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Direction Averaging and Suboptimal

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The effect of direction averaging in generating suboptimal algorithms for three-axis attitude determination, first proposed by Brozenec and Bender in 1994, is examined for attitude determination systems consisting of: (1) two star trackers; (2) a star tracker and a single-direction sensor; and (3) a single star tracker. It is shown for star trackers with fields of view smaller than 10 deg that little accuracy is lost for the first system or for the second system when the star tracker is paired with a single-direction sensor of comparable accuracy. When a star tracker is paired with a sensor of much smaller accuracy, such as an infra-red horizon scanner, a three-axis magnetometer or a coarse Sun sensor, the loss in attitude accuracy about the star-tracker boresight can be very significant. Applications to the StarNav multi-FOV star trackers are discussed. The Brozenec-Bender methodology is applied to attitude estimation from a single star tracker by partitioning the field of view.

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INTRODUCTION

A number of algorithms have been proposed for the computation of the three-axis attitude which minimizes the cost function

$$L(A) = \frac{1}{2} \sum_{k=1}^{N} a_k \left| \hat{\mathbf{W}}_k - A \hat{\mathbf{V}}_k \right|^2 , \tag{1}$$

where A is the direction-cosine matrix [1],[1] $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$, are directions (lines of sight, observation vectors) observed in the spacecraft body frame, $\hat{\mathbf{V}}_k$, $k = 1, \ldots, N$, are the corresponding directions known, say, in an inertial frame (the reference vectors), and a_k, $k = 1, \ldots, N$, are a set of positive weights, assumed to sum to unity. A caret in this work will be used to denote a unit vector. This cost function was first proposed by G. Wahba [2] in 1965 and has been the starting point of many algorithms [3],[2] of which the most popular has been the QUEST algorithm [4].

Many solutions to the Wahba problem begin with Davenport's q-algorithm [7]. Davenport showed that the Wahba cost function could be recast as

$$L(A) = \text{constant} - \text{tr}\left(B^T A\right) , \tag{2}$$

where

$$B \equiv \sum_{k=1}^{N} a_k \, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T . \tag{3}$$

and where tr(·) denotes the trace operation, and recast further as the quadratic form

$$L(A) = \text{constant} - \bar{q}^T K \bar{q} , \tag{4}$$

where the 4×4 matrix K is given by

$$K = \begin{bmatrix} S - s\,I & \mathbf{Z} \\ \mathbf{Z}^T & s \end{bmatrix} , \tag{5}$$

and

$$S = B + B^T , \quad s = \text{tr}\, B , \tag{6ab}$$

$$\mathbf{Z} = [B_{23} - B_{32},\ B_{31} - B_{13},\ B_{12} - B_{21}]^T . \tag{6c}$$

Here $\bar{q}$ denotes the quaternion of rotation [1].

Minimization of $L(A)$ leads to an eigenvalue equation for K, namely

$$K \bar{q}^* = \lambda_{\text{max}} \, \bar{q}^* , \tag{7}$$

[1] Publications by the author of the present work can be downloaded from the author's website.
[2] Ref. 3 gives a masterful overview of the many solutions to the Wahba problem and new theoretical results. On its numerical results for the QUEST algorithm and their interpretation see Refs. 5 and 6.

where the asterisk denotes the optimal value and $\lambda_{\max}$ is the largest eigenvalue of K.

The QUEST algorithm [4] uses a very efficient method for both the determination of the maximum eigenvalue $\lambda_{\max}$ and the construction of the optimal quaternion. In addition, it offered a model covariance matrix based on the simple measurement model

$$\hat{\mathbf{W}}_k = A\,\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k\,, \tag{8}$$

with the measurement error $\Delta\hat{\mathbf{W}}_k$ having approximate first and second moments

$$E\{\Delta\hat{\mathbf{W}}_k\} = \mathbf{0}\,, \tag{9}$$

$$E\{\Delta\hat{\mathbf{W}}_k\,\Delta\hat{\mathbf{W}}_k^T\} = \sigma_k^2\,[I - (A\,\hat{\mathbf{V}}_k)\,(A\,\hat{\mathbf{V}}_k)^T]\,, \tag{10}$$

where $E\{\cdot\}$ denotes the expectation, and I is the 3×3 identity matrix. This leads to the result

$$P_{\theta\theta} = \left[\sum_{k=1}^{N} \frac{1}{\sigma_k^2}\left(I - \hat{\mathbf{W}}_k^{\text{true}}\,\hat{\mathbf{W}}_k^{\text{true}\,T}\right)\right]^{-1}\,, \tag{11}$$

and

$$\hat{\mathbf{W}}_k^{\text{true}} \equiv A\,\hat{\mathbf{V}}_k\,, \tag{12}$$

provided that the weights a_k, $k = 1, \ldots, N$, are chosen to be proportional to $1/\sigma_k^2$. Note that in actual computations we must replace $\hat{\mathbf{W}}_k^{\text{true}}$ by $\hat{\mathbf{W}}_k$, because the former is not known in general. Since we will be interested in calculating quantities only to lowest nonvanishing order in $\Delta\hat{\mathbf{W}}_k$ this replacement will not lead to important errors in general.

The covariance matrix in equation (11) is defined in terms of error angles. If A_{true} is the true attitude, and A^* is the estimated attitude, then the 3×1 array of attitude error angles

$$\Delta\boldsymbol{\theta}^* \equiv [\Delta\theta_1^*,\ \Delta\theta_2^*,\ \Delta\theta_3^*]^T \tag{13}$$

are defined by

$$A^* = C(\Delta\boldsymbol{\theta}^*)\,A_{\text{true}}\,, \tag{14}$$

where

$$C(\boldsymbol{\theta}) = I + \frac{\sin(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|}\,[[\,\boldsymbol{\theta}\,]] + \frac{1-\cos(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|^2}\,[[\,\boldsymbol{\theta}\,]]^2 \tag{15}$$

is the formula for a proper orthogonal matrix parameterized by the rotation vector [1] and

$$[[\,\boldsymbol{\theta}\,]] \equiv \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix}\,. \tag{16}$$

Note that for $|\Delta\boldsymbol{\theta}| << 1$ we have that

$$C(\Delta\boldsymbol{\theta}) = I + [[\,\Delta\boldsymbol{\theta}\,]] + O(|\Delta\boldsymbol{\theta}|^2)\,. \tag{17}$$

The attitude covariance matrix is defined as

$$P_{\theta\theta} \equiv E\{\Delta\boldsymbol{\theta}^{*}\Delta\boldsymbol{\theta}^{*T}\}\,. \tag{18}$$

Markley has developed an equally efficient algorithm FOAM [8], which works directly in terms of the direction-cosine matrix.

