Consider an orthogonal matrix C defined by its row vectors,

$$\mathsf{C}^{\mathrm{T}} \equiv [\,\hat{\mathbf{c}}_1 \vdots \hat{\mathbf{c}}_2 \vdots \hat{\mathbf{c}}_3\,]\,, \tag{56}$$

where $\mathcal{C} = \{\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\}$, is an orthonormal triad of column vectors. Then,

$$\mathsf{C}\,[\mathbf{u}\times]\,\mathsf{C}^{\mathrm{T}} = \begin{bmatrix} \hat{\mathbf{c}}_1^{\mathrm{T}} \\ \hat{\mathbf{c}}_2^{\mathrm{T}} \\ \hat{\mathbf{c}}_3^{\mathrm{T}} \end{bmatrix} [\,\mathbf{u}\times\hat{\mathbf{c}}_1 \vdots \mathbf{u}\times\hat{\mathbf{c}}_2 \vdots \mathbf{u}\times\hat{\mathbf{c}}_3\,]\,, \tag{57}$$

and

$$\begin{aligned}\left(\mathsf{C}\,[\mathbf{u}\times]\,\mathsf{C}^{\mathrm{T}}\right)_{ij} &= \hat{\mathbf{c}}_i^{\mathrm{T}}\left(\mathbf{u}\times\hat{\mathbf{c}}_j\right) = -\left(\hat{\mathbf{c}}_i\times\hat{\mathbf{c}}_j\right)^{\mathrm{T}}\mathbf{u} \\ &= -(\pm)\sum_{k=1}^{3}\epsilon_{ijk}\,\mathbf{c}_k^T\mathbf{u}\,, \end{aligned} \tag{58}$$

where the plus (minus) sign holds if the $\hat{\mathbf{c}}_k$, $k = 1,2,3$, constitute a right-handed (left-handed) orthonormal triad, that is, if C is proper (improper) orthogonal. Thus,

$$\left(\mathsf{C}\,[\mathbf{u}\times]\,\mathsf{C}^{\mathrm{T}}\right)_{ij} = -(\det\mathsf{C})\sum_{k=1}^{3}\epsilon_{ijk}\,\mathbf{c}_k^T\mathbf{u}\,, \tag{59}$$

or

$$\mathsf{C}\,[\mathbf{u}\times]\,\mathsf{C}^{\mathrm{T}} = -(\det\mathsf{C})\begin{bmatrix} 0 & \hat{\mathbf{c}}_3^{\mathrm{T}}\mathbf{u} & -\hat{\mathbf{c}}_2^{\mathrm{T}}\mathbf{u} \\ -\hat{\mathbf{c}}_3^{\mathrm{T}}\mathbf{u} & 0 & \hat{\mathbf{c}}_1^{\mathrm{T}}\mathbf{u} \\ \hat{\mathbf{c}}_2^{\mathrm{T}}\mathbf{u} & -\hat{\mathbf{c}}_1^{\mathrm{T}}\mathbf{u} & 0 \end{bmatrix}. \tag{60}$$

But

$$\mathsf{C}\,\mathbf{u} = \begin{bmatrix} \hat{\mathbf{c}}_1^{\mathrm{T}} \\ \hat{\mathbf{c}}_2^{\mathrm{T}} \\ \hat{\mathbf{c}}_3^{\mathrm{T}} \end{bmatrix}\mathbf{u} = \begin{bmatrix} \hat{\mathbf{c}}_1^{\mathrm{T}}\mathbf{u} \\ \hat{\mathbf{c}}_2^{\mathrm{T}}\mathbf{u} \\ \hat{\mathbf{c}}_3^{\mathrm{T}}\mathbf{u} \end{bmatrix}, \tag{61}$$

so that, finally,

$$\mathsf{C}\,[\mathbf{u}\times]\,\mathsf{C}^{\mathrm{T}} = (\det\mathsf{C})\,[(\mathsf{C}\mathbf{u})\times]\,. \tag{62}$$

As an immediate consequence of (62), the attitude matrix $\mathsf{A}(\hat{\mathbf{n}},\theta)$ given an axis column vector $\hat{\mathbf{n}}$ and an angle of rotation θ [2, 3],

$$\mathsf{A}(\hat{\mathbf{n}},\,\theta) = \mathsf{I}_{3\times3} - (\sin\theta)\,[\hat{\mathbf{n}}\times] + (1-\cos\theta)\,[\hat{\mathbf{n}}\times]^2\,, \tag{63}$$

satisfies

$$\mathsf{C}\,\mathsf{A}(\hat{\mathbf{n}},\,\theta)\,\mathsf{C}^{\mathrm{T}} = \mathsf{A}(\mathsf{C}\hat{\mathbf{n}},\,\theta)\,. \tag{64}$$

Equation (62) implies also

$$\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,[{}^{\mathcal{A}}\mathbf{u}\times] = [{}^{\mathcal{B}}\mathbf{u}\times]\,\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,, \tag{65}$$

where $\mathsf{A}^{\mathcal{B}/\mathcal{A}}$ is the attitude matrix of basis $\mathcal{B}$ relative to basis $\mathcal{A}$. Equation (64) is presented often with the matrix C^{T} transposed to the right-hand side of the equation as C.

The Kinematics of the Attitude Matrix

For an arbitrary column vector ${}^{\mathcal{E}}\mathbf{u}$, examine the column-vector vector product ${}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\hat{\mathbf{e}}_i$, where ${}^{\mathcal{E}}\hat{\mathbf{e}}_i$ is an element of the autorepresentation ${}^{\mathcal{E}}\mathcal{E}$ of the right-handed orthonormal basis $\mathcal{E}$, defined by (4). We note for $i = 1$ that

$$\begin{aligned}
{}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\hat{\mathbf{e}}_1 &= [{}^{\mathcal{E}}\mathbf{u}\times]\,{}^{\mathcal{E}}\hat{\mathbf{e}}_1 \\
&= \begin{bmatrix} 0 & -{}^{\mathcal{E}}u_3 & {}^{\mathcal{E}}u_2 \\ {}^{\mathcal{E}}u_3 & 0 & -{}^{\mathcal{E}}u_1 \\ -{}^{\mathcal{E}}u_2 & {}^{\mathcal{E}}u_1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ {}^{\mathcal{E}}u_3 \\ -{}^{\mathcal{E}}u_2 \end{bmatrix} \\
&= -[{}^{\mathcal{E}}\mathbf{u}\times]_{11}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_1 - [{}^{\mathcal{E}}\mathbf{u}\times]_{12}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_2 - [{}^{\mathcal{E}}\mathbf{u}\times]_{13}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_3 \\
&= -\sum_{j=1}^{3} [{}^{\mathcal{E}}\mathbf{u}\times]_{1j}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_j \,, \qquad (66)
\end{aligned}$$

and similarly for ${}^{\mathcal{E}}\hat{\mathbf{e}}_2$ and ${}^{\mathcal{E}}\hat{\mathbf{e}}_3$. Thus,

$${}^{\mathcal{E}}\mathbf{u} \times {}^{\mathcal{E}}\hat{\mathbf{e}}_i = [{}^{\mathcal{E}}\mathbf{u}\times]\,{}^{\mathcal{E}}\hat{\mathbf{e}}_i = -\sum_{j=1}^{3} [{}^{\mathcal{E}}\mathbf{u}\times]_{ij}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_j \,, \qquad i = 1,2,3 \,. \qquad (67)$$

