The Journal of the Astronautical Sciences, Vol. 55, No. 4, October–December 2007, pp. 463–478

Effective Direction Measurements for Spacecraft Attitude:

I. Equivalent Directions

Malcolm D. Shuster[1]

My derivation was from ancestors
Who stood equivalent to mighty kings

William Shakespeare (1564–1616)
and George Wilkins (fl. 1607)
Pericles, Prince of Tyre, Act V, scene i

Abstract

The equivalent directions are a set of three unit vectors, conforming to the QUEST measurement model, which can lead in maximum-likelihood estimation to any given attitude estimate and attitude covariance matrix. Since they conform to the QUEST measurement model (i.e., are "QUEST-like"), the equivalent directions can be used not only in general estimation problems but also directly in the Wahba problem. It can be shown that three equivalent directions can always be found, and that they are unique within signs if the eigenvalues of the attitude co-information matrix are non-degenerate, but that the equivalent inverse variances for these equivalent directions may not always be non-negative, hence, not always physically meaningful. This can occur, for example, for attitude covariance matrices computed by the TRIAD algorithm for QUEST-like inputs. The equivalent inverse variances for attitude covariance matrices computed from original QUEST-like measurements will always be non-negative. The connection of the equivalent directions and inverse variances to Markley's SVD algorithm is presented. While not a practical vehicle for data fusion, the equivalent directions and their equivalent inverse variances can provide a useful tool for the analysis of attitude systems in data fusion problems and in mission design.

Introduction

In a previous work [1], the Wahba problem [2–4][2] was extended to include attitude estimates together with the related (body-referenced) attitude estimate-error

[1]Director of Research, Acme Spacecraft Company, 13017 Wisteria Drive, Box 328, Germantown, Maryland 20874. E-mail: mdshuster@comcast.net. Website: http://home.comcast.net/~mdshuster.

covariance matrices (hereafter, simply "attitude covariance matrices") as an equivalent attitude profile matrix B or Davenport matrix K. In this *generalized* Wahba problem, it was possible to include an initial condition as well as estimated attitudes (for example, from star trackers) without the need to reprocess the original data leading to that estimate.[3] In the present work, we examine a related, essentially inverse process, that of finding a minimal set of effective direction measurements conforming to the QUEST measurement model [3], their associated (non-random) reference directions, and their effective variance parameters, which in maximum-likelihood estimation lead to a given attitude estimate and attitude covariance matrix—in other words, the *Inverse Wahba Problem.* The effective (unit) vector measurements we call *equivalent directions* (or *equivalent direction measurements*), the corresponding reference directions *equivalent reference directions,* and the effective inverse variances *equivalent inverse variances.* The set of three equivalent measurement (or observation) directions, three equivalent reference directions, and three equivalent inverse variances we call the *equivalent-direction representation.* Thus, the equivalent-direction representation consists of[4]

$$\hat{\mathbf{W}}_i^{\text{eq true}}, \hat{\mathbf{V}}_i^{\text{eq}}, (1/\sigma_i^2)^{\text{eq}}, \qquad i = 1, 2, 3$$

We have not yet shown that there are exactly three. We shall see the reason for the superscript "true" below.

Like the attitude profile matrix B and the Davenport matrix K, the equivalent-direction representation is an *enhanced representation,* that is, it contains both the attitude estimate and the attitude covariance matrix. The four obvious enhanced representations of the attitude estimate and attitude covariance matrix and their most direct interrelationships are[5]

$$K \leftrightarrow B \leftrightarrow \{A^{*\prime}, P_{\tilde{\epsilon}\tilde{\epsilon}}\} \leftrightarrow \{\hat{\mathbf{W}}_i^{\text{eq true}}, \hat{\mathbf{V}}_i^{\text{eq}}, (1/\sigma_i^2)^{\text{eq}} \,|\, i = 1, 2, 3\}$$

