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Filter QUEST or REQUEST

Malcolm D. Shuster*
Acme Spacecraft Company
Germantown, Maryland 20874

Introduction

In 1989, the author published a implementation of the Wahba Problem [1] for dynamical systems as a sequential filter and smoother [2], These were called Filter QUEST and Smoother QUEST. The implementations of the filter and smoother were based on the Rauch-Tung-Striebel formulation [3] of the Kalman filter as a maximum-likelihood estimator assuming Gaussian noise. Smoother QUEST, naturally, was a Rauch-Tung-Striebel smoother. Filter QUEST was a candidate algorithm for the onboard attitude determination system of the Midcourse Space Experiment (MSX) [4], in planning in the late 1980s. Unfortunately, it failed to meet the accuracy specification by a factor of 2 and was removed from consideration. In the end that mission adopted the QUEST Filter algorithm[1] [5, 6], which was the standard attitude Kalman filter [7] but used the single-frame star-tracker attitude quaternion (computed using QUEST [8]) as an effective attitude measurement, rather than processing individual measurements of star directions. The QUEST Filter has since become the attitude Kalman filter implementation of choice for near-Earth and deep-space missions employing star trackers [9].

In 1996, the REQUEST algorithm [10], closely related to Filter QUEST, was published. The publication of the REQUEST smoother [11], equally closely related to Smoother QUEST, occurred in the following year. The present work examines the similarities of these later algorithms with Filter QUEST and Smoother QUEST.

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*Director of Research. mdshuster@comcast.net. website: http://home.comcast.net/~mdshuster.
[1]Not to be confused with the Filter QUEST algorithm [2].

The Wahba Problem and QUEST

Both Filter QUEST and REQUEST are based on the QUEST solution of the Wahba Problem. We shall review only the most important aspects of the Wahba problem and QUEST needed for an understanding of the present work.

The Wahba problem posed effectively as optimization criterion the minimization of a least-square cost function of the form

$$J(A) = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{\sigma_k^2} |\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k|^2 \tag{1}$$

Here A is the attitude matrix, that is, the direction-cosine matrix, $\hat{\mathbf{W}}_k$, $k = 1, \ldots, n$, are the observation vectors, the measured vectors observed with respect to the spacecraft body frame, $\hat{\mathbf{V}}_k$, $k = 1, \ldots, n$, are the reference vectors, the same vectors but with components given with respect to the inertial reference frame, and σ_k, $k = 1, \ldots, n$, are the variance parameters of the QUEST measurement model [8, 12]. Reference [12] showed also that the Wahba cost function was the data-dependent part or the negative-log-likelihood function [13] for the QUEST measurement model.

The Wahba cost function can be written also as

$$J(A) = \text{const} - g(A) \tag{2}$$

where $g(A)$ is the gain function

$$g(A) = \text{tr}\left(B^T A\right) \tag{3}$$

with B, the attitude profile matrix, given by

$$B = \sum_{k=1}^{n} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \tag{4}$$

The value of A which minimizes $J(A)$ maximizes $g(A)$. All solutions of the Wahba problem begin by constructing B. The many solution methods of the Wahba problem have been reviewed by Markley and Mortari [14].[2]

QUEST [8] is a special case of the Davenport q-algorithm. Davenport [8, 14, 17] showed that the gain function could be rewritten in terms of the quaternion $\bar{q}$ as

$$g(\bar{q}) \equiv g(A(\bar{q})) = \bar{q}^T K \bar{q} \tag{5}$$

where the Davenport matrix K is given by

$$K = \begin{bmatrix} S - sI_{3\times 3} & \mathbf{Z} \\ \mathbf{Z}^\mathrm{T} & s \end{bmatrix} \tag{6}$$

[2] On the simulation results of reference [14], note references [15] and [16].

with

$$S \equiv B + B^{\mathrm{T}}, \quad s \equiv \mathrm{tr}\, B \quad \text{and} \quad \mathbf{Z} \equiv [B_{23} - B_{32},\, B_{31} - B_{13},\, B_{12} - B_{21}]^{\mathrm{T}} \tag{7abc}$$

As a result, the minimization of $J_A(A)$ or, equivalently, the maximization of $g_A(A)$, could be accomplished by finding the solution of the characteristic-value problem

$$K\bar{q}^* = \lambda_{\max}\bar{q}^* \tag{8}$$

subject to the constraint $\bar{q}^{\mathrm{T}}\bar{q} = 1$. Here, $\lambda_{\max}$ is the largest characteristic value of the 4×4 real-symmetric matrix K and is also the maximum value of $g_{\bar{q}}(\bar{q})$ and $g_A(A)$. The QUEST algorithm is distinguished by a very fast method for computing $\bar{q}^*$ from K. The details of the QUEST calculation are not interesting for the present work. Of particular interest, however, is the fact that K is a linear homogeneous function of B, that is

$$K(B_1 + B_2) = K(B_1) + K(B_2) \tag{9}$$

Filter QUEST

The Filter QUEST algorithm takes advantage of the form of B given by equation (4). If the measurements are accumulated sequentially, then defining the "current" B_k as

$$B_k = \sum_{i=1}^{k} \frac{1}{\sigma_i^2}\, \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T \tag{10}$$

