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An Improvement to the QUEST Algorithm[1]

Yang Cheng[2] and Malcolm D. Shuster[3]

What a profound significance small things assume.
Marcel Proust (1871–1922)

Abstract

A numerical improvement is made to the QUEST algorithm. The significance of the expanded and partially-factored forms of the overlap characteristic polynomial for the computation of the maximum overlap (gain function) is examined in detail. The implementation of the partially-factored form of the QUEST characteristic polynomial greatly improves the numerical properties of that polynomial. Analyses are presented also for the other fast solutions of the Wahba problem.

Introduction

For more than two decades, the QUEST algorithm [2, 3] has been the most widely and most frequently implemented algorithm for batch three-axis attitude estimation. It is also an important component of many attitude Kalman filters [4, 5], where it serves as a preprocessor of star-tracker data. QUEST has gone through very few changes since 1979, when it was first implemented for the Magsat Mission [6]. The only change has been a rearrangement of terms by Markley (unpublished and only effecting the computer code) to improve numerical significance in calculating the attitude profile matrix B (see below). For nearly three decades, QUEST has supported hundreds of missions, both in Earth orbit and at the far reaches of the solar system, all without a single known anomaly.

QUEST's unblemished record has been challenged recently [7, 8] by the claim that in certain cases, it is highly inaccurate and not robust. From references 7 and 8, one might infer that QUEST is even a danger to spacecraft missions, an inference at variance with

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[1]This work is a revision and abridgement of an earlier conference article [1].

[2]Research Assistant Professor, Department of Mechanical and Aerospace Engineering, University at Buffalo, State University of New York, Amherst, New York, 14260. email: cheng3@buffalo.edu. website: www.acsu.buffalo.edu/∼cheng3.

[3]Director of Research, Acme Spacecraft Company, 13017 Wisteria Drive, Box 328, Germantown, Maryland 20874. email: mdshuster@comcast.net website: http://home.comcast.net/∼mdshuster

QUEST's superb performance for nearly 30 years. Reference [8] claims also that QUEST is slower than the ESOQ2 algorithm of Mortari [9, 10], which also requires qualification.[4]

We dispute the claim in references [7] and [8] that the QUEST algorithm is inaccurate or non-robust. QUEST is made to appear inaccurate and and not robust only for a test scenario (scenario 2) of references [7] and [8] and for a simplified version of QUEST specially created for tho tests of reference [7] and [8]. In addition, the attitude estimate for the data of scenario 2 is so inherently inaccurate, *no matter what the attitude estimation algorithm*, that its use might degrade attitude control or scientific data analysis unacceptably [1]. Furthermore, in practice, this scenario is easy to recognize automatically before the attitude computation and would be suppressed automatically by an intelligently designed attitude determination system before that computation. The original implementation of the QUEST algorithm developed for the Magsat mission [1], for which QUEST was originally created, contained tests to flag such scenarios.[5]

The contention that QUEST is a poor performer is based on the fact that the characteristic polynomial for the maximum overlap characteristic value $\lambda_{\rm max}$ (see below) does not behave well for solution by Newton-Raphson iteration [14] of the QUEST characteristic polynomial in scenario 2 of references [7] and [8]. This has been known since 1993 [12] and was not a cause for concern then for the reasons given in the previous paragraph. There is, in fact, no requirement that QUEST perform Newton-Raphson iterations of the characteristic equation in order to calculate $\lambda_{\rm max}$, as noted from the earliest days [2, 3]. For data including a measurement of at least one direction with arc-second accuracy, $\lambda_{\rm max}$ is known to eleven decimal places from a knowledge of the sensor accuracies alone. The purpose of the Newton-Raphson iteration in QUEST was not, in fact, to compute $\lambda_{\rm max}$ but rather to compute an auxiliary test parameter TASTE [6, 13, 15, 16] used in the Magsat and later missions for data validation. The TASTE test would not be useful in scenario 2 because of the typically large modeling errors for coarse sensors.