The Attitude Covariance Matrix

The central object in constructing the equivalent-direction representation is the attitude covariance matrix. Mathematically, as in references [1] and [8], the attitude covariance matrix $P_{\tilde{\epsilon}\tilde{\epsilon}}$ is defined as the covariance matrix of the attitude increment error vector $\tilde{\boldsymbol{\epsilon}}^*$

$$P_{\tilde{\epsilon}\tilde{\epsilon}} \equiv E\{\tilde{\boldsymbol{\epsilon}}^* \tilde{\boldsymbol{\epsilon}}^{*\text{T}}\} \tag{1}$$

where $E\{\cdot\}$ denotes the expectation, and the attitude increment vector is defined by

$$A = e^{[[\boldsymbol{\epsilon}]]} A^{\text{true}} \tag{2}$$

[2]Reference [4] provides a masterful review of the solutions to the Wahba problem and new theoretical results. On its numerical results and their interpretation, see references [5] and [6].

[3]This aspect of the generalized Wahba problem was first presented in 1989 [7]. Reference [1] presents a much more extensive treatment.

[4]Following references [1] and [8], we denote 3×1 column vectors by bold sans-serif letters. Realizations of a random variable are denoted by a prime, the true values by the superscript "true." Unlike in reference [1], we will not denote every random variable by the superscript "r.v." in order not to overburden our notation. The attitude estimator A^* and later the attitude error estimator $\tilde{\boldsymbol{\epsilon}}^*$ are obviously random variables. As a rule, we will prefer to write an equation which holds *mutatis mutandis* both for random variables and for their realizations by that for random variables, since it is less ambiguous to determine thence the homologous equation for the realizations than *vice versa.*

[5]The attitude matrix A may be taken as the proxy for any representation of the attitude. In practical applications, the representation of choice is often the quaternion [8].

and $[[\mathbf{u}]]$ denotes the 3×3 antisymmetric matrix constructed according to[6]

$$[[\mathbf{u}]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{3}$$

If follows from equations (1) and (2) that $\tilde{\boldsymbol{\epsilon}}^*$ and $P_{\tilde{\epsilon}\tilde{\epsilon}}$ are referred to the spacecraft body-axes. The estimator $\tilde{\boldsymbol{\epsilon}}^*$ and estimate $\tilde{\boldsymbol{\epsilon}}^{*\prime}$ satisfy

$$A^* = e^{[[\tilde{\boldsymbol{\epsilon}}]]} A^{\text{true}} \quad \text{and} \quad A^{*\prime} = e^{[[\tilde{\boldsymbol{\epsilon}}^{*\prime}]]} A^{\text{true}} \tag{4}$$

As in reference [1], the tilde indicates that $\tilde{\boldsymbol{\epsilon}}$ is measured from the true attitude.

We make no assumptions about the statistical nature or dimensions of the original attitude measurements, other than that they are sufficient to produce an attitude estimate. There is no reason to believe that $\tilde{\boldsymbol{\epsilon}}^*$ is Gaussian, although in practice we expect $\tilde{\boldsymbol{\epsilon}}^*$ to be Gaussian if the measurements are Gaussian. The effective measurements which we construct in this work are simply a minimum set of QUEST-like (see below) measurements which in maximum-likelihood estimation reproduces a given attitude estimate and attitude covariance matrix.

The Equivalent-Direction Model

Our goal is to determine a model for the equivalent directions which conform to the QUEST measurement model, that is, are "QUEST-like," and reproduce the given $A^{*\prime}$ and $P_{\tilde{\epsilon}\tilde{\epsilon}}$. We treat the problem first in a purely mathematical manner. The possible origin of $A^{*\prime}$ and $P_{\tilde{\epsilon}\tilde{\epsilon}}$ in real mission data is immaterial to the present discussion. The complications inherent in the treatment of "mission" data will be presented briefly later in this work.