Equation (67) should be compared with equation (103) of [2].

It follows, for the physical basis $\mathcal{E} = \{\,\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3\,\}$, that

$$\boldsymbol{u} \times \hat{\boldsymbol{e}}_i = \{\,\boldsymbol{u}\times\,\}\,\hat{\boldsymbol{e}}_i = -\sum_{j=1}^{3} [{}^{\mathcal{E}}\mathbf{u}\times]_{ij}\,\hat{\boldsymbol{e}}_j \,, \qquad i = 1,2,3 \,, \qquad (68)$$

where $\{\,\boldsymbol{u}\times\,\}$ is the dyadic defined by (25). It follows, in particular, for a time-varying basis $\mathcal{E}(t)$ that

$$\frac{d}{dt}\hat{\boldsymbol{e}}_i(t) = \boldsymbol{\omega}^{\mathcal{E}}(t) \times \hat{\boldsymbol{e}}_i(t) = -\sum_{j=1}^{3} [{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}}(t)\times]_{ij}\,\hat{\boldsymbol{e}}_j(t) \,, \qquad i = 1,2,3 \,, \qquad (69)$$

and

$$\begin{aligned}
\frac{d}{dt}\,{}^{\mathcal{E}'}\hat{\mathbf{e}}_i(t) &= [{}^{\mathcal{E}'}\boldsymbol{\omega}^{\mathcal{E}}(t)\times]\,{}^{\mathcal{E}'}\hat{\mathbf{e}}_i(t) \\
&= -\sum_{j=1}^{3} [{}^{\mathcal{E}'}\boldsymbol{\omega}^{\mathcal{E}}(t)\times]_{ij}\,{}^{\mathcal{E}'}\hat{\mathbf{e}}_j(t) \,, \qquad i = 1,2,3 \,, \qquad (70)
\end{aligned}$$

where $\mathcal{E}'(t)$ is any other time-varying basis. The central expression and right-hand side of (67) should be compared with equations (96) and (97) of [2].

Consider now the attitude matrix relating a physical basis $\mathcal{A}$ to a physical basis $\mathcal{B}$

$$\mathsf{A}_{ij}^{\mathcal{B}/\mathcal{A}}(t) = \hat{\boldsymbol{b}}_i(t) \cdot \hat{\boldsymbol{a}}_j(t)\,, \tag{71}$$

where both $\mathcal{A}(t) = \{\hat{\boldsymbol{a}}_1(t), \hat{\boldsymbol{a}}_2(t), \hat{\boldsymbol{a}}_3(t)\}$ and $\mathcal{B}(t) = \{\hat{\boldsymbol{b}}_1(t), \hat{\boldsymbol{b}}_2(t), \hat{\boldsymbol{b}}_3(t)\}$ are time-varying. From (71), it follows, suppressing the time variable for clarity, that

$$\begin{aligned}\frac{d}{dt}\,\mathsf{A}_{ij}^{\mathcal{B}/\mathcal{A}} &= \left(\frac{d}{dt}\,\hat{\boldsymbol{b}}_i\right) \cdot \hat{\boldsymbol{a}}_j + \hat{\boldsymbol{b}}_i \cdot \left(\frac{d}{dt}\,\hat{\boldsymbol{a}}_j\right) \\ &= -\sum_{k=1}^{3} [{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}}\times]_{ik}\,\hat{\boldsymbol{b}}_k \cdot \hat{\boldsymbol{a}}_j - \sum_{k=1}^{3} [{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times]_{jk}\,\hat{\boldsymbol{a}}_i \cdot \hat{\boldsymbol{b}}_k \\ &= -\sum_{k=1}^{3} [{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}}\times]_{ik}\,\mathsf{A}_{kj}^{\mathcal{B}/\mathcal{A}} + \sum_{k=1}^{3} \mathsf{A}_{ik}^{\mathcal{B}/\mathcal{A}}\,[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times]_{kj}\,,\end{aligned} \tag{72}$$

or

$$\frac{d}{dt}\,\mathsf{A}^{\mathcal{B}/\mathcal{A}} = -[{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}}\times]\,\mathsf{A}^{\mathcal{B}/\mathcal{A}} + \mathsf{A}^{\mathcal{B}/\mathcal{A}}\,[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times]\,. \tag{73}$$

Recalling (62) and (65), we have

$$\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times] = [\mathsf{A}^{\mathcal{B}/\mathcal{A}}\;{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times]\,\mathsf{A}^{\mathcal{B}/\mathcal{A}} = [{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}}\times]\,\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,, \tag{74}$$

and

$$\begin{aligned}\frac{d}{dt}\,\mathsf{A}^{\mathcal{B}/\mathcal{A}} &= -\left[\left({}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}} - {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}}\right)\times\right]\mathsf{A}^{\mathcal{B}/\mathcal{A}} \\ &= -\left[{}^{\mathcal{B}}\!\left(\boldsymbol{\omega}^{\mathcal{B}} - \boldsymbol{\omega}^{\mathcal{A}}\right)\times\right]\mathsf{A}^{\mathcal{B}/\mathcal{A}} \\ &= -\mathsf{A}^{\mathcal{B}/\mathcal{A}}\left[{}^{\mathcal{A}}\!\left(\boldsymbol{\omega}^{\mathcal{B}} - \boldsymbol{\omega}^{\mathcal{A}}\right)\times\right].\end{aligned} \tag{75}$$

The Angular Velocity

The very beginning of this article, in (16), presents the angular velocity of a reference frame as absolute, that is, not relative to any other frame. Readers disturbed by this notion may find solace in this section, in which we show, among other things, that the absolute angular velocity is, *mirabile dictu*, equal the angular velocity relative to a non-rotating frame.