Another important result in the development of solutions to the Wahba problem was to show that if the measurement model of equations (8) through (10) is accepted and the measurement errors are assumed to be Gaussian as well, then maximum-likelihood estimation [9] of the attitude leads directly to the Wahba cost function [10]. This put the Wahba problem on a firm statistical footing. The QUEST algorithm has supported numerous spacecraft missions, beginning with the Magsat mission in 1979. It has the additional advantage of providing a useful figure of merit as additional output, which allows data rejection to be automated easily.

THE PSEUDO-MEASUREMENT

Brozenec and Bender [11] have presented a method for decreasing the computational burden for QUEST when attitude was determined from multiple star-direction data from a star tracker and a second sensor mounted on the spacecraft. The authors argued that because star trackers generally have very small fields of view (generally on the order of ±5 deg/axis), the measurements will be closely clustered. As a result, the star direction measurements will provide much less information on the attitude of the spacecraft about the star-tracker boresight compared with that about the other two axes, a phenomenon generally known as *geometric dilution of precision* (GDOP). Since, a second star tracker or other accurate vector sensor was assumed to be present, this second sensor would provide a great deal of information about the attitude of the spacecraft about the first star tracker's boresight and vice versa if the second sensor is also a star tracker. Hence, the authors argued, it was reasonable to simply average over the directions measured in the star tracker at any one time, and use the direction of this average as an effective measurement for the attitude. In this way one discards any information about the attitude about the star tracker boresight. That information, as we have said, is assumed to be minuscule compared to equivalent information provided by the other sensor.

Thus, specializing now to the case where one has two star trackers, one defines

$$\widehat{\hat{\mathbf{W}}}_l \equiv \text{unit}\left(\sum_{k=1}^{N_l} \hat{\mathbf{W}}_{l,k}\right), \quad \widehat{\hat{\mathbf{V}}}_l \equiv \text{unit}\left(\sum_{k=1}^{N_l} \hat{\mathbf{V}}_{l,k}\right), \quad l = 1, 2, \tag{19}$$

where N_l are the number of directions observed by star tracker "l," and unit$(\cdot)$ is the function which generates a unit vector in the same direction as its argument if non-vanishing. The spacecraft attitude was determined by finding the optimal attitude from the Wahba cost function

$$L(A) = \frac{1}{2}\sum_{l=1}^{2} a_l \left|\widehat{\overline{\mathbf{W}}}_l - A\widehat{\overline{\mathbf{V}}}_l\right|^2 , \tag{20}$$

with

$$a_l = \frac{N_l}{N_1 + N_2} \quad l = 1,\, 2\,. \tag{21}$$

Thus, no matter how many stars are observed in each tracker, the QUEST algorithm, or any other optimal algorithm using line-of-sight data, is applied only to the two effective observations, rather than to $(N_1 + N_2)$ individual star observations. The weighting of the two terms is based on the assumption that the two star trackers have the same accuracy and that the individual measurements of each star tracker have a uniform circle of error. When the measurements do not all have the same circle of error, the *three* N_l in equation (21) should be replaced by $\sum_{k=1}^{N_l}(1/\sigma_{k,l}^2)$, since the weights ought to reflect not only the relative quantity but also the relative quality of the measurements.

An important point is that the QUEST algorithm has a simple closed-form solution for $\lambda_{\rm max}$ when there are only two vector measurements [4], which would make the Brozenec-Bender solution atttractive if there is no loss in averaging all of the measurements in one star tracker.

The present work will examine the performance of the Brozenec-Bender approach and present the results of a detailed covariance analysis. If the $(N_1 + N_2)$ line-of-sight measurements were entered directly as inputs into the Wahba cost function, then the covariance of the resulting attitude would be simply

$$P_{\theta\theta}^{\rm QUEST} = \left[\sum_{l=1}^{2}\sum_{k=1}^{N_l} \frac{1}{\sigma_{l,k}^2}\left(I - \hat{\mathbf{W}}_{l,k}^{\rm true}\,\hat{\mathbf{W}}_{l,k}^{{\rm true}\,T}\right)\right]^{-1} . \tag{22}$$

The $\sigma_{l,k}$, we have said, are assumed to be equal to a common value σ. The corresponding attitude covariance matrix for the Brozenec-Bender algorithm is more complex and will occupy the next section.

COVARIANCE ANALYSIS OF THE BROZENEC-BENDER PSEUDO-MEASUREMENT WITH TWO STAR TRACKERS

Because the Wahba problem yields the maximum-likelihood estimate of the attitude given measurements obeying the QUEST model, its attitude covariance matrix can be computed simply from the Hessian matrix of the Wahba cost function[3] [10].

$$\left(P_{\theta\theta}^{\rm QUEST}\right)^{-1} = \frac{1}{\sigma_{\rm tot}^2}\, E\left\{\frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\, L\left(C(\boldsymbol{\theta})A_{\rm true}\right)\right\}\bigg|_{\boldsymbol{\theta}=\mathbf{0}} , \tag{23}$$

[3]The vactor $\sigma_{\rm tot}^{-2}$ arises from the fact that we have used unit-norm weights in L, so that it differs by a factor from the negative-log-likelihood function [10].

provided we choose

$$a_{l,k} = \frac{\sigma_{\text{tot}}^2}{\sigma_{l,k}^2}\,, \qquad \text{with} \qquad \frac{1}{\sigma_{\text{tot}}^2} = \sum_{l=1}^{2}\sum_{k=1}^{N_l} \frac{1}{\sigma_{l,k}^2}\,, \tag{24ab}$$

and the single summation over k in equation (1) is replaced with a double summation over l and k. The evaluation of equation (23) leads directly to the result given in equation (22).