Since the equivalent observation directions (or direction measurements) must be QUEST-like, they satisfy by definition

$$\hat{\mathbf{W}}_i^{\text{eq}} = A^{\text{eq true}} \hat{\mathbf{V}}_i^{\text{eq}} + \Delta \hat{\mathbf{W}}_i^{\text{eq}}, \qquad i = 1, \ldots, n_{\text{eq}} \tag{5}$$

and

$$\Delta \hat{\mathbf{W}}_i^{\text{eq}} \sim \mathcal{N}(\mathbf{0}, R_i^{\text{eq}}), \qquad i = 1, \ldots, n_{\text{eq}} \tag{6a}$$

$$R_i^{\text{eq}} = (\sigma_i^2)(I - \hat{\mathbf{W}}_i^{\text{eq true}} \hat{\mathbf{W}}_i^{\text{eq true}\mathrm{T}}) \qquad i = 1, \ldots, n_{\text{eq}} \tag{6b}$$

The equivalent direction measurements $\hat{\mathbf{W}}_i^{\text{eq}}$ and the equivalent measurement noise vectors $\Delta \hat{\mathbf{W}}_i^{\text{eq}}$ are random, but the reference directions $\hat{\mathbf{V}}_i^{\text{eq}}$, are nonrandom. The index i runs from 1 to n_{eq}, the minimal number of equivalent direction measurements. We choose the realization $\hat{\mathbf{W}}_i^{\text{eq}\prime}$ of the equivalent direction for the given $A^{*\prime}$ to correspond to

$$\Delta \hat{\mathbf{W}}_i^{\text{eq}\prime} = \mathbf{0} \qquad i = 1, \ldots, n_{\text{eq}} \tag{7}$$

This defines the random $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, \ldots, n_{\text{eq}}$, and the realizations $\hat{\mathbf{W}}_i^{\text{eq}\prime}$, $i = 1, \ldots, n_{\text{eq}}$, to be used in later calculations.

[6]Some authors prefer to use $[\mathbf{u}\times] \equiv -[[\mathbf{u}]]$.

The Inverse Wahba Problem

It follows from equation (7) that

$$\hat{\mathbf{W}}_i^{\mathrm{eq}\,\prime} = A^{\mathrm{eq\,true}}\hat{\mathbf{V}}_i^{\mathrm{eq}} \equiv \hat{\mathbf{W}}_i^{\mathrm{eq\,true}}, \qquad i = 1, \ldots, n_{\mathrm{eq}} \tag{8}$$

for the values that we will substitute in later calculations for the original data. Therefore, we must have

$$A^{\mathrm{eq\,true}} = A^{*\prime} \tag{9}$$

if $\hat{\mathbf{W}}_i^{\mathrm{eq}\,\prime}$ and $\hat{\mathbf{V}}_i^{\mathrm{eq}}$, $i = 1, \ldots, n_{\mathrm{eq}}$ are to reproduce $A^{*\prime}$. What were realizations of random variables for the original measured directions which led to $A^{*\prime}$ are now embodied in the truth model $A^{\mathrm{eq\,true}}$ for the equivalent direction measurements.[7]

Now we have the connection between $\hat{\mathbf{W}}_i^{\mathrm{eq\,true}}$, $i = 1, \ldots, n_{\mathrm{eq}}$ and $\hat{\mathbf{V}}_i^{\mathrm{eq}}$, $i = 1, \ldots, n_{\mathrm{eq}}$. It remains to determine $\hat{\mathbf{W}}_i^{\mathrm{eq\,true}}$, $i = 1, \ldots, n_{\mathrm{eq}}$, the corresponding equivalent inverse variances[8] $1/(\sigma_i^2)^{\mathrm{eq}}$, $i = 1, \ldots, n_{\mathrm{eq}}$, and n_{eq}.