We call the physical vector

$$\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \equiv \boldsymbol{\omega}^{\mathcal{B}} - \boldsymbol{\omega}^{\mathcal{A}} \tag{76}$$

the *physical relative angular velocity vector* of frame $\mathcal{B}$ relative to frame $\mathcal{A}$. and we rewrite (75) as

$$\frac{d}{dt}\,\mathsf{A}^{\mathcal{B}/\mathcal{A}} = -[{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}\times]\,\mathsf{A}^{\mathcal{B}/\mathcal{A}} = -\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}\times]\,. \tag{77}$$

Equation (77) is often called the kinematic equation for the attitude matrix.

We can define the representation of the relative angular velocity with respect to any basis as

$$ {}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \equiv \mathsf{A}^{\mathcal{C}/\mathcal{B}}\, {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} . \tag{78} $$

Generally, the only useful values of ${}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$ are ${}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$ and ${}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$. We could also define

$$ {}^{\mathcal{C}}\mathsf{A}_{ij}^{\mathcal{B}/\mathcal{A}} = {}^{\mathcal{C}}\mathbf{b}_i \cdot {}^{\mathcal{C}}\mathbf{a}_j , \tag{79} $$

or

$$ {}^{\mathcal{C}}\mathsf{A}^{\mathcal{B}/\mathcal{A}} \equiv \mathsf{C}\,\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,\mathsf{C}^{\mathrm{T}} = \mathsf{A}({}^{\mathcal{C}}\hat{\mathbf{n}}^{\mathcal{B}/\mathcal{A}}, \theta) , \tag{80} $$

with C now right-handed orthonormal. ${}^{\mathcal{C}}\mathsf{A}^{\mathcal{B}/\mathcal{A}}$, like ${}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$, is useful only for $\mathcal{C} = \mathcal{A}$ or $\mathcal{C} = \mathcal{B}$, and we note

$$ {}^{\mathcal{A}}\mathsf{A}^{\mathcal{B}/\mathcal{A}} = {}^{\mathcal{B}}\mathsf{A}^{\mathcal{B}/\mathcal{A}} = \mathsf{A}^{\mathcal{B}/\mathcal{A}} , \tag{81} $$

as shown in [2].

From $\boldsymbol{\omega}^{\mathcal{N}} = \mathbf{0}$ for an irrotational frame $\mathcal{N}$, it follows from (76) that

$$ \boldsymbol{\omega}^{\mathcal{E}/\mathcal{N}} = \boldsymbol{\omega}^{\mathcal{E}} , \tag{82} $$

and the absolute angular velocity vector may be considered as the angular velocity vector relative to an irrotational frame. Trivially,

$$ \boldsymbol{\omega}^{\mathcal{A}/\mathcal{A}} = \mathbf{0} , \tag{83} $$

which follows also from $\mathsf{A}^{\mathcal{A}/\mathcal{A}} = \mathsf{I}_{3\times 3}$. From (76), it follows that

$$ \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} = -\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}} , \tag{84} $$

and

$$ \boldsymbol{\omega}^{\mathcal{C}/\mathcal{A}} = \boldsymbol{\omega}^{\mathcal{C}/\mathcal{B}} + \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} . \tag{85} $$

The equation

$$ {}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{C}/\mathcal{A}} = {}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{C}/\mathcal{B}} + \mathsf{A}^{\mathcal{C}/\mathcal{B}}\, {}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} , \tag{86} $$

which follows from $\mathsf{A}^{\mathcal{C}/\mathcal{A}} = \mathsf{A}^{\mathcal{C}/\mathcal{B}}\,\mathsf{A}^{\mathcal{B}/\mathcal{A}}$, is sometimes called the transport equation for angular velocities. It appears in the inverse kinematic equation for the Euler angles [3].

From [2],

$$ \mathsf{A}^{\mathcal{B}/\mathcal{A}}(t) = \exp\{-[\boldsymbol{\theta}^{\mathcal{B}/\mathcal{A}}(t)\times]\} , \tag{87} $$

but

$$ \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}(t) \neq \frac{d}{dt}\,\boldsymbol{\theta}^{\mathcal{B}/\mathcal{A}}(t) \tag{88} $$

over a finite time interval unless $\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$ is constant in direction over that interval and $\mathcal{A}(t)$ and $\mathcal{B}(t)$ coincide for some value of t in that interval. This results from the fact that $[(d/dt)\,\boldsymbol{\theta}^{\mathcal{B}/\mathcal{A}}\times]$ does not commute in general with $[\boldsymbol{\theta}^{\mathcal{B}/\mathcal{A}}\times]$ otherwise.

It is interesting to examine the temporal derivatives of the representations of the physical basis vectors. We note trivially that

$$\frac{d}{dt}\,{}^{\mathcal{N}}\hat{\mathbf{n}}(t) = \mathbf{0}, \qquad \frac{d}{dt}\,{}^{\mathcal{B}}\hat{\mathbf{b}}(t) = \mathbf{0}, \tag{89}$$

and, equally trivially,

$$\frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{b}(t) = {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{B}}(t) \times {}^{\mathcal{N}}\mathbf{b}(t) = {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{N}}(t) \times {}^{\mathcal{N}}\mathbf{b}(t), \tag{90}$$

which is the representation with respect to $\mathcal{N}$ of (7). But from

$${}^{\mathcal{B}}\hat{\mathbf{n}}(t) = \mathsf{A}^{\mathcal{B}/\mathcal{N}}(t)\,{}^{\mathcal{N}}\hat{\mathbf{n}}(t), \tag{91}$$

we have

$$\frac{d}{dt}\,{}^{\mathcal{B}}\hat{\mathbf{n}}(t) = -{}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{N}}(t) \times {}^{\mathcal{B}}\hat{\mathbf{n}}(t). \tag{92}$$

If $\mathcal{B}(t_1) = \mathcal{N}$ for some time t_1, and $\boldsymbol{\omega}^{\mathcal{B}/\mathcal{N}}$ is constant, then at all times,

$${}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{N}}(t) = {}^{\mathcal{N}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{N}}(t). \tag{93}$$

The Transport Equation for Physical Vectors

In this section we examine the traditional development of attitude kinematics. The earlier presentation of the development of the transport equation for a column-vector representation from the equation of motion of a physical vector, although it leads more directly to the equations of motion for ${}^{\mathcal{E}}\mathbf{r}(t)$, is not the traditional method, the method found in textbooks. We present that method here.