Nonetheless, it is true that the numerical properties of the QUEST characteristic polynomial were inferior to those of the characteristic polynomial of the FOAM algorithm of Markley [12], if only for a very extreme and unrealistic case.[6] It turns out that there is a simple remedy that will eliminate any alleged poor numerical behavior of the QUEST characteristic polynomial, even in the extreme test scenario 2 of references [7] and [8], and even without the data checks which those works have omitted from the QUEST algorithm. The remedy for the QUEST characteristic polynomial, is simply to make the substitutions

$$\lambda^4 - (a+b)\lambda^2 + ab \;\rightarrow\; (\lambda^2 - a)(\lambda^2 - b) \tag{1a}$$

and

$$c = \det S + \mathbf{Z}^{\rm T} S \mathbf{Z} \rightarrow 8 \det B \tag{1b}$$

in equations (11) and (12c) below, respectively. Substitution (1a) is truly trivial; substitution (1b) less so, but the equality was known nearly three decades ago. While the

[4] Reference [8] provides a masterful overview of the Wahba problem. A brief summary of our criticisms of the numerical results of references [7] and [8] can be found in the section "Alternatives to QUEST," appearing on pages 678–679 of reference [6] and in more detail in reference [1]. For details of the speed issues, see reference [11].

[5] The presence of these internal tests is acknowledged in reference [12], which set the test parameter inappropriately. These tests can be found in the Magsat-mission FORTRAN code for QUEST, as modified slightly by Markley in 1987 and available on the website of the second author. They receive brief mention in reference [6], a revision of the author's Brouwer lecture [13]. These tests were not part of reference [3], which presented only the analytical aspects of the algorithm.

[6] Unrealistic only because one would not normally compute the attitude in those cases.

Newton-Raphson sequence for $\lambda_{\max}$ for the QUEST characteristic polynomial [2, 3, 8] is made convergent by these substitutions, this is not, as we have said, relevant to the accuracy of QUEST, as demonstrated in reference [1]. It is also noteworthy that if the reverse substitution of (1a) is made in algorithms using the FOAM characteristic polynomial, then, as we shall see, these display the same poor numerical properties under Newton-Raphson iteration as had been shown for the version of QUEST of references [7] and [8].[7]

The analysis of the numerical properties of the two characteristic polynomials and their consequences for attitude estimation are the main topic of this work.

The Characteristic Polynomials of the Wahba Problem

The modern fast batch attitude estimation algorithms all depend on the solution of the Wahba problem [17], namely, to find the attitude, expressed here by the attitude matrix A [18], which maximizes the gain matrix [3, 8]

$$g_A(A) = \mathrm{tr}\left[B^{\mathrm{T}}A\right] \tag{2}$$

with the attitude profile matrix B given by

$$B \equiv \sum_{k=1}^{n} a_k\, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^{\mathrm{T}} \tag{3}$$

where $\hat{\mathbf{W}}_k$, $k = 1, \ldots, n$, are a set of n measured directions. $\hat{\mathbf{V}}_k$, $k = 1, \ldots, n$, are a set of n corresponding reference directions, and a_k, $k = 1, \ldots, n$, are a set of positive weights. We will assume, without loss of generality, that these weights have unit sum.

The gain function can be written alternately in terms of the quaternion $\bar{q}$ [18] as [3]

$$g_{\bar{q}}(\bar{q}) \equiv g_A(A(\bar{q})) = \bar{q}^{\mathrm{T}} K \bar{q} \tag{4}$$

where the Davenport matrix K is given by

$$K = \begin{bmatrix} S - sI & \mathbf{Z} \\ \mathbf{Z}^{\mathrm{T}} & s \end{bmatrix} \tag{5}$$

with

$$S \equiv B + B^{\mathrm{T}}, \quad s \equiv \mathrm{tr}\, B \quad \text{and} \quad \mathbf{Z} \equiv [B_{23} - B_{32},\ B_{31} - B_{13},\ B_{12} - B_{21}]^{\mathrm{T}} \tag{6abc}$$

As a result, the minimization of $J_A(A)$ or, equivalently, the maximization of $g_A(A)$ and $g_{\bar{q}}(\bar{q})$, can be accomplished by finding the solution of the characteristic-value problem

$$K\bar{q}^* = \lambda_{\max}\bar{q}^* \tag{7}$$

where $\lambda_{\max}$ is the largest characteristic value of the 4×4 real symmetric matrix K and is also the maximum value of $g_{\bar{q}}(\bar{q})$ and $g_A(A)$. The maximization of $g_A(A)$ and $g_{\bar{q}}(\bar{q})$ has led to numerous solutions of the Wahba problem, of which the most prominent have been (in chronological order) Davenport's original q-method [3, 8, 19, 20], QUEST [3. 8],

[7] In reference [7], the ESOQ algorithm shows the same poor behavior, because it employs essentially the QUEST characteristic polynomial in that work.