We may write the inverse (body-referenced) covariance matrix, which, for linear Gaussian estimation problems,[9] is also the (body-referenced) attitude information matrix, as

$$P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}^{-1} = \frac{1}{\tau_1^2}\,\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^{\mathrm{T}} + \frac{1}{\tau_2^2}\,\hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^{\mathrm{T}} + \frac{1}{\tau_3^2}\,\hat{\mathbf{u}}_3\hat{\mathbf{u}}_3^{\mathrm{T}} \tag{10}$$

Here, $(1/\tau_i^2)$, $i = 1, 2, 3$, are (necessarily non-negative) characteristic values of the inverse attitude covariance matrix, and the $\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, are the associated characteristic unit vectors, which we can always choose to be a proper orthogonal set. Equally, the τ_i^2, $i = 1, 2, 3$, are the principal variances (eigenvariances, characteristic variances) of the attitude covariance matrix. For all cases of interest, since the attitude estimate exists, the inverse attitude covariance matrix must be of rank 3.

The attitude information matrix as a function of the equivalent observation directions and inverse variances for QUEST-like measurements is [1, 3, 7]

$$P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}^{-1} = \sum_{i=1}^{n_{\mathrm{eq}}} \frac{1}{(\sigma_i^2)^{\mathrm{eq}}}\left(I_{3\times 3} - (\hat{\mathbf{W}}_i^{\mathrm{eq\,true}})\,(\hat{\mathbf{W}}_i^{\mathrm{eq\,true}})^{\mathrm{T}}\right) \tag{11}$$

Clearly, from the comparison of equations (10) and (11)

$$n_{\mathrm{eq}} = 3 \tag{12}$$

[7] It is ultimately the value of the *realization* of the equivalent measurements which we must fix, since it is the realization $A^{*\prime}$ which we wish to reproduce. We could base our model on A^{true} and a nonzero value of $\Delta\hat{\mathbf{W}}_i^{\mathrm{eq}\,\prime}$, $i = 1, 2, 3$, rather than on $A^{\mathrm{eq\,true}} \equiv A^{*\prime}$, but this would be less convenient in practice. The truth model of the *artificial* equivalent directions is ours to define, as long as it accomplishes our ends, and we define it as simply as possible.

[8] It would be more correct to write $(1/\sigma_i^2)^{\mathrm{eq}}$, but that would make our expressions somewhat more cumbersome in some places. This should cause no confusion for the reader.

[9] From equation (5) we see that attitude estimation cannot be a linear Gaussian estimation problem in A or $\bar{q}$ or if the measurement noise is not Gaussian. However, the equivalent direction measurements are Gaussian by explicit construction, and to within deviations of order $|\tilde{\boldsymbol{\epsilon}}|^2$, the attitude estimation problem given the equivalent direction measurements is a linear Gaussian estimation problem in $\tilde{\boldsymbol{\epsilon}}$. We make no assertion that the original estimation problem which led to $A^{*\prime}$ be Gaussian.

The result follows also from simple counting.[10] Expanding equation (11), we obtain

$$P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} = \left(\frac{1}{(\sigma_2^2)^{\text{eq}}} + \frac{1}{(\sigma_3^2)^{\text{eq}}}\right)\hat{\mathbf{W}}_1^{\text{eq true}}\hat{\mathbf{W}}_1^{\text{eq true}\mathrm{T}} + \left(\frac{1}{(\sigma_3^2)^{\text{eq}}} + \frac{1}{(\sigma_1^2)^{\text{eq}}}\right)\hat{\mathbf{W}}_2^{\text{eq true}}\hat{\mathbf{W}}_2^{\text{eq true}\mathrm{T}} + \left(\frac{1}{(\sigma_1^2)^{\text{eq}}} + \frac{1}{(\sigma_2^2)^{\text{eq}}}\right)\hat{\mathbf{W}}_3^{\text{eq true}}\hat{\mathbf{W}}_3^{\text{eq true}\mathrm{T}} \tag{13}$$