The same is not true with Brozenec-Bender averaging, because the Wahba-like cost function of equation (20) does not arise from the maximum-likelihood estimate of the attitude given the Brozenec-Bender effective measurements. Thus, the attitude errors must be computed directly in terms of the measurement errors in the Brozenec-Bender effective measurement and the covariance computed from this. This computation is the subject of most of this work. (Ref. 4 computed the covariance matrix in this way not only for the TRIAD algorithm but also for the QUEST algorithm as well, because it was not realized by the author at the time that the Wahba cost function followed directly from the measurement error model used to calculate the attitude covariance matrix.)

We thus define *unnormalized* vectors in a manner similar to that of equation (19), namely

$$\overline{\mathbf{W}}_l \equiv \sum_{k=1}^{N_l} \hat{\mathbf{W}}_{l,k}\,, \quad \overline{\mathbf{V}}_l \equiv \sum_{k=1}^{N_l} \hat{\mathbf{V}}_{l,k}\,, \quad l = 1,\, 2. \tag{25}$$

Clearly,

$$\overline{\mathbf{W}}_l = A\overline{\mathbf{V}}_l + \Delta\overline{\mathbf{W}}_l\,, \quad l = 1,\, 2\,, \tag{26}$$

with

$$\Delta\overline{\mathbf{W}}_l = \sum_{k=1}^{N_l} \Delta\hat{\mathbf{W}}_{l,k}\,, \quad l = 1,\, 2\,. \tag{27}$$

Thus, given the QUEST model for the individual line-of-sight measurements, we have that $\Delta\mathbf{W}_l$ has mean zero and covariance matrix

$$R_{\overline{\mathbf{W}}_l} = \sum_{k=1}^{N_l} \sigma_{l,k}^2 \left(I - \hat{\mathbf{W}}_{l,k}^{\text{true}}\, \hat{\mathbf{W}}_{l,k}^{\text{true},T} \right)\,, \quad l = 1,\, 2\,. \tag{28}$$

From

$$\widehat{\overline{\mathbf{W}}}_l = \overline{\mathbf{W}}_l / |\overline{\mathbf{W}}_l|\,, \quad l = 1,\, 2\,, \tag{29}$$

it follows that to lowest order in $\Delta\overline{\mathbf{W}}_l$, $l = 1,\, 2$,

$$\Delta\widehat{\overline{\mathbf{W}}}_l = \frac{1}{|\overline{\mathbf{W}}_l|} \left(I - \widehat{\overline{\mathbf{W}}}_l \widehat{\overline{\mathbf{W}}}_l^T \right) \Delta\overline{\mathbf{W}}_l\,, \quad l = 1,\, 2\,. \tag{30}$$

Thus, the Brozenec-Bender effective measurement satisfies

$$\widehat{\overline{\mathbf{W}}}_l = A\widehat{\overline{\mathbf{V}}}_l + \Delta\widehat{\overline{\mathbf{W}}}_l\,, \quad l = 1,\, 2\,, \tag{31}$$

with[4]

$$E\{\,\Delta\widehat{\overline{\mathbf{W}}}_l\,\} = \mathbf{0}\,, \quad l = 1,\,2\,, \tag{32a}$$

$$E\{\,\Delta\widehat{\overline{\mathbf{W}}}_l \Delta\widehat{\overline{\mathbf{W}}}_l^T\,\} = R_{\widehat{\overline{\mathbf{W}}}_l}\,, \quad l = 1,\,2\,, \tag{32b}$$

and

$$R_{\widehat{\overline{\mathbf{W}}}_l} = \frac{1}{|\overline{\mathbf{W}}_l|^2}\left(I - \widehat{\overline{\mathbf{W}}}_l\widehat{\overline{\mathbf{W}}}_l^T\right) R_{\overline{\mathbf{W}}_l}\left(I - \widehat{\overline{\mathbf{W}}}_l\widehat{\overline{\mathbf{W}}}_l^T\right), \quad l = 1,\,2\,. \tag{33}$$

COVARIANCE ANALYSIS OF THE BROZENEC-BENDER ALGORITHM FOR TWO STAR TRACKERS

Now that we have a complete model for the Brozenec-Bender measurement, we may compute the spacecraft attitude. The mechanization of the QUEST algorithm is straightforward, has been described in detail elsewhere [4], and need not concern us here. What does concern us is the attitude error. To compute the attitude error, we are interested only in computing $C(\Delta\boldsymbol{\theta}^*) = A^* A_{\rm true}^{-1}$, after which we will extract $\Delta\boldsymbol{\theta}^*$ using equation (17). We can compute $C(\Delta\boldsymbol{\theta}^*)$ most easily be replacing $\hat{\hat{\mathbf{V}}}_l$ with $\widehat{\overline{\mathbf{W}}}_l^{\rm true}$ in equation (20), leading to

$$L\,(C(\Delta\boldsymbol{\theta})) = \frac{1}{2}\sum_{l=1}^{2} a_l\,\left|\widehat{\overline{\mathbf{W}}}_l - C(\Delta\boldsymbol{\theta})\widehat{\overline{\mathbf{W}}}_l^{\rm true}\right|^2, \tag{34}$$