Markley's SVD algorithm [8, 21] FOAM [8, 12], ESOQ [8, 22, 23] and ESOQ2 [8, 9, 10]. These are reviewed briefly in reference [8].

From equation (4), we see that if a value can be found for the maximum characteristic value of the Davenport matrix K, then the construction of the attitude, either as the attitude matrix (FOAM) or as the quaternion (QUEST, ESOQ, ESOQ2) can be accomplished easily. Thus, the central part of the fast attitude estimation algorithms has been the computation of $\lambda_{\rm max}$.

An important early result was that for unit-sum weights and the QUEST measurement model [16][8],

$$\lambda_{\rm max} = 1 - \frac{1}{2}\,\sigma_{\rm tot}^2\,\chi^2(2n-3) \tag{8}$$

where $\sigma_{\rm tot}^2$ is a cumulative variance characteristic of the measurement vectors [2, 3, 16], and $\chi^2(2n-3)$ is a χ^2 random variable with $2n-3$ degrees of freedom. For n direction measurements with a common accuracy σ in the QUEST measurement model, $\sigma_{\rm tot}^2$ has the value σ^2/n. Thus, for example, for a star tracker with a single-star direction accuracy of 1 arcsec and observing three stars, $\sigma_{\rm tot}^2$ will have the value $\sigma_{\rm tot}^2 \approx 7.72\times 10^{-12}$. It follows that $\lambda_{\rm max}$ differs from unity in this case by terms of order 10^{-11}.

As first noted in 1978 [2], this means that our zero-th-order approximation for $\lambda_{\rm max}$, namely

$$\lambda_{\rm max}(0) = \lambda_{\rm o} \equiv \sum_{k=1}^{n} a_k = 1 \tag{9}$$

should be adequate for computing the attitude. It also suggests that one may use this zero-th-order approximation as an excellent starting value for further refinement by differential correction.

Thus far, the instrument for this refinement for all of the fast algorithms has been the application of the Newton-Raphson method to the characteristic polynomial for λ, namely,

$$\psi(\lambda) \equiv \det\left[\lambda\, I_{4\times 4} - K\right] \tag{10}$$

For the QUEST algorithm, this has had the form [2, 3]

$$\psi_{\rm QUEST}(\lambda) = \lambda^4 - (a+b)\lambda^2 + ab - c\lambda + cs - d \tag{11}$$

with

$$a = s^2 - {\rm tr}\big({\rm adj}\,S\big)\,, \qquad b = s^2 + \mathbf{Z}^{\rm T}\mathbf{Z} \tag{12ab}$$

$$c = \det S + \mathbf{Z}^{\rm T} S \mathbf{Z}\,, \qquad d = \mathbf{Z}^{\rm T} S^2 \mathbf{Z} \tag{12cd}$$

The function "adj" denotes the matrix adjoint, and "det" the determinant.

Markley's FOAM algorithm [8, 12], like QUEST, begins with the approximation $\lambda_{\rm max} = \lambda_{\rm o}$ and then uses Newton-Raphson iteration to determine a more refined value of the maximum overlap characteristic value. The characteristic polynomial for the FOAM estimate (identical to that of QUEST for infinitely precise arithmetic) has the form

$$\psi_{\rm FOAM}(\lambda) = \left(\lambda^2 - \|B\|_F^2\right)^2 - 8\lambda\det B - 4\,\|{\rm adj}\,B\|_F^2 \tag{13}$$

[8]Although the result was published only relatively recently, it has been know since 1983. This relationship also appears in references [7] and [8], where it is given correct attribution.

where $\|\cdot\|_F$ denotes the Frobenius norm

$$\|M\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |M_{ij}|^2 \tag{14}$$

and M is any $n \times n$ matrix.

For equations (12) and (13) to be consistent, we must have

$$a = \|B\|_F^2 - e\,, \qquad b = \|B\|_F^2 + e \tag{15ab}$$

$$c = 8\det B\,, \qquad d = cs + 4\,\|\mathrm{adj}\,B\|_F^2 - e^2 \tag{15cd}$$

with

$$e = (b-a)/2 = \big(\mathbf{Z}^{\mathrm{T}}\mathbf{Z} + \mathrm{tr}(\mathrm{adj}\,S)\big)/2 \tag{16}$$

which provides an alternate means for calculating the coefficients of the QUEST parameters a, b, c, and d.