For any physical vector, we may write again (see (43))

$$\boldsymbol{r}(t) = \sum_{k=1}^{3} {}^{\mathcal{E}}r_k(t)\,\hat{\boldsymbol{e}}_k(t). \tag{94}$$

It follows again (see (44)) that

$$\frac{d}{dt}\boldsymbol{r}(t) = \sum_{k=1}^{3}\left(\frac{d}{dt}\,{}^{\mathcal{E}}r_k(t)\right)\hat{\boldsymbol{e}}_k(t) + \boldsymbol{\omega}^{\mathcal{E}}(t) \times \boldsymbol{r}(t) \equiv \boldsymbol{v}(t), \tag{95}$$

as in the earlier development above. Equation (95) is the *transport equation for a physical vector* $\boldsymbol{r}(t)$. Together with (18), it is the basis for the study of attitude dynamics. Again, $\boldsymbol{r}(t)$ need not be the position vector, and $\boldsymbol{v}(t) \equiv d\boldsymbol{r}(t)/dt$.

Equation (95) can be rewritten

$$\frac{d}{dt}\boldsymbol{r}(t) = \left(\frac{d}{dt}\right)^{\mathcal{E}}\boldsymbol{r}(t) + \boldsymbol{\omega}^{\mathcal{E}} \times \boldsymbol{r}(t) = \boldsymbol{v}(t), \tag{96}$$

where

$$\left(\frac{d}{dt}\right)^{\mathcal{E}} \boldsymbol{r}(t) \equiv \sum_{k=1}^{3} \left(\frac{d}{dt}\, {}^{\mathcal{E}}r_k(t)\right) \hat{\boldsymbol{e}}_k(t)\,. \tag{97}$$

Since $\boldsymbol{\omega}^{\mathcal{N}} = \boldsymbol{0}$, it follows that

$$\frac{d}{dt}\,\boldsymbol{r}(t) = \left(\frac{d}{dt}\right)^{\mathcal{N}} \boldsymbol{r}(t)\,. \tag{98}$$

The temporal derivative with respect to an irrotational reference frame is identical to the ordinary temporal derivative, as one would expect. The transport equation for physical vectors may be rewritten

$$\left(\frac{d}{dt}\right)^{\mathcal{N}} \boldsymbol{r}(t) = \left(\frac{d}{dt}\right)^{\mathcal{E}} \boldsymbol{r}(t) + \boldsymbol{\omega}^{\mathcal{E}} \times \boldsymbol{r}(t) = \boldsymbol{v}(t)\,. \tag{99}$$

Some authors call $(d/dt)^{\mathcal{E}}$ the *frame-dependent temporal derivative*, which we do not, because this phrase can also describe ${}^{\mathcal{E}}(\mathsf{d/dt})$. For $(d.dt)^{\mathcal{E}}$ we use only the appellation of the *temporal derivative with respect to a frame*. Equation (99) is the form of the transport equation that appears in most textbooks.

Noting that $\boldsymbol{r}(t)$ is an arbitrary physical vector in (99) and (99), we may write the operator equation

$$\left(\frac{d}{dt}\right)^{\mathcal{E}} = \frac{d}{dt} - \{\boldsymbol{\omega}^{\mathcal{E}}\times\}\,, \tag{100}$$

where $\{\boldsymbol{\omega}^{\mathcal{E}}\times\}$ is the dyadic defined by (25). Note the difference in sign of the omega-cross terms in (49) and (100), a reflection of the fact that if the physical basis vectors rotate in one sense, then the column-vector representations of other physical vectors "rotate" in the opposite sense. (See Figures 3 and 4 of [2].) This effective difference in sign can lead to error if $\boldsymbol{r}(t)$ is equated with ${}^{\mathcal{E}}\mathbf{r}(t)$.

To be more rigorous, we should note that $(d/dt)^{\mathcal{E}}$ is an operator on a physical vector space, and, therefore, we should write in preference

$$\left(\frac{d}{dt}\right)^{\mathcal{E}} = \mathcal{I}_{(3)}\,\frac{d}{dt} - \{\boldsymbol{\omega}^{\mathcal{E}}\times\}\,, \tag{101}$$

where $\mathcal{I}_{(3)}$, the identity dyadic in three dimensions, is defined in (21). (See our remarks following (52).) The temporal derivative with respect to a frame is examined in detail in a later section. There is no standard notation for the temporal derivative with respect to a frame. Reference [5], for example, uses the notation $\{\mathrm{d}/\mathrm{d}t\}_{\mathcal{E}}$, [6] prefers ${}^{\mathcal{E}}\mathrm{d}/\mathrm{d}t$. Some authors distinguish between $(d/dt)^{\mathcal{N}}$ and $(d/dt)^{\mathcal{B}}$, by placing a solid or empty circle over the vector.

Noting (100) or (101), it is easy to show that

$${}^{\mathcal{E}}\!\left(\left[\left(\frac{d}{dt}\right)^{\mathcal{E}}\right]^{n} \boldsymbol{r}(t))\right) = \left(\frac{d}{dt}\right)^{n} {}^{\mathcal{E}}\mathbf{r}(t)\,. \tag{102}$$

The representation of (96) with respect to $\mathcal{E}$ is (55). Thus, we achieve the same final result for the transport equation for column-vector representations, whether we employ ${}^{\mathcal{E}}(\mathsf{d}/\mathsf{dt})$ or $(d/dt)^{\mathcal{E}}$ in the development. Column vectors, we mention again, are what we must use in computer programing.

The Temporal Derivatives

As generally described in textbooks that treat rotational mechanics, the temporal derivative with respect to a frame differentiates only the components of a vector and not the basis vectors.

The *temporal derivative with respect to a frame* for a frame $\mathcal{E}(t) = \{\hat{\boldsymbol{e}}_1(t), \hat{\boldsymbol{e}}_2(t), \hat{\boldsymbol{e}}_3(t)\}$ is defined so that (recall (95))

$$\left(\frac{d}{dt}\right)^{\mathcal{E}} \boldsymbol{r}(t) = \sum_{k=1}^{3} \left(d\,{}^{\mathcal{E}}\mathbf{r}(t)/dt\right)_k \hat{\boldsymbol{e}}_k(t)\,. = \sum_{k=1}^{3} \left(\frac{d}{dt}\left(\hat{\boldsymbol{e}}_k^{\dagger}\boldsymbol{r}(t)\right)\right)_k \hat{\boldsymbol{e}}_k(t)\,, \tag{103}$$

which is equivalent to the operator equation

$$\left(\frac{d}{dt}\right)^{\mathcal{E}} \equiv \sum_{k=1}^{3} \hat{\boldsymbol{e}}_k(t)\,\frac{\overrightarrow{d}}{dt}\;\hat{\boldsymbol{e}}_k^{\dagger}(t)\,, \tag{104}$$

Equation (104) defines the temporal derivative with respect to a frame without needing to include an arbitrary physical vector $\boldsymbol{r}(t)$ in the definition.