Substituting equation (17) and minimizing over $\Delta\boldsymbol{\theta}$ leads straightforwardly to

$$\Delta\boldsymbol{\theta}^* = \left[\sum_{l=1}^{2} a_l\left(I - \widehat{\overline{\mathbf{W}}}_l\widehat{\overline{\mathbf{W}}}_l^T\right)\right]^{-1}\sum_{l=1}^{2} a_l\,[[\,\widehat{\overline{\mathbf{W}}}_l\,]]\,\Delta\widehat{\overline{\mathbf{W}}}_l\,, \tag{35}$$

whence the attitude covariance matrix for the Brozenec-Bender algorithm is given by

$$P_{\theta\theta}^{\rm BB} = \left[\sum_{l=1}^{2} a_l\left(I - \widehat{\overline{\mathbf{W}}}_l\widehat{\overline{\mathbf{W}}}_l^T\right)\right]^{-1} \sum_{l=1}^{2} a_l^2\,[[\,\widehat{\overline{\mathbf{W}}}_l\,]]\,R_{\widehat{\overline{\mathbf{W}}}_l}\,[[\,\widehat{\overline{\mathbf{W}}}_l\,]]^T\left[\sum_{l=1}^{2} a_l\left(I - \widehat{\overline{\mathbf{W}}}_l\widehat{\overline{\mathbf{W}}}_l^T\right)\right]^{-1}, \tag{36}$$

which should be compared with the result for the QUEST algorithm in equation (21).

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[4]Equation (32a) cannot be true exactly because of the norm constraint, but it possesses sufficient veracity for our purposes. The reader is referred to Ref. 10 for a complete explanation.

It follows from the Cramér-Rao Theorem [9] that

$$P_{\theta\theta}^{\text{QUEST}} \leq P_{\theta\theta}^{\text{BB}} \,. \tag{37}$$

The important question is how large is the difference between the two attitude covariance matrices. To answer this question, we examine the two covariances in a simple model, in which the star trackers are assumed to have a circular field of view of angular radius ρ and the stars are distributed uniformly over the field of view of each sensor.[5] We will assume further that one star tracker has its boresight along the spacecraft x-axis and the other about the spacecraft y-axis. In the frame of each of the star trackers, the boresight will be taken to be the z-axis. We assume, as in Equation (21) that the two star trackers are characterized by the same variance σ^2, which is the same for all observations in the fields of view of the two star trackers. For each of the two star trackers the boresight is chosen to be the z-axis of that sensor's reference frame. The chosen example will present the Brozenec-Bender algorithm to best advantage and also allow us to develop simple closed-form expressions for the attitude covariance matrices for the Brozenec-Bender algorithm and QUEST.

In the limit that N_1 and N_2 are very large we may replace the summation over the observations to good approximation by an integral. Thus, if $f(\hat{\mathbf{W}})$ is any function of the observations we may write

$$\sum_{k=1}^{N_l} f(\hat{\mathbf{W}}_{l,k}) \rightarrow \frac{N_l}{\Omega} \int_0^{2\pi} \int_0^{\rho} f\big(\hat{\mathbf{W}}(\vartheta,\, \varphi)\big) \, \sin\vartheta \, d\vartheta \, d\varphi \,, \quad l = 1,\, 2 \,, \tag{38}$$

with Ω the solid angle subtended by the star tracker field of view,

$$\Omega = 2\pi(1 - \cos\rho) \,, \tag{39}$$

and

$$\hat{\mathbf{W}}(\vartheta,\, \varphi) = \begin{bmatrix} \sin\vartheta \, \cos\varphi \\ \sin\vartheta \, \sin\varphi \\ \cos\vartheta \end{bmatrix} . \tag{40}$$

With these substitutions, and assuming the distribution of observed vectors to be uniform in the star tracker field of view, the inverse covariance matrix for each star tracker (in that star tracker's reference frame) using the QUEST algorithm for computing the attitude is

$$\left(P_{\theta\theta}^{\text{QUEST}} \right)_l^{-1} = \frac{N_l}{\sigma^2} \, \text{diag}(a,\, a,\, b) \,, \quad l = 1,\, 2 \,, \tag{41}$$

where

$$\text{diag}(d_1,\, d_2,\, d_3) \equiv \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} , \tag{42}$$

[5]A circular field of view is actually not very far-fetched. The very large fields of view investigated here could be possible only by means of an optical system, which would create a circular image on the CCD plane.

and

$$a = (4 + \cos\rho + \cos^2\rho)/6\,, \tag{43a}$$

$$b = (2 - \cos\rho - \cos^2\rho)/3\,. \tag{43b}$$

Note that as $\rho \to 0$ we have that $a \to 1$ and $b \to 0$.

The star tracker boresights are along the body x- and y-axes, respectively. Thus, the covariance matrix above must be transformed for each star tracker from sensor to body coordinates. Let us denote the two star trackers by sc1 and sc2, respectively. The sensor alignment matrix [12, 13] has been defined according to

$$\hat{\mathbf{W}}_{l,k} = S_l\,\hat{\mathbf{U}}_{l,k}\,, \tag{44}$$

where $\hat{\mathbf{W}}_{l,k}$ is the representation of the observation in the spacecraft body frame, $\hat{\mathbf{U}}_{l,k}$ is the representation of the same observation in the sensor frame, and S_l is the alignment matrix of sensor l, a proper orthogonal matrix. The transformation of only one column vector into another does not completely specify the transformation matrix. However, since we assume the field of view of each star trackers to be circular, the ambiguity is inconsequential. Thus, we may chose as the two alignment matrices

$$S_{\rm sc1} = R(\hat{\mathbf{2}},\,-\pi/2) \qquad \text{and} \qquad S_{\rm sc2} = R(\hat{\mathbf{1}},\,\pi/2)\,. \tag{45}$$

with

$$\hat{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{2}} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{3}} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{46abc}$$

and $R(\hat{\mathbf{n}},\,\theta)$ denotes the rotation about the axis $\hat{\mathbf{n}}$ through an angle θ.