The ESOQ algorithm writes the characteristic equation somewhat differently. For this algorithm [22],

$$\psi_{\mathrm{ESOQ}}(\lambda) = \lambda^4 + a_{\mathrm{ESOQ}}\lambda^3 + b_{\mathrm{ESOQ}}\lambda^2 + c_{\mathrm{ESOQ}}\lambda + d_{\mathrm{ESOQ}} \tag{17}$$

with

$$a_{\mathrm{ESOQ}} = \mathrm{tr}K = 0\,, \qquad b_{\mathrm{ESOQ}} = -2\,(\mathrm{tr}B)^2 + \mathrm{tr}(\mathrm{adj}\,S) - \mathbf{Z}^{\mathrm{T}}\mathbf{Z} \tag{18ab}$$

$$c_{\mathrm{ESOQ}} = -\mathrm{tr}(\mathrm{adj}\,K)\,, \qquad d_{\mathrm{ESOQ}} = \det K \tag{18cd}$$

These coefficients are identical with the corresponding coefficients of the QUEST characteristic polynomial. This is true also to high numerical precision, as shown by the results of reference [7]. Equations (17) and (18) were used in the original publication of the ESOQ algorithm [22] for the closed-form solution for λ_{max} in terms of surds. In reference [7], λ_{max} is determined using a Newton-Raphson sequence for λ_{max} for this same characteristic polynomial, which is essentially the QUEST characteristic polynomial. In reference [8], the ESOQ algorithm uses the FOAM characteristic polynomial and Newton-Raphson iteration. There are, thus, three ESOQ algorithms.

The Extreme Test Scenario 2 of Markley and Mortari

References [7] and [8] use a very extreme example to test the accuracy of the algorithms. In this test scenario, scenario 2, which examines the case where there are three measured directions, one along the spacecraft body x-axis and two in the body xy-plane separated from the negative body x-axis by approximately 4.5 degrees. The exact values of the three directions are

$$\hat{\mathbf{W}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{W}}_2 = \begin{bmatrix} -0.99712 \\ 0.07584 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \hat{\mathbf{W}}_3 = \begin{bmatrix} -0.99712 \\ -0.07584 \\ 0 \end{bmatrix} \tag{19abc}$$

The values of the components have been chosen so that each column vector has exactly unit norm. The three measurements of scenario 2 of references [7] and [8] are modeled according to the QUEST measurement model with variance parameters given by[9]

$$\sigma_1 = 1 \text{ arcsec}, \qquad \sigma_2 = \sigma_3 = 1 \text{ deg} \tag{20abc}$$

From reference [3] or [24], the inverse covariance covariance matrix is given by

$$P_{\theta\theta}^{-1} = \text{diag}\left[\frac{2\sin^2\alpha}{\sigma_2^2}, \left(\frac{1}{\sigma_1^2} + \frac{2\cos^2\alpha}{\sigma_2^2}\right), \left(\frac{1}{\sigma_1^2} + \frac{2}{\sigma_2^2}\right)\right] \tag{21}$$

where α is the angle between $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{W}}_2$. Substituting the values from equations (19) and (20) leads to attitude estimate error levels about each axis of

$$\sigma_x \approx 9.32 \text{ deg} \qquad \text{and} \qquad \sigma_{yz} \equiv \sqrt{\sigma_y^2 + \sigma_z^2} \approx 1.41 \text{ arcsec} \tag{22ab}$$

in agreement with equation (90) of reference [8]. Note that $\sigma_{yz} \approx \sqrt{2}\,\sigma_1$, as expected. From equation (21), σ_y and σ_z should be equal to σ_1 within about one part in 12,000,000. Note also that for scenario 2 of references [7] and [8], the attitude accuracy about the x-axis is absurdly bad compared to that about the other two axes, $\sigma_x/\sigma_y = \sigma_x/\sigma_z = 34,000$. No real mission would tolerate such a disparity.