The expression in (104) is a dyadic. The dual vector $\hat{\boldsymbol{e}}_k^{\dagger}$ picks out the kth component from the vector, the d/dt differentiates this component, and the vector $\hat{\boldsymbol{e}}_k(t)$ and the summation assemble the final physical vector. The arrow above d/dt indicates that the temporal derivative acts not only on $\hat{\boldsymbol{e}}_k^{\dagger}(t)$ but on anything that should follow it, even beyond a delimiter or on a following unwritten but implied physical-vector function. When there is no arrow, the action of the temporal derivative is limited by any surrounding delimiter, as in the (103) above. Equation (104) applied to a time-dependent physical vector $\boldsymbol{r}(t)$ yields (103).

If a basis $\mathcal{E}'(t)$ is related to the basis $\mathcal{E}(t)$ by a constant proper orthogonal transformation, that is, if the right-handed orthonormal basis $\mathcal{E}'(t)$ is at rest relative to $\mathcal{E}(t)$, then

$$\left(\frac{d}{dt}\right)^{\mathcal{E}'} = \left(\frac{d}{dt}\right)^{\mathcal{E}}\,, \tag{105}$$

which is related to (36), a similar invariance for the physical angular-velocity vector.

We note also that, similarly to (53)

$$\left(\left[\frac{d}{dt}\right]^n\right)^{\mathcal{E}} = \left[\left(\frac{d}{dt}\right)^{\mathcal{E}}\right]^n, \tag{106}$$

for any positive integer n.

For $\mathcal{N}$ a (physical) irrotational orthonormal basis, we have immediately, noting (42),

$$\left(\frac{d}{dt}\right)^{\mathcal{N}} = \sum_{k=1}^{3} \hat{\boldsymbol{n}}_k \, \frac{\overrightarrow{d}}{dt} \, \hat{\boldsymbol{n}}_k^{\dagger} = \sum_{k=1}^{3} \hat{\boldsymbol{n}}_k \, \hat{\boldsymbol{n}}_k^{\dagger} \, \frac{d}{dt} = \frac{d}{dt} \,, \tag{107}$$

the ordinary temporal derivative, since $\hat{\boldsymbol{n}}_k^{\dagger}$, like $\hat{\boldsymbol{n}}_k$, must be constant in time. Note that (107) does not state that ordinary temporal differentiation is, in origin or in essence. just differentiation with respect to an irrotational frame. Equation (107) is not true for the temporal derivative with respect to the body frame, because the $\hat{\boldsymbol{b}}_k$, $k = 1, 2, 3$, are time-dependent. In general, noting that $\overrightarrow{d/dt}$ in (104) operates both on $\hat{\boldsymbol{e}}^{\dagger}(t)$ and on any quantity that follows,

$$\begin{aligned}
\left(\frac{d}{dt}\right)^{\mathcal{E}} &= \frac{d}{dt} + \sum_{k=1}^{3} \hat{\boldsymbol{e}}_k(t) \left(\frac{d\hat{\boldsymbol{e}}_k^{\dagger}(t)}{dt} \right) \\
&= \frac{d}{dt} + \sum_{k=1}^{3} \hat{\boldsymbol{e}}_k(t) \left(\frac{d\hat{\boldsymbol{e}}_k(t)}{dt} \right)^{\dagger} \\
&= \frac{d}{dt} + \sum_{k=1}^{3} \hat{\boldsymbol{e}}_k(t) \left(\boldsymbol{\omega}^{\mathcal{E}}(t) \times \hat{\boldsymbol{e}}_k(t) \right)^{\dagger} \\
&= \frac{d}{dt} - \{ \boldsymbol{\omega}^{\mathcal{E}}(t) \times \} \,,
\end{aligned} \tag{108}$$

which is the same as (100).

It follows from (108) and (76) that

$$\left(\frac{d}{dt}\right)^{\mathcal{B}} = \left(\frac{d}{dt}\right)^{\mathcal{A}} - \{ \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \times \} \,. \tag{109}$$

Let us examine ${}^{\mathcal{A}}(d/dt)^{\mathcal{B}}$, the representation with respect to a basis $\mathcal{A}$ of $(d/dt)^{\mathcal{B}}$. The ith component of ${}^{\mathcal{A}}\{(d/dt)^{\mathcal{B}}\boldsymbol{r}(t)\}$ is

$$\begin{aligned}
{}^{\mathcal{A}}\!\left(\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{B}} \mathsf{r}(t) \right)_i &= \hat{\boldsymbol{a}}_i^{\dagger}(t) \left(\sum_{k=1}^{3} \hat{\boldsymbol{b}}_k \, \frac{\overrightarrow{d}}{dt} \, \hat{\boldsymbol{b}}_k^{\dagger} \right) \boldsymbol{r}(t) \\
&= \sum_{k=1}^{3} (\hat{\boldsymbol{a}}_i(t) \cdot \hat{\boldsymbol{b}}_k) \, \frac{d}{dt} \, {}^{\mathcal{B}}r_k(t) \\
&= \left(\mathsf{A}^{\mathcal{A}/\mathcal{B}}(t) \, \frac{\overrightarrow{d}}{dt} \, \mathsf{A}^{\mathcal{B}/\mathcal{A}}(t) \, {}^{\mathcal{A}}\mathsf{r}(t) \right)_i \\
&= \left[{}^{\mathcal{A}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{B}} {}^{\mathcal{A}}\mathsf{r}(t) \right]_i \,.
\end{aligned} \tag{110}$$

Thus,

$$ {}^{\mathcal{A}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{B}} = \mathsf{A}^{\mathcal{A}/\mathcal{B}}(t) \, \frac{\overrightarrow{d}}{dt} \, \mathsf{A}^{\mathcal{B}/\mathcal{A}}(t) \,. \tag{111}$$

The matrix representation of the temporal differentiation operator with respect to a frame first transforms the representation of the column vector from the frame of representation to the frame of differentiation, then carries out the ordinary (frame-independent) differentiation, and, finally, transforms the differentiated column vector back to the frame of representation. This sequence of steps should be compared with that for the temporal derivative operator in (104). Carrying out the differentiation of the rightmost direction-cosine matrix of (111) yields

$$ {}^{\mathcal{A}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{B}} = \frac{d}{dt} + [{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}}(t)\times] . \tag{112} $$