Comparing equation (13) with equation (10), we have immediately that

$$\hat{\mathbf{W}}_i^{\text{eq true}} = \hat{\mathbf{u}}_i, \qquad i = 1, 2, 3 \tag{14a}$$

$$\frac{1}{(\sigma_i^2)^{\text{eq}}} = \frac{1}{2}\left(\frac{1}{\tau_1^2} + \frac{1}{\tau_2^2} + \frac{1}{\tau_3^2}\right) - \frac{1}{\tau_i^2}, \qquad i = 1, 2, 3 \tag{14b}$$

and from equation (8),

$$\hat{\mathbf{V}}_i^{\text{eq}} = A^{\text{eq true}\mathrm{T}}\hat{\mathbf{W}}_i^{\text{eq true}} = A^{*\prime\mathrm{T}}\hat{\mathbf{W}}_i^{\text{eq true}}, \qquad i = 1, 2, 3 \tag{14c}$$

Equations (14) are the solution of the inverse Wahba problem and the solution for the equivalent-direction representation. Clearly, if we estimate the attitude and compute the attitude covariance given the quantities in equations (14), we will obtain

$$A^{\text{eq}*\prime} = A^{\text{eq true}} = A^{*\prime} \quad \text{and} \quad P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{eq}} = P_{\tilde{\epsilon}\tilde{\epsilon}} \tag{15}$$

Q.E.D.

Note that given equation (8) one could write $\hat{\mathbf{W}}_i^{\text{eq}\prime}$, $i = 1, 2, 3$, as the equivalent observation directions in the equivalent-direction representation in place of $\hat{\mathbf{W}}_i^{\text{eq true}}$, but the latter choice seems less ambiguous, because $\hat{\mathbf{W}}_i^{\text{eq}\prime}$, $i = 1, 2, 3$, could be misinterpreted to mean any realization of equation (5). Note also that the equivalent directions cannot reproduce necessarily the probability distribution of the attitude estimator. It has been constructed solely to produce *one* realization of that estimator and the attitude covariance matrix.

Define the attitude co-information matrix $D_{\tilde{\epsilon}\tilde{\epsilon}}$ from reference [1] as

$$D_{\tilde{\epsilon}\tilde{\epsilon}} \equiv \frac{1}{2}(\text{tr}\, P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1})I_{3\times3} - P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} = \sum_{i=1}^{3}\frac{1}{(\sigma_1^2)^{\text{eq}}}\hat{\mathbf{W}}_i^{\text{eq true}}\hat{\mathbf{W}}_i^{\text{eq true}\mathrm{T}} \tag{16}$$

with "tr" denoting the trace operation. Then, within the model of equations (5) and (6), the three equivalent true observation directions $\hat{\mathbf{W}}_i^{\text{eq true}}$, $i = 1, 2, 3$, are the characteristic unit vectors of the body-referenced attitude co-information matrix, the three equivalent reference directions $\hat{\mathbf{V}}_i^{\text{eq}}$, $i = 1, 2, 3$, are the three corresponding characteristic unit vectors of the space-referenced attitude co-information matrix, and the three equivalent inverse variances are the corresponding three characteristic values of the attitude co-information matrix, but only for the equivalent direction measurements, for which $A^{\text{eq true}}$ is the "true" attitude.[11]

[10]The given attitude estimate has three independent parameters, the given attitude covariance matrix six, making a total of nine independent parameters, just the total number of parameters in the orthonormal triad of equivalent observation directions (3), the equivalent reference-direction triad (3), and the three equivalent variances. If there were only two equivalent directions, then there would be only eight independent parameters, not quite enough.

[11]Note that characteristic vectors of the attitude co-information matrix are also characteristic values of the attitude information matrix and (because it will always exist in cases where the attitude is observable) of the attitude covariance matrix. Note also that in data fusion it is $\hat{\mathbf{W}}_i^{\text{eq true}}$, $i = 1, 2, 3$, which one will substitute as the realizations of the effective measurement, because these quantities already embody the realization of the measurement noise of the original measurements which led to $A^{*\prime}$.