Numerical Precision and the QUEST Characteristic Polynomial

The cause of the lack of convergence in scenario 2 of references [7] and [8] in the Newton-Raphson sequence for λ_{max} from the QUEST characteristic polynomial, occurs only when this polynomial is written in the *expanded* form

$$\psi_{\text{QUEST-exp}}(\lambda) = \lambda^4 - (a+b)\lambda^2 + ab - c\lambda + cs - d \tag{23a}$$

One can also write the QUEST characteristic polynomial in the equivalent *partially-factored* form

$$\psi_{\text{QUEST-fac}}(\lambda) = (\lambda^2 - a)(\lambda^2 - b) - c\lambda + cs - d \tag{23b}$$

Analytically, the two forms are identical and yield the same result for infinitely precise arithmetic. For finite 64-bit arithmetic (IEEE double precision [25]) and for the parameter values of scenario 2, they do not. This minute difference in the form of the overlap characteristic equation is the cause for claims by reference [8] that QUEST does not perform as well as the other fast algorithms. We emphasize, however, that, for scenarios such as scenario 2 of references [7] and [8], one should not calculate λ_{max} by performing Newton-Raphson iterations using the characteristic polynomial, but should use the value λ_{o}. One accomplishes this in QUEST by setting NEWT $= 0$ in the QUEST program, written by the second author in 1979 and revised slightly by Markley in 1987. NEWT is the input variable

[9]Note that in references [7] and [8], as in the present work, the observed measurement vectors have been chosen to have exact constant values. As a result, random noise must be added to the reference vectors instead. We point out that for vector magnetometer data, it is generally the reference vectors which are more poorly known than the observed magnetic field vector. Reference [3] shows how to treat the presence of random noise on both the observation and reference measurements.

in QUEST that sets the maximum number of Newton-Raphson iterations in the calculation of λ_{max}.

To see how the different forms of the QUEST characteristic polynomial lead to large errors in the Newton-Raphson sequence for λ_{max} for scenario 2 of references [7] and [8], examine the first correction to λ_{max} in that sequence. For scenario 2 of references [7] and [8], the coefficient c in equations (12) vanishes identically, because the measurements are coplanar. The two forms of the QUEST characteristic equation become for scenario 2 of references [7] and [8]

$$\psi_{\text{QUEST-exp}}(\lambda) = \lambda^4 - (a+b)\lambda^2 + ab - d \tag{24a}$$

$$\psi_{\text{QUEST-fac}}(\lambda) = (\lambda^2 - a)(\lambda^2 - b) - d \tag{24b}$$

For both characteristic polynomials, the first correction in the Newton-Raphson sequence is

$$\Delta\lambda_{\text{max}}(1) = -\psi(1)/\psi'(1) \tag{25a}$$

$$\lambda_{\text{max}}(1) = \lambda_{\text{o}} + \Delta\lambda_{\text{max}}(1) \tag{25b}$$

where $\psi'(\lambda)$ denotes the first derivative of $\psi(\lambda)$. For the partially-factored form

$$\begin{aligned}\psi_{\text{QUEST-fac}}(1) &= (1-a)(1-b) - d \\ &= (1 - 0.9999999974609042)(1 - 0.9999999971236696) \\ &\qquad - 5.946137136732494 \times 10^{-18} \\ &= (2.5390958 \times 10^{-9})\,(2.8763304 \times 10^{-9}) \\ &\qquad - 5.946137136732494 \times 10^{-18} \\ &= 1.3571414 \times 10^{-18} \end{aligned} \tag{26a}$$

$$\begin{aligned}\psi'_{\text{QUEST-fac}}(1) &= 2(1-a) + 2(1-b) \\ &= 2(2.5390958 \times 10^{-9}) + 2(2.8763304 \times 10^{-9}) \\ &= 1.0830853 \times 10^{-8} \end{aligned} \tag{26b}$$

In IEEE double-precision arithmetic [25], both $\psi_{\text{QUEST-fac}}(1)$ and $\psi'_{\text{QUEST-fac}}(1)$ as calculated above have up to seven significant figures, and consequently, $\Delta\lambda_{\text{max}}$ calculated from equation (25a) will have seven significant figures. Note, however, that equations (25) do not include the cumulative round-off error from the many arithmetic operations in the computations. Therefore, the number of significant figures is likely closer to five. However, the magnitude of $\Delta\lambda_{\text{max}}(1)$ is approximately 10^{-10}, so that the contribution of $\Delta\lambda_{\text{max}}(1)$ can increase the numerical error in λ_{max} by only 10^{-15}, which is close to the limit of IEEE double-precision representation. The numerical error in this case is far smaller than the statistical error. Thus, we must conclude that the total error in λ_{max}, statistical plus numerical is still about 10^{-11} for scenario 2 for the factored form of the characteristic polynomial. This is true for all algorithms.