Applying these to equation (41) leads to

$$\left(P_{\theta\theta}^{\rm QUEST}\right)^{-1} = S_{\rm sc1}\left(P_{\theta\theta}^{\rm QUEST}\right)_{\rm sc1}^{-1} S_{\rm sc1}^T + S_{\rm sc2}\left(P_{\theta\theta}^{\rm QUEST}\right)_{\rm sc2}^{-1} S_{\rm sc2}^T \tag{47a}$$

$$= \frac{N_1}{\sigma^2}\,\mathrm{diag}(b,\,a,\,a) + \frac{N_2}{\sigma^2}\,\mathrm{diag}(a,\,b,\,a) \tag{47b}$$

For the Brozenec-Bender algorithm, we obtain straightforwardly *in the individual star-tracker frames* (boresight $=$ $\hat{\mathbf{z}}$)[6]

[6]Note that when we write "boresight $=$ $\hat{\mathbf{z}}$" we are stating that the *physical* boresight of the star tracker is the *physical* z-axis of the sensor (it would be more exact to add the appropriate subscript to $\hat{\mathbf{z}}$ since there are four physical z-axes in our problem (sc1, sc2, the spacecraft body, and the frame of the reference vectors), but when we write $\widehat{\mathbf{W}}=\hat{\mathbf{3}}$ we are stating that the numerical value of the *representation* of the boresight vector is given by equation (46c), which is not the same. The boresight of the first star tracker is always $\hat{\mathbf{z}}_{\rm sc1}$ but in body coordinates its representation is $\hat{\mathbf{1}}$. We could write $\hat{\mathbf{z}}_{\rm sc1}$ as a general notation for the representation of the z-axis of the first star tracker to be consistent with equation (46c) – in Ref. 1 we do just that – but in the present work we stick with more familiar if less exact notation, particularly since we have avoided ever using bold Roman letters for anything but the representations of vectors (i.e., 3×1 numerical arrays) so that there is never any confusion.

$$\overline{\mathbf{W}} = N_l \left(\frac{1+\cos\rho}{2}\right) \hat{\mathbf{3}}\,, \qquad \text{and} \qquad \widehat{\overline{\mathbf{W}}} = \hat{\mathbf{3}}\,, \tag{48}$$

and

$$R_{\overline{\mathbf{W}}_l} = N_l\, \sigma^2\, \mathrm{diag}(a,\, a,\, b)\,, \tag{49a}$$

$$R_{\widehat{\overline{\mathbf{W}}}_l} = \frac{\sigma^2}{N_l} \left(\frac{2}{1+\cos\rho}\right)^2 \mathrm{diag}(a,\, a,\, 0)\,. \tag{49b}$$

From this it follows that in the spacecraft body frame

$$\sum_{l=1}^{2} a_l^2\, [[\,\widehat{\overline{\mathbf{W}}}_l\,]]\, R_{\widehat{\overline{\mathbf{W}}}_l}\, [[\,\widehat{\overline{\mathbf{W}}}_l\,]]^T = \frac{\sigma^2}{(N_1+N_2)^2} \left(\frac{2}{1+\cos\rho}\right)^2 \big(N_1\, \mathrm{diag}(0,\, a,\, a) + N_2\, \mathrm{diag}(a,\, 0,\, a)\big)\,. \tag{50}$$

Likewise,

$$\left[\sum_{l=1}^{2} a_l \left(I - \widehat{\overline{\mathbf{W}}}_l \widehat{\overline{\mathbf{W}}}_l^T\right)\right] = \frac{1}{N_1+N_2} \big(N_1\, \mathrm{diag}(0,\, a,\, a) + N_2\, \mathrm{diag}(a,\, 0,\, a)\big)\,, \tag{51}$$

whence the inverse attitude covariance for the Brozenec-Bender algorithm is easily shown to be

$$\left(P_{\theta\theta}^{\mathrm{BB}}\right)^{-1} = \frac{1}{\sigma^2} \left(\frac{1+\cos\rho}{2}\right)^2 \frac{1}{a}\, \mathrm{diag}(N_2,\, N_1,\, N_1+N_2)\,, \tag{52}$$

which should be compared with equation (47b) above. The modification of these results for the case that the $\sigma_{l,k}$ are not all equal to a universal value should be obvious from the remarks following equation (20).

If we consider the special case $N_1 = N_2 = N$, we obtain the simple results

$$\left(P_{\theta\theta}^{\mathrm{QUEST}}\right)^{-1} = \frac{N}{\sigma^2}\, \mathrm{diag}(a+b,\, a+b,\, 2a)\,, \tag{53a}$$

$$\left(P_{\theta\theta}^{\mathrm{BB}}\right)^{-1} = \frac{N}{\sigma^2} \left(\frac{1+\cos\rho}{2}\right)^2 \frac{1}{a}\, \mathrm{diag}(1,\, 1,\, 2)\,. \tag{53b}$$

For $\rho << 1$ these reduce to

$$P_{\theta\theta}^{\mathrm{QUEST}} = \frac{\sigma^2}{N}\, \mathrm{diag}[1-\rho^2/4,\, 1-\rho^2/4,\, (1+\rho^2/4)/2]\,, \tag{54a}$$

$$P_{\theta\theta}^{\mathrm{BB}} = \frac{\sigma^2}{N}\, (1+\rho^2/4)\, \mathrm{diag}[1,\, 1,\, 1/2]\,. \tag{54b}$$

Thus, the fractional loss in accuracy is on the order of ρ^2, but for typical star-tracker fields of view, ρ is only 5. deg, leading to a loss of accuracy of only about one percent. This is certainly negligible. For $\rho = \pi/2$, corresponding to a star tracker whose field of view encompasses half the celestial sphere, we have

$$P_{\theta\theta}^{\text{QUEST}} = \frac{3}{4}\frac{\sigma^2}{N}\,\text{diag}[1,\ 1,\ 1]\,, \tag{55a}$$

$$P_{\theta\theta}^{\text{BB}} = \frac{8}{3}\frac{\sigma^2}{N}\,\text{diag}[1,\ 1,\ 1/2]\,, \tag{55b}$$

so that the QUEST algorithm is better (in variance) by a factor of from 1.77 to 3.55 about any axis. This, however, is a very unusual case and clearly outside the expected range of application of the Brozenec-Bender algorithm. For $\rho = \pi$, the full sky case, the covariance of the Brozenec-Bender algorithm is infinite, because the $\overline{\mathbf{W}}_l$ vanish in our example. (If there is no average star direction, then it cannot be used in an attitude determination algorithm.)