Similarly to (109),

$$ {}^{\mathcal{C}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{B}} = {}^{\mathcal{C}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{A}} + [\,{}^{\mathcal{C}}\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}}(t)\times\,] , \tag{113} $$

Note that ${}^{\mathcal{A}}(\mathsf{d/dt})$ and ${}^{\mathcal{A}}(\mathsf{d/dt})^{\mathcal{B}}$ are useful only when applied to column-vector representations with respect to the basis $\mathcal{A}$. It follows from (112) that

$$ {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{N}} = {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right) , \tag{114} $$

and, therefore, noting (113),

$$ {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right) = \mathsf{A}^{\mathcal{E}/\mathcal{N}}(t)\, \frac{\overrightarrow{d}}{dt}\, \mathsf{A}^{\mathcal{N}/\mathcal{E}}(t) . \tag{115} $$

We now have expressions for all three auxiliary temporal derivatives in terms of $\overrightarrow{d/dt}$. It follows trivially that

$$ {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{E}} = \frac{d}{dt} . \tag{116} $$

The representation of (96) with respect of $\mathcal{E}(t)$ leads directly in a formal manner to (55), making $({}^{\mathcal{A}}\mathsf{d/dt})^{\mathcal{B}}$ a sort of missing link between $(\mathsf{d/dt})^{\mathcal{B}}$ and ${}^{\mathcal{A}}(\mathsf{d/dt})$. The four temporal derivatives are summarized in Table 1.

In summary, we note the following results for the action of the various temporal derivatives on an element $\hat{\boldsymbol{e}}_k(t)$, $k = 1, 2, 3$, of the tight-handed orthonormal basis $\mathcal{E}(t)$ and for any other right-handed orthonormal basis $\mathcal{B}$.

$$ \frac{d}{dt}\hat{\boldsymbol{e}}_k(t) = \boldsymbol{\omega}^{\mathcal{E}} \times \hat{\boldsymbol{e}}_k(t) , \tag{117} $$

$$ \frac{d}{dt}\,{}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) = \mathbf{0} , \tag{118} $$

$$ {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right) {}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) = {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}/\mathcal{N}} \times {}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) , \tag{119} $$

$$ \left(\frac{d}{dt}\right)^{\mathcal{E}} \hat{\boldsymbol{e}}_k(t) = \boldsymbol{0} , \tag{120} $$

$$ {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{B}} {}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) = {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}/\mathcal{B}} \times {}^{\mathcal{E}}\hat{\mathbf{e}}_k(t) \,. \tag{121} $$

Equations (117) and (118) are just (18) and (33), respectively. Equation (119) follows trivially from (32). Equation (120) follows easily from the definition, (104). Equation (121) follows directly from (112) and (118).

Note that while ${}^{\mathcal{E}}(\mathsf{d/dt})$ and ${}^{\mathcal{E}}(\mathsf{d/d})^{\mathcal{B}}$ act on the column vector ${}^{\mathcal{E}}\hat{\mathbf{e}}_k(t)$ externally, internally inside the operator, it is really the physical vector $\hat{\boldsymbol{e}}(t)$ that is being differentiated. Likewise, while $(d/dt)^{\mathcal{E}}$ acts on the physical vector $\hat{\boldsymbol{e}}_k(t)$ externally, internally inside the operator, it is really the column vector ${}^{\mathcal{E}}\hat{\mathbf{e}}(t)$ that is being differentiated.

·	Ordinary	WRT a Frame
Physical	$\frac{d}{dt}$	$\left(\frac{d}{dt}\right)^{\mathcal{E}} = \frac{d}{dt} - \{\boldsymbol{\omega}^{\mathcal{E}} \times\}$
Matricial	${}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right) = \frac{d}{dt} + [{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}} \times]$	${}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right)^{\mathcal{E}'} = \frac{d}{dt} + [{}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}/\mathcal{E}'} \times]$

Table 1. Ordinary Temporal Derivatives and Temporal Derivatives with Respect to (WRT) a Frame. The matricial operators are the matrix representations of the physical operators. $\mathcal{E}(t)$ is a rotating frame. $\mathcal{E}'(t)$ is any other frame. Note that all four temporal derivatives become identical when $\mathcal{E}$ and $\mathcal{E}'$ are irrotational frames.

Attitude Dynamics

We can apply the same methods developed for the kinematic equation of motion to the development of the dynamical equation of motion for column-vector representations. Within the context of this article, the dynamical equation of motion is just one more temporal differential equation.

We know that the physical equation of motion for the angular momentum of a rigid system of particles can be written as

$$ \frac{d}{dt}\boldsymbol{L}(t) = \boldsymbol{N}(t) \,, \tag{122} $$

where $\boldsymbol{L}(t)$ is the total physical angular momentum vector of the system and $\boldsymbol{N}(t)$ is the total physical torque vector. For the representation ${}^{\mathcal{E}}\mathbf{L}(t)$ with respect to a basis $\mathcal{E}(t)$, we can apply (48) and (49) to (122)

$$ \frac{d}{dt}\,{}^{\mathcal{E}}\mathbf{L}(t) + {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}/\mathcal{N}}(t) \times {}^{\mathcal{E}}\mathbf{L}(t) = A^{\mathcal{E}/\mathcal{N}}\,\frac{d}{dt}\,{}^{\mathcal{N}}\mathbf{L}(t) \,, \tag{123} $$

where again $\mathcal{N}$ is an irrotational basis. Thus, for irrotational- and body-referenced representations of the angular momentum vector, we obtain the familiar Euler equations,

$$\frac{d}{dt}\,{}^{\mathcal{N}}\mathsf{L}(t) = {}^{\mathcal{N}}\mathsf{N}(t)\,, \tag{124}$$

$$\frac{d}{dt}\,{}^{\mathcal{E}}\mathsf{L}(t) + {}^{\mathcal{E}}\boldsymbol{\omega}^{\mathcal{E}/\mathcal{N}} \times {}^{\mathcal{E}}\mathsf{L}(t) = {}^{\mathcal{E}}\mathsf{N}(t)\,. \tag{125}$$

For a rigid body, $\mathcal{E}$ is usually $\mathcal{B}$, the body frame. For any physical vector function of time $\boldsymbol{f}(t)$, we have that the representation of $d\boldsymbol{f}(t)/dt$ with respect to $\mathcal{E}(t)$ is

$${}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\,\mathsf{f}(t)\right) = {}^{\mathcal{E}}\!\left(\frac{\mathsf{d}}{\mathsf{dt}}\right){}^{\mathcal{E}}\mathsf{f}(t)\,, \tag{126}$$

so that ${}^{\mathcal{E}}(\mathsf{d}/\mathsf{dt})$ truly is the representation of d/dt with respect to $\mathcal{E}(t)$.