At best, IEEE double-precision floating-point numbers have 16.8 significant digits, but only for numbers whose binary mantissa can be represented exactly by 52 bits. Otherwise, the number of significant digits is more often only 16. Note that while the significance of $\Delta\lambda$ is only about seven significant digits it is smaller than λ_o by ten orders of magnitude, and the significance of $\lambda(1)$ is 16 significant digits.

Examine now the same calculation for the expanded form of the QUEST characteristic equations. In that case, $\psi'_{\text{QUEST-exp}}(1)$ will have the same value as $\psi'_{\text{QUEST-fac}}(1)$ in equation (26b) to eight significant figures, but

$$\begin{aligned}\psi_{\text{QUEST-exp}}(1) &= (1 + ab - d) - (a + b)\\ &= 1.9999999945845738 - 1.9999999945845738\\ &= \bigcirc \times 10^{-16}\end{aligned} \tag{27}$$

where $\bigcirc$ denotes a number of order 1 with no significant figures. Clearly, the Newton-Raphson approximation applied to the partially-factored form of the QUEST characteristic equation will yield a result valid to eleven significant figures, while that applied to the expanded form will yield a barely numerically significant result.

We may treat the FOAM characteristic equation in the same way

$$\psi_{\text{FOAM-fac}}(\lambda) = \left(\lambda^2 - \|B\|_F^2\right)^2 - 8\lambda \det B - 4\,\|\text{adj}\, B\|_F^2 \tag{28a}$$

as in equation (13), and

$$\psi_{\text{FOAM-exp}}(\lambda) = \lambda^4 - 2\|B\|_F^2\,\lambda^2 - 8(\det B)\,\lambda + \|B\|_F^4 - 4\,\|\text{adj}\, B\|_F^2 \tag{28b}$$

and we will find that the partially-factored FOAM characteristic polynomial (the form that has always been used in FOAM) performs well in the Newton-Raphson expansion, but the expanded form does not. Similar results will occur for higher-order terms in the Newton-Raphson sequence for λ_{max}.

Besides the polynomial $\lambda^4 - (a + b)\lambda^2 + ab$, there is a second source of numerical imprecision. The forms for c given by equations (12c) and (15c) are identical algebraically, but the form given by equation (15c) is more precise numerically. Therefore, in conjunction with the partially-factored form of the QUEST characteristic polynomial, we should also employ equation (15c) for c.

Numerical Results

To see that the use of the partially-factored form of the QUEST characteristic polynomial allows the convergence of the Newton-Raphson sequence for λ_{max} for scenario 2 of references [7] and [8], we have recomputed Table 2 of references [7] and [8] but with all characteristic polynomials in partially-factored form. We have not included results for ESOQ1.1 or ESOQ2.1 which are not important for the discussion and are unchanged from the values in references [7] and [8]. We have, however, given the results for the four iterative algorithms for up to five iterations, the better to observe the convergence properties. As can seen from our Table 1, all algorithms perform equally admirably. We repeat, once again, however, that the performance of the QUEST algorithm as designed in 1979 and modified slightly by Markley in 1987, is not affected by the better convergence properties of the partially-factored QUEST characteristic polynomial. Again, we emphasize, the QUEST entries in Table 2 of references [7] and [8] are for the QUEST algorithm created for

TABLE 1. Estimation Results for Scenario 2 of references [7] and [8] for Partially-Factored Characteristic Polynomials[12]

Algorithm	Iterations	ΔJ	σ_x (deg)	σ_{yz} (arcsec)
q-Davenport	—	—	9.30	1.43
M-SVD	—	1.40 e-5	9.30	1.43
FOAM	0	—	9.42	1.41
	1	0.102	9.42	1.41
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
QUEST of references [7] and [8]	0	—	9.25	1.43
	1	0.102	9.29	1.43
	2	0.953 e-3	9.30	1.43
	3	1.33 e-5	9.30	1.43
	4	1.33 e-5	9.30	1.43
	5	1.33 e-5	9.30	1.43
ESOQ	0	—	9.25	1.43
	1	0.102	9.29	1.43
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
ESOQ2	0	—	9.42	1.43
	1	0.102	9.30	1.43
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43

those works. The QUEST algorithm designed by the second author in 1979 would simply have flagged the result as unusable.