The efficacy of the Brozenec-Bender algorithm when the field of view of the star tracker is small has been demonstrated for two star trackers.

BROZENEC-BENDER AVERAGING WITH ONE STAR TRACKER AND ONE SINGLE-VECTOR SENSOR

Let us consider now the alternate case where the first sensor is a CCD star tracker with a circular field of view of radius ρ and single-direction standard deviation σ_1 and with generally $N_1 = N$ stars in the field of view. We will assume as a typical value $\rho = 5$ deg and $\sigma_1 = 10$ arc seconds or approximately 50 microradians, and $N = 10$. Sensor 2 is a single-direction sensor with standard deviation σ_2 and, clearly, $N_2 = 1$. If Sensor 2 is a precise Sun sensor then we can expect σ_2 to have values close to 10 arc seconds or 50 microradians. Otherwise, if Sensor 2 is a coarse sensor, its accuracy will be taken as 0.3 deg or approximately 5 milliradians. For definiteness, we will assume that Sensor 1, the CCD star tracker, has its boresight aligned with the spacecraft body x-axis, while Sensor 2 measures a single vector along the spacecraft body y-axis.

The computation of the spacecraft covariance matrix follows procedures similar to those of the previously considered case. For the application of the QUEST algorithm to all of the data without averaging we have for the inverse covariance matrix

$$\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1} = \frac{N}{\sigma_1^2}\,\text{diag}(b,\ a,\ a) + \frac{1}{\sigma_2^2}\,\text{diag}(1,\ 0,\ 1)\,, \tag{56}$$

showing clearly the two contributions to the inverse covariance matrix. Note that the inverse covariance (information) for each sensor will be smallest about the boresight, hence, about the spacecraft body x-axis for the star tracker (Sensor 1) and about the spacecraft body y-axis for the single-direction sensor (Sensor 2). With Brozenec-Bender averaging, however, we obtain a slightly less transparent expression.

We note first that relative weights of the two sensors, according to the earlier discussion will be

$$a_1 = \frac{N/\sigma_1^2}{N/\sigma_1^2 + 1/\sigma_2^2}, \qquad a_2 = \frac{1/\sigma_2^2}{N/\sigma_1^2 + 1/\sigma_2^2}. \tag{57}$$

The pseudo-measurement covariance matrix for the star tracker (with respect to sensor coordinates) is again following Equations (28), (33) and (49)

$$R_{\widehat{\mathbf{W}}_1} = \frac{\sigma_1^2}{N}\left(\frac{2}{1+\cos\rho}\right)^2 \operatorname{diag}(a,\, a,\, 0), \tag{58}$$

with a as in Equation (43a), which we write (in body coordinates) as

$$R_{\widehat{\mathbf{W}}_1} = \sigma_{\text{eff}}^2 \operatorname{diag}(0,\, 1,\, 1), \tag{59}$$

with

$$\sigma_{\text{eff}}^2 \equiv \frac{\sigma_1^2}{N}\left(\frac{2}{1+\cos\rho}\right)^2 a = \beta\left(\frac{\sigma_1^2}{N}\right), \tag{60}$$

and trivially for the single-direction sensor (again in body coordinates)

$$R_{\widehat{\mathbf{W}}_2} = \sigma_2^2 \operatorname{diag}(1,\, 0,\, 1). \tag{61}$$

Note that for the specified star tracker $\beta \approx 1$. Equation (50) for the present case becomes equivalently

$$\sum_{l=1}^{2} a_l^2\, [[\widehat{\mathbf{W}}_l]]\, R_{\widehat{\mathbf{W}}_l}\, [[\widehat{\mathbf{W}}_l]]^T = a_1^2\, \sigma_{\text{eff}}^2 \operatorname{diag}(0,\, 1,\, 1) + a_2^2\, \sigma_2^2 \operatorname{diag}(1,\, 0,\, 1). \tag{62}$$

Evaluating Equation (36) in this case leads after some manipulation to

$$\left(P_{\theta\theta}^{\text{BB}}\right)^{-1} = \operatorname{diag}\left[\frac{1}{\sigma_2^2},\, \frac{1}{\sigma_{\text{eff}}^2},\, \frac{1}{a_1^2\, \sigma_{\text{eff}}^2 + a_2^2\, \sigma_2^2}\right]. \tag{63}$$

It will be useful to define

$$c = \frac{N\sigma_2^2}{\sigma_1^2}. \tag{64}$$

Then

$$a_1 = \frac{c}{1+c}, \quad \text{and} \quad a_2 = \frac{1}{1+c}, \tag{65}$$

and also

$$\sigma_{\text{eff}}^2 = \frac{\beta}{c}\, \sigma_2^2. \tag{66}$$

For $\sigma_2 = 0.3$ deg we have then $c \approx 100,000$, while for $\sigma_2 = 3$ arc seconds we have instead $c \approx 1$.

Comparing the diagonal elements of the inverse covariance matrix we obtain

$$\left[\left(P_{\theta\theta}^{\mathrm{QUEST}}\right)^{-1}\right]_{22} \Big/ \left[\left(P_{\theta\theta}^{\mathrm{BB}}\right)^{-1}\right]_{22} = \beta a\,, \tag{67a}$$

$$\left[\left(P_{\theta\theta}^{\mathrm{QUEST}}\right)^{-1}\right]_{33} \Big/ \left[\left(P_{\theta\theta}^{\mathrm{BB}}\right)^{-1}\right]_{33} = \frac{(1+ac)(1+\beta c)}{(1+c)^2}\,. \tag{67b}$$

Note that for $\rho = 5$ deg we have $a = 0.988$, $\beta = 1.0002$ and $\beta a = 1.000005$ so that the attitude accuracy about the y-axis is not affected adversely by Brozenec-Bender averaging, independent of the nature of Sensor 2. For $c = 100,000$ (Sensor 2 is, say, an infra-red horizon scanner) the right member of Equation (67b) differs from unity again by terms of order 10^{-5}. For $c = 1$ (Sensor 2 is a precise Sun sensor), the right member of Equation (67b) becomes $(\beta a + \beta + a + 1)/4$, which is equally close to unity. Thus, the attitude determination accuracy about any axis perpendicular to the star-tracker boresight is not sensitive to the nature of Sensor 2 or to Brozenec-Bender averaging.