The Kinematics of the Attitude Dyadic

The attitude dyadic is given by [2] as

$$\mathcal{A}^{\mathcal{B}/\mathcal{A}} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{B}/\mathcal{A}}\,\hat{\boldsymbol{a}}_i\,\hat{\boldsymbol{a}}_j^{\dagger} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathsf{A}_{ij}^{\mathcal{B}/\mathcal{A}}\,\hat{\boldsymbol{b}}_i\,\hat{\boldsymbol{b}}_j^{\dagger} = \sum_{k=1}^{3} \hat{\boldsymbol{a}}_k\,\hat{\boldsymbol{b}}_k^{\dagger}\,. \tag{127}$$

It follows from (23) and (24) that

$$\frac{d}{dt}\mathcal{A}^{\mathcal{B}/\mathcal{A}} = \{\boldsymbol{\omega}^{\mathcal{A}}\times\}\,\mathcal{A}^{\mathcal{B}/\mathcal{A}} - \mathcal{A}^{\mathcal{B}/\mathcal{A}}\,\{\boldsymbol{\omega}^{\mathcal{B}}\times\}\,, \tag{128}$$

which should be compared with (73).

Since $\mathcal{A}^{\mathcal{B}/\mathcal{A}}$ is invertible, we can write

$$\frac{d}{dt}\mathcal{A}^{\mathcal{B}/\mathcal{A}} = -\mathcal{A}^{\mathcal{B}/\mathcal{A}}\,\Psi^{\mathcal{B}/\mathcal{A}}\,, \tag{129}$$

with

$$\Psi^{\mathcal{B}/\mathcal{A}} \equiv -\left(\mathcal{A}^{\mathcal{B}/\mathcal{A}}\right)^{\dagger}\left(\frac{d}{dt}\mathcal{A}^{\mathcal{B}/\mathcal{A}}\right)\,. \tag{130}$$

From

$$\left(\mathcal{A}^{\mathcal{B}/\mathcal{A}}\right)^{\dagger}\mathcal{A}^{\mathcal{B}/\mathcal{A}} = \mathcal{I}_{(3)}\,. \tag{131}$$

It follows that

$$\Psi^{\mathcal{B}/\mathcal{A}} = -\left(\Psi^{\mathcal{B}/\mathcal{A}}\right)^{\dagger} = -\Psi^{\mathcal{A}/\mathcal{B}}\,, \tag{132}$$

and, therefore, there exists a physical vector $\boldsymbol{\psi}^{\mathcal{B}/\mathcal{A}}$, for which

$$\Psi^{\mathcal{B}/\mathcal{A}} = \{\boldsymbol{\psi}^{\mathcal{B}/\mathcal{A}}\times\}\,. \tag{133}$$

From (129),

$$\Psi^{\mathcal{B}/\mathcal{A}} = \{\boldsymbol{\omega}^{\mathcal{B}}\times\} - \left(\mathscr{A}^{\mathcal{B}/\mathcal{A}}\right)^{\dagger} \{\boldsymbol{\omega}^{\mathcal{A}}\times\}\,\mathscr{A}^{\mathcal{B}/\mathcal{A}} . \tag{134}$$

Up to now, the development of the kinematic equation for the attitude dyadic looks very much like that for the attitude matrix. However, noting (25), writing

$$\{\boldsymbol{\omega}^{\mathcal{A}}\times\} = \sum_{i=1}^{3}\sum_{j=1}^{3} [{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times]_{ij}\,\hat{\boldsymbol{a}}_i\,\hat{\boldsymbol{a}}_j^{\dagger} , \tag{135}$$

and, inserting (136) into (134), we obtain

$$\begin{aligned}\Psi^{\mathcal{B}/\mathcal{A}} &= \{\boldsymbol{\omega}^{\mathcal{B}}\times\} - \sum_{i=1}^{3}\sum_{j=1}^{3} [{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\times]_{ij}\,\hat{\boldsymbol{b}}_i\,\hat{\boldsymbol{b}}_j \\ &= \sum_{i=1}^{3}\sum_{k=1}^{3} \left[\left({}^{\mathcal{B}}\boldsymbol{\omega}^{\mathcal{B}} - {}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{A}}\right)\times\right]_{ij}\,\hat{\boldsymbol{b}}_i\,\hat{\boldsymbol{b}}_j^{\dagger} ,\end{aligned} \tag{136}$$

or

$$\begin{aligned}\boldsymbol{\psi}^{\mathcal{B}/\mathcal{A}} &= \boldsymbol{\omega}^{\mathcal{B}} - \sum_{k=1}^{3} {}^{\mathcal{A}}\omega_k^{\mathcal{A}}\,\hat{\boldsymbol{b}}_k \\ &= \boldsymbol{\omega}^{\mathcal{B}} - \mathscr{A}^{\mathcal{A}/\mathcal{B}} \sum_{k=1}^{3} {}^{\mathcal{A}}\omega_k^{\mathcal{A}}\,\hat{\boldsymbol{a}}_k \\ &= \boldsymbol{\omega}^{\mathcal{B}} - \mathscr{A}^{\mathcal{A}/\mathcal{B}}\boldsymbol{\omega}^{\mathcal{A}} ,\end{aligned} \tag{137}$$

which should be compared with

$$\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} = \boldsymbol{\omega}^{\mathcal{B}} - \boldsymbol{\omega}^{\mathcal{A}} = \boldsymbol{\omega}^{\mathcal{B}} - \sum_{k=1}^{3} {}^{\mathcal{B}}\omega_k^{\mathcal{A}}\,\hat{\boldsymbol{b}}_k . \tag{138}$$

Comparison of (137) and (134) leads to the relation

$$\mathscr{A}^{\mathcal{A}/\mathcal{B}}\,\{\boldsymbol{u}\times\}\,\left(\mathscr{A}^{\mathcal{A}/\mathcal{B}}\right)^{\dagger} = \left\{\left(\mathscr{A}^{\mathcal{A}/\mathcal{B}}\boldsymbol{u}\right)\times\right\} . \tag{139}$$

for an arbitrary physical vector $\boldsymbol{u}$.