We have also tried starting the Newton-Raphson sequence not at λ_o but at $\lambda_o - 3/2$ following equation (8) (see also equation (29) below). The results were significantly different from that for $\lambda_{max}(0) = \lambda_o$ but equally accurate. The fact that there was any difference at all is a criticism of scenario 2 of references [7] and [8] as a valid test scenario.

Note the increasing values of σ_x as a function of iteration number. This increasing nature for ESOQ to be due to a greater sensitivity on the angle of rotation as it became "close"

[12] We use the designations 'q-Davenport" and "M-SVD" to avoid confusing Davenport's original q-algorithm with the QUEST algorithm and others which are also implementations of Davenport's q-algorithm and to avoid confusing Markley's SVD algorithm from the SVD algorithm of Numerical Linear Algebra [26].

TABLE 2. Estimation Results for Scenario 2 of references [7] and [8] for Expanded Characteristic Polynomials

Algorithm	Iterations	ΔJ	σ_x (deg)	σ_{yz} (arcsec)
q-Davenport	—	—	9.30	1.43
M-SVD	—	1.40 e-5	9.30	1.43
FOAM	0	—	9.42	1.43
	1	471	108.	1.43
	2	6620	109.	1.43
	3	3430	105.	1.43
	4	5950	104.	1.43
	5	3470	104.	1.43
QUEST of references [7] and [8]	0	—	9.25	1.43
	1	767	54.6	1.43
	2	3130	54.9	1.43
	3	1730	54.3	1.43
	4	2660	53.6	1.43
	5	2410	53.4	1.43
ESOQ	0	—	9.25	1.43
	1	471	57.0	1.43
	2	6620	56.3	1.43
	3	3430	56.4	1.43
	4	5950	56.2	1.43
	5	3470	56.5	1.43
ESOQ2	0	—	9.42	1.43
	1	471	95.5	1.43
	2	6620	95.4	1.43
	3	3430	93.1	1.43
	4	5950	91.9	1.43
	5	3470	90.0	1.43

to 180 deg. This same sensitivity is also the case for the QUEST algorithm, for which the singularity in rotation angle is also at 180 deg. This problem has been shown [1] to be an artifact of scenario 2 of references [7] and [8].

To illustrate the effect of using an expanded form of the characteristic polynomial more dramatically, we have repeated the calculations of our Table 1 for scenario 2 of references [7] and [8] but with both the QUEST and the FOAM characteristic polynomials in expanded form. The results are shown in our Table 2. As we see, all of the iterative fast algorithms perform uniformly horribly in scenario 2 of references [7] and [8] when the overlap characteristic polynomial is in expanded form. All four iterative algorithms, however, perform well using the value λ_o in place of λ_{max}.

While our mathematical analysis above has used unit-sum weights, in constructing the tables we have used $a_k = 1/\sigma_k^2$, $k = 1, 2, 3$, where σ_k is the accuracy parameter of the

QUEST measurement model [3]. These weights sum not to unity but to $1/\sigma_{\text{tot}}^2$. For this choice of weights, we have

$$E\{J(\bar{q}^*)\} = \frac{1}{2}\lambda_{\text{o}}\sigma_{\text{tot}}^2\, E\{\chi^2(2n-3)\} = 3/2 \tag{29}$$

for scenario 2 of references [7] and [8]. This makes it easier to assess the number of significant figures in λ_{max} (including the effects of random measurement noise) from the values of ΔJ.

A very important point to raise again is that for all four iterative algorithms, λ_{o} is on the order of 10^{10}. Thus, in the Newton-Raphson sequence for each algorithm $\lambda_{\text{max}}(0) = \lambda_{\text{o}}$ has more than ten significant figures while $\lambda_{\text{max}}(1)$, the result of the first iteration of the Newton-Raphson sequence, has about seven significant digits for scenario 2 of references [7] and [8], which should be enough to calculate the attitude accurately, but, obviously, from Table 2, it is not. Scenario 2 of references [7] and [8] has many diseases.