The situation changes for the component about the star-tracker boresight. In that case we find

$$\left[\left(P_{\theta\theta}^{\mathrm{QUEST}}\right)^{-1}\right]_{11} \Big/ \left[\left(P_{\theta\theta}^{\mathrm{BB}}\right)^{-1}\right]_{11} = 1 + bc\,, \tag{67c}$$

where b was defined in Equation (43b). For $\rho = 5$ deg we have $b = 4 \times 10^{-3}$, so that the ratio of the inverse covariances is 400 when $c = 100,000$ and 1.004 when $c = 1$. Brozenec-Bender averaging leads to little loss in attitude determination accuracy about the boresight when the single-vector sensor is of the same accuracy roughly as the star tracker but a considerable degradation of the attitude accuracy when the single-vector sensor is not very accurate.

APPLICATION TO THE StarNav STAR TRACKER

Cost factors generally prohibit the implementation of more than a single star tracker on a spacecraft. However, an interesting idea has been proposed by Mortari, Pollack and Junkins [14] of adding a prism to a star tracker so that a single star tracker can have two orthogonal fields of view. If we assume that the field of view is the usual 8 deg × 8 deg, then, by matching areas, the equivalent ρ is 0.787 rad, which leads to

$$P_{\theta\theta}^{\mathrm{QUEST}} = \frac{\sigma^2}{N}\,\mathrm{diag}[0.9985,\ 0.9985,\ 0.5008]\,, \tag{68a}$$

$$P_{\theta\theta}^{\mathrm{BB}} = \frac{\sigma^2}{N}\,\mathrm{diag}[1.0015,\ 1.0015,\ 0.5008]\,. \tag{68b}$$

where we have assumed that the two focal panes intersect in the y-axis. The difference in accuracy (in standard deviation) is only a negligible 0.15 percent.

The newest StarNav III multiple-FOV star tracker will have three FOV with optical axes coplanar and separated by 120 deg [15].

APPLICATION TO A SINGLE STAR TRACKER

We may apply the Brozenec-Bender methodology to an attitude determination system consisting of a single star tracker by partitioning its field of view into two segments of equal area and treating each segment as an independent sensor. Thus, if a star tracker with a rectangular field of view has dimensions (full width) of $2\alpha \times 2\beta$, we can compute the expected inverse covariance using the above methods to obtain

$$\left(P_{\theta\theta}^{\mathrm{QUEST}}(N,\,\alpha,\,\beta)\right)^{-1} = \frac{N}{\sigma^2}\begin{bmatrix} 1-\alpha^2/3 & 0 & 0 \\ 0 & 1-\beta^2/3 & 0 \\ 0 & 0 & (\alpha^2+\beta^2)/3 \end{bmatrix} . \tag{69}$$

assuming again that the stars are distributed uniformly in the field of view. Typically, $\alpha = \beta \approx 4$ deg. We shall assume in our example that $\alpha = \beta$.

If we now partition the field of view by the y axis, we will have two adjacent fields of view of dimensions $\alpha \times 2\alpha$ with optical axes at offsets of $\pm\alpha/2$ from the optical axis of the star tracker. Applying the Brozenec-Bender methodology, equations (25) through (33) lead to within terms of order α^2

$$\overline{\mathbf{W}}_1 = (N/2)\,S_1\,\hat{\mathbf{3}}\,, \qquad \overline{\mathbf{W}}_2 = (N/2)\,S_2\,\hat{\mathbf{3}}\,, \tag{70ab}$$

$$R_{\widehat{\overline{\mathbf{W}}}_1} = S_1\,R_{\widehat{\overline{\mathbf{W}}}_o}\,S_1^T\,, \qquad R_{\widehat{\overline{\mathbf{W}}}_2} = S_2\,R_{\widehat{\overline{\mathbf{W}}}_o}\,S_2^T\,, \tag{71ab}$$

where $R_{\widehat{\overline{\mathbf{W}}}_o}$ is the expected measurement covariance matrix for a field of view of dimensions $\alpha \times 2\alpha$ centered on the origin. To lowest nonvanishing order in α

$$\left(R_{\widehat{\overline{\mathbf{W}}}_o}\right) = \frac{2\sigma^2}{N}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{72}$$

and we have assumed that there are $N_1 = N_2 = N/2$ star observations in each $\alpha \times 2\alpha$ partition of the field of view, so that also $a_1 = a_2 = 1/2$. We may write the alignment matrices to within terms of order α^2 as

$$S_1 = S_2^T = R(\hat{\mathbf{2}},\,\alpha/2) = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{bmatrix} , \tag{73}$$

with

$$s = \frac{\alpha/2}{\sqrt{1+\alpha^2/4}} \approx \alpha/2\,, \qquad c = \frac{1}{\sqrt{1+\alpha^2/4}} \approx 1\,. \tag{74ab}$$

The evaluation of equation (36) for this case is straightforward and leads to lowest order in α to

$$P_{\theta\theta}^{\mathrm{BB}} = \frac{\sigma^2}{N}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4/\alpha^2 \end{bmatrix} \tag{75}$$

This should be compared with the similar expression for the covariance matrix for the QUEST estimate, which can be obtained by inverting equation (69) with $\alpha = \beta$. To lowest order in α this is

$$P_{\theta\theta}^{\text{QUEST}} = \frac{\sigma^2}{N}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/(2\alpha^2) \end{bmatrix} . \tag{76}$$