Properties of the Equivalent-Direction Representation

How Many Equivalent Directions?

Examine the attitude co-information matrix for the special case that all of the original measurements were QUEST-like. We can also write

$$D_{\bar{\epsilon}\bar{\epsilon}} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k^{\text{true}} \hat{\mathbf{W}}_k^{\text{true}\,\mathrm{T}} \tag{17}$$

where $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N \geq 2$, are the N original QUEST-like measurements, each with variance parameter σ_k^2, $k = 1, \ldots, N$. The attitude co-information matrix in this case is clearly positive-semidefinite and at least of rank 2 if the attitude is to be observable. (An attitude co-information matrix of rank 1 is possible only if the true values of the N original QUEST-like measurement vectors are all mutually parallel or antiparallel, in which case the attitude is not observable.) Thus, there can be at most one equivalent inverse variance which can vanish, and that only in the case that the N original QUEST-like measurement vectors are all coplanar. If there are three linearly-independent (true) direction measurements, that is, not coplanar, then the attitude co-information matrix must be rank 3, i.e., positive definite. In that case, all three equivalent inverse variances are positive. In the case where one equivalent inverse variance vanishes, there are, *effectively,* only two equivalent directions, but *formally,* there are always three.

For non-QUEST-like measurements, it is possible that the attitude co-information matrix be full rank (that is, rank equal to 3) even when there are only two direction measurements. The predicted directions of the succeeding article [9] will have this property.

Two for Two

As an explicit example of a case when there are only two nonvanishing equivalent inverse variances, suppose that the attitude is estimated for the Wahba problem with two QUEST-like measurements. Let $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{W}}_2$ be the two noncollinear QUEST-like measurements with respective (positive) variance parameters σ_1^2 and σ_2^2. The three equivalent inverse variances are easily calculated as $1/(\sigma_a^2)^{\text{eq}}$, $1/(\sigma_b^2)^{\text{eq}}$ and $1/(\sigma_c^2)^{\text{eq}}$, with

$$\frac{1}{(\sigma_a^2)^{\text{eq}}} = \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) + \frac{1}{2}\sqrt{\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^2 - \frac{4\beta^2}{\sigma_1^2\sigma_2^2}} \tag{18a}$$

$$\frac{1}{(\sigma_b^2)^{\text{eq}}} = \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) - \frac{1}{2}\sqrt{\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^2 - \frac{4\beta^2}{\sigma_1^2\sigma_2^2}} \tag{18b}$$

$$\frac{1}{(\sigma_c^2)^{\text{eq}}} = 0 \tag{18c}$$

where

$$\beta \equiv \left|\hat{\mathbf{W}}_1^{\text{true}} \times \hat{\mathbf{W}}_2^{\text{true}}\right| \tag{19}$$

We have used subscripts a, b, c to avoid confusion with the standard deviations σ_1 and σ_2 associated with the original direction measurements. The three equivalent inverse variances are always non-negative in this case, and an equivalent-direction representation does indeed exist. Note that *two* of the equivalent inverse variances vanish when the two original measurements are collinear, as expected.

The Curse of Negativity

If the original measurements were not QUEST-like, or if they were QUEST-like, and the estimation process were not maximum-likelihood estimation, then the attitude co-information matrix may not be positive semi-definite. Thus, one of the three equivalent inverse variances may be negative. No more than one equivalent inverse variance can be negative. To see this, note that

$$\frac{1}{(\sigma_1^2)^{\mathrm{eq}}} < 0 \quad \text{and} \quad \frac{1}{(\sigma_2^2)^{\mathrm{eq}}} < 0 \quad \text{implies} \quad \frac{1}{\tau_3^2} < 0 \tag{20}$$

which is impossible. Negative effective variances may still be used *within the Wahba problem,* since the three equivalent directions and variances will lead to the correct attitude profile matrix, which is physical even if the equivalent measurements are not.[12] Their use in *sequential* estimation processes, especially in the Kalman filter, could prove disastrous, however.[13] It is mathematically impossible to have a measurement with a covariance matrix which is not positive-semidefinite. We should think of the set of equivalent direction "measurements" with a negative equivalent inverse variance as only a different representation of the attitude profile matrix B and not as possible data.