A Comment on Newton-Raphson Iteration

We remarked in the paragraph following equation (8) that in infinitely precise arithmetic

$$\lambda_{\text{max}} - \lambda_{\text{o}} = O(10^{-11}) \tag{30}$$

Our numerical analyses of the partially factored and expanded characteristic polynomials for scenario 2 of references [7] and [8] showed that

$$\lambda_{\text{max}} - \lambda_{\text{fac}} = O(10^{-15}) \tag{31}$$

and

$$\lambda_{\text{max}} - \lambda_{\text{exp}} = O(10^{-8}) \tag{32}$$

for all algorithms in IEEE double-precision arithmetic and with the weights in the Wahba cost function normalized so that $\lambda_{\text{o}} = 1$. Note that the attitude error levels using λ_{o} or λ_{fac} still cannot be less than 10 deg about the worst axis or 1 arcsec about the other two axes for scenario 2 due to the geometry of the scenario and the measurement noise error levels. Thus, the calculation of λ_{max} by Newton-Raphson iteration does not yield a more accurate result for λ_{max} than the *a priori* value of λ_{max} obtainable from the accuracies alone for scenario 2. This is true for all fast algorithms, whether or not they use the QUEST or the FOAM characteristic polynomial and whether or not these polynomials are in partially factored of in expanded form. It was pointed out in reference [3], 27 years ago, that the Newton-Raphson iteration of λ_{max} was not required for an accurate attitude estimate. In fact, the reason for performing a Newton-Raphson solution of the QUEST characteristic polynomial was to obtain the parameter TASTE, which was used for data validation [27, 28]. The attention of references [7] and [8] to the Newton-Raphson sequence for λ_{max} is, thus, a distraction from a true examination of the accuracy and robustness of the QUEST and other algorithms. That eight significant digits in λ_{max} are insufficient for an adequate estimate of the attitude is indicative that scenario 2 of references [7] and [8] is at the knife edge of numerical problems for the Wahba problem in general.

Discussion and Conclusions

We have seen that the poor convergence properties of the QUEST characteristic polynomials in the Newton-Raphson sequence for the largest root is due to the use of the expanded form of the characteristic polynomial for λ_{max}, and from the use of a less compact expression for the coefficient c in that polynomial. For the partially-factored form of the QUEST characteristic polynomial, the convergence properties become identical to those of the FOAM characteristic polynomial in its usual partially-factored form. Both characteristic polynomials display non-convergent Newton-Raphson sequences when the expanded forms are used.

We note once more that the QUEST algorithm does not depend for its performance on the convergence of the Newton-Raphson sequence. The poor properties for QUEST shown in Table 2 of references [7] and [8] are only for the QUEST algorithm of references [7] and [8], which insists that a Newton-Raphson iteration be performed even though these iterations has been shown to be unnecessary [2, 3]. Nonetheless, the partially-factored form of the QUEST characteristic polynomial leads to a version of the QUEST algorithm which is numerically more flexible. Nonetheless, the evaluation of the accuracy and robustness of the QUEST algorithm on the basis of the behavior of the Newton-Raphson iterations of the characteristic polynomial is wrong and a distraction from the examination of the true accuracy and robustness of the QUEST algorithm.

Even though the modified QUEST characteristic polynomial performs better numerically, and performs well even for the sinister scenario 2 of references [7] and [8], this does not obviate the need for data validation tests. The version of QUEST implemented in the Magsat mission contained a test for poor observability of the attitude (the FIBBL test), as confirmed by reference [12]. Such a test would have flagged the data of scenario 2 of references [7] and [8].[10] One could, of course, make a minor modification to QUEST so that a Newton-Raphson iteration of the characteristic polynomial would be suppressed for the data of scenario 2. Such a modification is no greater than the modification made to QUEST for the tests in references [7] and [8]. The better numerical performance of the modified QUEST characteristic polynomial is not a panacea against the ills of poor data or negligent system design. If the attitude computed from a scenario like test scenario 2 of references [7] and [8] is not suppressed by the attitude determination system, it might cause significant problems downstream.

In summary, the contention of Markley and Mortari that the QUEST algorithm is less robust and less accurate than the algorithms developed by Markley and Mortari, even without the minor modification of the QUEST characteristic polynomial, cannot be supported. That contention is based on the insistence of those authors that the solutions of the Wahba problem implement a Newton-Raphson iteration of the overlap eigenvalue λ_{max}, which has long been known to be unnecessary. With a trivial rearrangement of terms in the QUEST characteristic polynomial, the claim of reference [7] and [8] for the poor convergence of the Newton-Raphson sequence for λ_{max} in QUEST, even for scenario 2, is also not true.

The QUEST algorithm has always been robust, accurate and, as shown in reference [11] and even in references [7] and [8], very fast.

[10]This flagging of bad data did not occur in the tests of reference [12], because the value of FIBBLwas chosen inappropriately.

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