The standard deviation about the optical axis for the method inspired by the Brozenec-Bender methodology is larger than the Cremér-Rao lower bound by a factor in standard deviation of $\sqrt{8/3}$ or 1.63. This might be a price worth paying if the computational burden were much smaller. However, the general solution for the partitioned field of view would be the attitude matrix which minimized the cost function

$$J^{\text{BB}} \equiv \frac{1}{2}\sum_{l=1}^{2}[\widehat{\widehat{\mathbf{W}}}_l - A\widehat{\widehat{\mathbf{V}}}_l]^T \left(R_{\widehat{\widehat{\mathbf{W}}}_l}\right)^{\#} [\widehat{\widehat{\mathbf{W}}}_l - A\widehat{\widehat{\mathbf{V}}}_l] \tag{77}$$

where # denotes the pseudo-inverse, and now we must compute $\widehat{\widehat{\mathbf{W}}}_l$ and $R_{\widehat{\widehat{\mathbf{W}}}_l}$, $l = 1, 2$, from the data. The solution of equation (77) is straightforward but considerably more burdensome than calculating the more accurate QUEST estimate from the N star directions, especially when one considers the large burden of computing the centroid directions and equivalent centroid-direction covariance matrices. It is interesting to note that to lowest order in α the Brozenec-Bender methodology applied to our example leads to the same estimate as would have been obtained by applying the QUEST algorithm to effective measurements $\widehat{\widehat{\mathbf{W}}}_1$ and $\widehat{\widehat{\mathbf{W}}}_2$ with QUEST-measurement-model accuracies of $N\sigma^2/2$. This will not be true in general. Although the Brozenec-Bender methodology does not lead to a sufficiently accurate nor simpler algorithm for estimating the attitude from the data of a single star tracker, it is interesting that it does so well.

DISCUSSION

The reasons for this great disparity in accuracy about the star tracker boresight in equation (67c) can be understood more simply than from the above derivation. The geometric dilution of precision (GDOP) factor of a sensor with a narrow field of view is approximately $1/\sin(\alpha)$, where α is the half-cone angle of the sensor. For a typical star tracker with a field of view 8 deg × 8 degrees, $\alpha \approx \text{FOV}/\sqrt{(12)} \approx 4$ deg, and the GDOP factor will be about 25. Thus, if the attitude accuracy of the star tracker is 3 arcsec per star and the star tracker measures typically 9 stars, the attitude accuracy about the average star direction will be

$$\frac{\text{GDOP} \times \sigma}{\sqrt{N}} = 25 \text{ arcsec} . \tag{78}$$

This is considerably better than the accuracy of one of the three coarse sensors listed above, which is typically only about 0.5 deg. Thus, in this case, the Brozenec-Bender algorithm results in a a worsening of the attitude accuracy about the average star direction by two orders of magnitude. The Brozenec-Bender algorithm should not be used in this case. However, if the second sensor is a precise Sun sensor of accuracy 5.0 arcsec, the Brozenec-Bender algorithm can be used with assurance.

For two-star-tracker attitude determination systems, it would seem that the Brozenec-Bender algorithm presents a real computational savings, since the construction of the Davenport profile matrix, B of equation (3), requires roughly $12N$ floating point multiplications if the standard deviations are not equal. However, there is a significant sacrifice if one adopts such a course. This is the loss of the QUEST output variable TASTE as a figure of merit for the optimization. This quantity is expected to have a mean value of $2N - 3$ and a variance of $2(2N - 3)$. Very large deviations from the mean usually indicate misidentified stars or other bad data. Since TASTE is computed at almost no additional computation burden by QUEST, it provides the simplest and cheapest means of data validation. It is, in fact, the introduction of the TASTE variable, which greatly streamlined attitude mission support for the Magsat mission and cut processing time by a factor of, perhaps, 12 [16] and not QUEST's remarkable speed, which makes the QUEST algorithm so attractive. It seems unwise to give up this attribute of QUEST to lessen the computational burden of one small part of the attitude determination system.

Note also that the geometrical dilution of precision is not a “killing” effect. For the example above the attitude accuracy about the boresight is poorer that that about an axis perpendicular to the boresight by only a factor of about 8. Thus, if the single-vector sensor has an error level of 0.5 arc minute (rather than 0.3 deg) one will gain a factor of two in attitude accuracy about the star tracker boresight by executing the complete QUEST computation. If the single-attitude sensor has an error level greater than a few arc minutes, than the accuracy of the data discarded by the Brozenec-Bender method will overwhelm that of the single-vector sensor for all three axes.

For a multi-field-of-view star tracker, like the StarNav star trackers under development, the Brozenec-Bender methodology may be a very useful path, especially when these star tracker can observe sometimes 50 stars in each field of view. Its application to the partitioning of the field of view of a single star-tracker, however, leads to a significant loss of accuracy.

Brozenec-Bender centroiding can be incorporated in a more complete attitude estimator which does not discard data and permits three-axis attitude estimation with a single star tracker [17]. Such an algorithm (SCAD) begins with the centroiding to create effective measurements $\widehat{\widehat{\mathbf{W}}}_1$ and $\widehat{\widehat{\mathbf{W}}}_2$ but uses the remaining data to determine the rotation about the centroid vector. Such an algorithm works quite well, but not well enough to be a replacement for QUEST.

CONCLUSION

The Brozenec-Bender algorithm, while not adequate when a star-tracker is paired with a coarse attitude sensor, nonetheless performs extremely well when the attitude sensors consist of two star trackers or a star tracker and a second sensor of comparable accuracy. By employing such a method, however, one gives up other features of the QUEST algorithm, which may be more significant than speed. For a multi-field-of-view star tracker, however, like StarNav, the Brozenec-Bender method offers real advantages. The field of view of a single star tracker can be partitioned to take advantage of the Brozenec-Bender methodology, but there is significant loss in accuracy about the star-tracker boresight.

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