We observe that

$$\begin{aligned}\frac{d}{dt}\mathscr{A}^{\mathcal{B}/\mathcal{A}} &= -\mathscr{A}^{\mathcal{B}/\mathcal{A}}\,\{\boldsymbol{\psi}^{\mathcal{B}/\mathcal{A}}\times\} \\ &= -\mathscr{A}^{\mathcal{B}/\mathcal{A}}\,\left\{\left(\boldsymbol{\omega}^{\mathcal{B}} - \sum_{k=1}^{3} {}^{\mathcal{A}}\omega_k^{\mathcal{A}}\,\hat{\boldsymbol{b}}_k\right)\times\right\} ,\end{aligned} \tag{140}$$

but

$$\boldsymbol{\psi}^{\mathcal{B}/\mathcal{A}} \neq \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} . \tag{141}$$

The correspondence of $\boldsymbol{\psi}^{\mathcal{B}/\mathcal{A}}$ to $\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$ is complex, but we know from the comparison of the right-hand sides of (128) and (77) that the representation of $(d/dt)\mathscr{A}^{\mathcal{B}/\mathcal{A}}$ with respect to the prior basis $\mathcal{A}$ or with respect to the posterior basis $\mathcal{B}$ must be $(d/dt)\,\mathsf{A}^{\mathcal{B}/\mathcal{A}}$ without an "omega-cross" term. It follows that the representation of $\mathscr{A}^{\mathcal{B}/\mathcal{A}}\{\psi^{\mathcal{B}/\mathcal{A}}\times\}$ with respect to the basis $\mathcal{A}$ must be $\mathsf{A}^{\mathcal{B}/\mathcal{A}}\,[{}^{\mathcal{A}}\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}\times]$.

Discussion

Our study of attitude kinematics has given equal attention to physical vectors and to column-vector representations. We have treated in detail material normally passed over briefly in textbooks. In the context of sharply distinguished physical vectors and their column-vector representations, we have examined two approaches to the development of the transport equation for column vectors. In the first, which is new in this article, the transition from physical vectors to column vectors is made as soon as possible; in the second it is not made until the very end. Both approaches are correct. The difference is largely one of order. However, the analytical properties of column vectors are more pedestrian and better known than hose of physical vectors, and the study of attitude kinematics is simpler if one treats column vectors rather than if one treats physical vectors. Reference [3] treats only column vectors in the section on attitude kinematics. The treatment of the attitude dyadic, a physical transformation, is complex and unintuitive compared to that of the attitude matrix.

In spacecraft mission support work and in simulation in general, the attitude matrix is replaced generally by the attitude quaternion [3], or, for short time intervals, by the rotation vector [3].

Our treatment has given greater emphasis to the basic ordinary temporal derivative and has presented also three other forms of the temporal derivative, one acting only on column vectors; the other two acting only on physical vectors. The ordinary temporal derivative is not the derivative with respect to an absolute frame. It is, intrinsically, not the derivative with respect to any frame at all.

The genesis of the temporal derivative with respect to a frame occurred in the late eighteenth and early nineteenth centuries, at a time when matrix algebra had not yet been invented. Matrices, determinants, and their use in the solution of linear equations had been discovered by the Chinese more than 2000 years ago [12], but until the second half of the nineteenth century their use in the West was confined largely to display. The name "matrix" originated only in 1850 with James Joseph Sylvester [13]. (*Matrix* means "womb" in Latin, and this fact elicited the complaint of embarrassed Oxford mathematician Charles L. Dodgson, better known to the world as Lewis Carroll, the author of *Alice in Wonderland*, who proposed the more properly Victorian word "block.") Matrix Algebra, in particular, the multiplication rule for matrices, was the invention of Arthur Cayley in 1855 [14], but matrices did not "catch on" in Physics for almost a half century after Cayley's discovery. The use of vectors was made popular by Gibbs only in 1904 [15], and the first serious book in English on Matrix Analysis appeared only well into the twentieth century [16]. The most important results of rotational mechanics and the temporal derivative with respect to a frame were developed in a world devoid of the enormous mathematical apparatus taken for granted by engineers and physicists today, a world in which it did not make sense to speak of the identification of physical vectors with their column-vector representations, because these concepts had not found root in current thinking. This was a world in which the notation for a function, $f(x)$, had been invented (by Euler) in living memory. Two centuries ago, certainly, all column-vectors were perceived as "inertial," and the temporal derivative with respect to a frame was, from the perspective of this article, a device for representing the temporal derivative of ${}^{\mathcal{E}}\mathbf{r}(t)$ as a physical vector in terms of irrotational axes.

The world of early nineteenths-century mathematics and physics is not our world. We cannot with certainty understand the minds of the great mathematicians and physicists of that era. Part of the persistence of the temporal derivative with respect to a frame in rotational mechanics, this writer believes, is due to intellectual inertia. Our world progresses very slowly. In 1900, when Einstein graduated from the Eidgenossische Technische Hochschule in Zurich, Maxwell's equations, published four decades earlier. were still not taught there [17].

Another reason for the persistence of the temporal derivative with respect to a frame is that Physics looks upon the world from a largely inertial perspective and physicists, in general, do identify physical vectors with column vectors. Only in rotational mechanics, where one must treat representations with respect to different reference frames, is this identification a problem, but only a very small problem, because the temporal derivative with respect to a frame, as we have seen, is a rigorous concept. The Theory of Relativity, of course, deals almost entirely with column-vector representations in space-time, although coordinate-free approaches also exist [18], but in the Theory of Relativity, the finiteness of the speed of light precludes the existence of rigid bodies. In Quantum Mechanics, obviously, one never considers a coordinate system fixed in an elementary particle, because the intrinsic spin of an elementary particle is not associated with describable internal rotational motion, which, furthermore, can lead to angular momenta of only integral multiples of $\hbar$. In Classical Physics, the minor difficulty associated with multiple bases is confined to rotational mechanics. Not surprisingly, physicists are content to maintain the *status quo* of the presentation of rotational mechanics (for most of them, a dead topic) rather than rewrite the textbooks. The aesthetic desires of a small minority in Astronautics cannot alter that situation. It goes without saying that Astrodynamics is highly influenced by Classical Mechanics. Thus, the two approaches to rotational mechanics presented in this article, which differ largely only in when the transition to column matrices is enacted, will likely endure for some time.

To borrow a phrase from Shakespeare's *Romeo and Juliet*, attitude kinematics and dynamics is characterized by "two households, both alike in dignity" [19]. We know how that story ends.

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