We can calculate B_k, $k = 1, \ldots, n$, sequentially as

$$B_{\mathrm{o}} = 0 \tag{11a}$$

$$B_k = B_{k-1} + \Delta B_k\,, \quad k = 1, \ldots, n \tag{11b}$$

with

$$\Delta B_k = \frac{1}{\sigma_k^2}\, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T\,, \quad k = 1, \ldots, n \tag{12}$$

If the attitude is dynamic and changing according to

$$A_k = \Phi_{k-1} A_{k-1}\,, \qquad k = 1, \ldots, n \tag{13}$$

then it follows from the invariance of equation (3) under a proper orthogonal transformation of the body coordinate axes that B must be changing in a corresponding fashion. In this case, we may obtain (non-static) $A^*_{k,k-1}$, $A^*_{k,k}$ and their associated covariance matrices as a function of k by the application of the QUEST algorithm to the sequence of attitude profile matrices[3]

$$B_{\mathrm{o,o}} = 0 \tag{14a}$$

$$B_{k,k-1} = \Phi_{k-1}\, B_{k-1,k-1}\,, \quad k = 1, \ldots, n \tag{14b}$$

$$B_{k,k} = B_{k,k-1} + \Delta B_k\,, \quad k = 1, \ldots, n \tag{14c}$$

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[3]It is not necessary to calculate these quantities for every $B_{k,k-1}$ and $B_{k,k}$.

If there is *a priori* knowledge of the attitude consisting of an initial estimate $A^*(-)$ and associated attitude estimate error covariance matrix $P_{\theta\theta}(-)$, then the sequence of attitude profile matrices is initialized as [12]

$$B_{\rm o,o} \equiv B(-) = \left[\frac{1}{2}\mathrm{tr}\left(P_{\theta\theta}^{-1}(-)\right) I_{3\times 3} - P_{\theta\theta}^{-1}(-)\right] A^*(-) \tag{15}$$

For the definition of $P_{\theta\theta}(-)$, see reference [12]. There is no straightforward way to include process noise in the Wahba problem. Reference [2] approximated process noise by replacing equation (14b) with

$$B_{k,k-1} = \alpha\,\Phi_{k-1}\,B_{k-1,k-1}\,, \quad k = 1,\,\ldots\,,\,n \tag{16}$$

where α, $0 < \alpha \leq 1$ is a fading-memory factor. Equations (14) through (16) constitute the Filter QUEST algorithm. Reference [2] gave an algorithm for calculating α heuristically from the steady-state predicted and updated covariance matrices of the standard Kalman filter. For a simple model Filter QUEST performed almost as well as the standard attitude Kalman filter. For more realistic data, however, the estimate error levels (in standard deviation) for Filter QUEST were about twice those of the standard attitude Kalman filter and outside mission requirements, and so Filter QUEST was abandoned for practical mission support.

REQUEST

In the REQUEST algorithm [10] was published seven years after Filter QUEST. Reference [10] remarks that the update step for the attitude profile matrix B can be written equivalently in terms of the Davenport matrix K as[4]

$$K_{k,k} = K_{k,k-1} + \Delta K_k\,, \quad k = 1,\,\ldots\,,\,n \tag{17}$$

with

$$\Delta K_k = \frac{1}{\sigma_k^2}\begin{bmatrix} \Delta S_k - \Delta s_k I_{3\times 3} & \Delta\mathbf{Z}_k \\ \Delta\mathbf{Z}_k^{\rm T} & \Delta s_k \end{bmatrix} \tag{18}$$

which is obvious from equation (9). Since the gain function of equation (5) is invariant under a proper orthogonal transformation of the spacecraft-body coordinate axes, one has for the prediction step for the sequential construction of K

$$K_{k,k-1} = \alpha\,\{\bar\varphi_{k-1}\}_L\,K_{k-1,k-1}\{\bar\varphi_{k-1}\}_L^T\,, \quad k = 1,\,\ldots\,,\,n \tag{19}$$

where $\bar\varphi_k$ is the quaternion [18] corresponding to Φ_k. Here, the 4×4 proper orthogonal matrix $\{\bar p\}_L$ is defined by [18]

$$\bar p \circ \bar q = \{\bar p\}_L\,\bar q \tag{20}$$

[4]Reference [10] presents REQUEST in more cumbersome form than this.

and "$\circ$" is the binary operation of quaternion composition (multiplication), and quarternion multiplication satisfies [18]

$$A(\bar{p})\, A(\bar{q}) = A(\bar{p} \circ \bar{q}) \tag{21}$$

where A is the attituee matrix. Explicitly [18],

$$\{\, \bar{p} \,\}_L \equiv \begin{bmatrix} p_4 & p_3 & -p_2 & p_1 \\ -p_3 & p_4 & p_1 & p_2 \\ p_2 & -p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{bmatrix} \tag{22}$$

Equation (18) does not, in fact, present REQUEST exactly as in reference [10]. Reference [10] prefers to use unit-sum weights a_k, $k = 1, \ldots, n$, rather than inverse-variances, with the result that REQUEST must give an additional non-linear recursion relation for the a_k. Beyond this, reference [10] simply repeats the derivation of Filter QUEST in reference [2] in terms of K and presents a less informative simulation than did reference [2].

A Closer Look at Filter QUEST and REQUEST

The similarity of the Filter QUEST and REQUEST algorithm is closer than presented in the previous section. In order to investigate the similarity, we must consider in each case where the filter ends and QUEST begins.

Let us examine QUEST alone. We may divide the operations in QUEST into three broad steps: (1) the computation of B from the vector data; (2) the computation of K from B; and (3) the computation of the attitude quaternion from K. Thus, we may summarize QUEST as

$$\{\hat{\mathbf{W}}_k,\, \hat{\mathbf{V}}_k,\, \sigma_k^2 \,|\, k = 1, \ldots, n\} \rightarrow B \rightarrow K \rightarrow \bar{q}^* \tag{23}$$

The steps in Filter QUEST and REQUEST may each be separated into two groups of operations: B-filter and B-QUEST for Filter QUEST and K-filter and K-QUEST for REQUEST. With these distinctions, one may analyze the steps of Filter QUEST and REQUEST as follows (we present only the update steps for clarity). For k, $k = 1, \ldots, n$,

Filter QUEST

$$\text{B-filter;} \quad \{\hat{\mathbf{W}}_k,\, \hat{\mathbf{V}}_k,\, \sigma_k^2,\, B_{k-1}\} \rightarrow \Delta B_k,\; B_{k-1} \rightarrow B_k \tag{24a}$$

$$\text{B-QUEST:} \quad B_k \rightarrow K_k \rightarrow \bar{q}_k^* \tag{24b}$$

REQUEST

$$\text{K-filter;} \quad \{\hat{\mathbf{W}}_k,\, \hat{\mathbf{V}}_k,\, \sigma_k^2,\, K_{k-1}\} \rightarrow \Delta B_k,\; K_{k-1} \rightarrow \Delta K_k,\; K_{k-1} \rightarrow K_k \tag{25a}$$

$$\text{K-QUEST:} \quad K_k \rightarrow \bar{q}_k^* \tag{25b}$$

The sequence of steps in Filter QUEST and REQUEST is *identical* except for the simple substitution (part of the middle rightarrow in equation (25a))

$$K(B_{k,k-1} + \Delta B_k) \rightarrow K(B_{k,k-1}) + K(\Delta B_k) \tag{26}$$

in REQUEST. The substitution is justified by equation (9). The right member of equation (9), however, imposes a larger computational burden than the left. In addition, the single multiplication of 3×3 matrices (27 scalar multiplications) in the prediction step of Filter QUEST (equation (16)) is replaced by two multiplications of 4×4 matrices (128 scalar multiplication) in the prediction step of REQUEST (equation (19)). Otherwise, the operations in the two algorithms are not only mathematical equivalent but identical. The principal difference between Filter QUEST and REQUEST is the imaginary boundary where one stops calling the mathematical operations "filter steps" and begins calling them "QUEST steps."

The REQUEST Smoother [11] was published in 1997 and bears the same relation to Smoother QUEST as REQUEST does to Filter QUEST.

Summary and Discussion

The relationship of Filter QUEST and REQUEST has been examined in detail. The REQUEST algorithm has been shown to be not only mathematically equivalent to the Filter QUEST algorithm but, apart from trivial differences, which make REQUEST slower, the two algorithms are essentially *identical*, analytically and computationally. A similar assertion can be made for the Smoother QUEST algorithm and the corresponding REQUEST Smoother algorithm. A recent survey article on sequential attitude estimation [19] remarks that Filter QUEST and REQUEST are mathematically *equivalent*. It is not uncommon for algorithms to be mathematically equivalent. There are, for example, several dozen non-sequential implementations of the Wahba problem [14], all of which are mathematically equivalent. What does not seem to be widely recognized, however, is that the two algorithms are essentially mathematically and computationally identical, the point revealed by the present work.

References [10] and [11] claim the superiority of REQUEST and REQUEST Smoother on the grounds that the Davenport matrix K is more important to QUEST than is the attitude profile matrix B. This is hardly true. Both matrices are indispensable to QUEST, and the attitude profile matrix is not less important. In particular, the attitude covariance matrix can be computed directly from A^* and B and only clumsily from A^* and K (via A^* and $B(K)$). The sequentialization of QUEST is also simpler in terms of B than in terms of K and requires far far fewer floating-point operations as well. We note also, that references [10] and [11], although claiming that the Davenport matrix K is more basic, begin their developments with the attitude profile matrix B.

Again, the implementation of the Wahba problem in QUEST has two parts: (1) the computation of the attitude profile matrix B from the input direction data, and (2) the computation of the attitude quaternion from B.

The only real difference between Filter QUEST and REQUEST consists of the transposition by REQUEST of those operations which compute K_k from B_k from the second part of the Filter QUEST program to the first.

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