It is trivially obvious that if one equivalent inverse variance vanishes, then the other two must be positive if the attitude is observable. As a corollary, noting equation (20), if one equivalent inverse variance is negative, the other two must be positive.

A negative equivalent inverse variance occurs, for example, when the characteristic values of the attitude information matrix are in the ratio 5 : 2 : 1, in which case the corresponding characteristic values of the attitude co-information matrix are in the ratio −1 : 2 : 3, which means that one equivalent QUEST-like measurement has negative variance.[14]

Degeneracy

It is easily verified that a degeneracy in the attitude covariance matrix will lead to a degeneracy in the equivalent inverse variances. As an extreme case, if the attitude covariance matrix is a multiple of the 3×3 identity matrix, then the equivalent inverse variances will also be equal, and the observation directions will be arbitrary, except that they must form an orthonormal triad.

[12]F. Landis Markley, recalling the golden days of our youth as theoretical physicists, has suggested that these negative-variance "pseudo-measurements" be referred to as "ghost" measurements in analogy with a similar phenomenon in Quantum Field Theory calculations.

[13]Consider, for example, the case where one of the equivalent variances were negative and we processed the associated equivalent direction and its now negative-semidefinite information matrix as the first measurement in an information Kalman filter. We assume that there is no initial attitude information. The result for the attitude estimate following this first update would be an attitude estimation-error information matrix which is negative-semidefinite. We cannot write the probability density function for an attitude estimate with negative-definite information matrix. When one of the equivalent variances is negative, then only the complete set of three equivalent directions measurements and equivalent inverse variances can be used in an estimation problem and then only to construct the attitude profile matrix B. All other uses should be avoided. We must be careful not to mistake the matrix equations of the Kalman filter for fundamental estimation theory, which lies rather in the underlying probability density functions.

[14]For similar reasons one cannot have an inertia tensor with principal moments in the ratio 5 : 2 : 1. Unlike the similar situation for the inertia tensor, characteristic attitude inverse variances in the ratio 5 : 2 : 1 are not unphysical; they simply cannot be modeled by equivalent QUEST-like observation directions with non-negative inverse variances. Principal variances in the ratio 5 : 2 : 1 can be obtained, for example, from the application of the TRIAD algorithm to QUEST-like measurements, as we shall see below.

The Wahba Cost Function

For the equivalent directions as sole inputs, the Wahba cost function is not only minimized for $A = A^{*\prime}$ but vanishes at that value.

The Equivalent Direction Representation and the Singular-Value Decomposition

The singular-value decomposition (SVD) of the attitude profile matrix B provides a simple computational method for finding the equivalent directions and their associated equivalent variances.[15] According to the Singular-Value Decomposition Theorem [10] B can be decomposed as[16]

$$B = D_{\tilde{\epsilon}\tilde{\epsilon}}\, A^{*\prime} = B^{\mathrm{eq}} = \sum_{i=1}^{3} \frac{1}{(\sigma_i^2)^{\mathrm{eq}}}\, \hat{\mathbf{W}}_i^{\mathrm{eq\ true}} \hat{\mathbf{V}}_i^{\mathrm{eq}\,\mathrm{T}} = W\, S\, V^{\mathrm{T}} \tag{21}$$

where S is diagonal and positive-semidefinite with diagonal elements s_1, s_2, s_3, and W and V are orthogonal. Equivalently, we may write

$$B^{\mathrm{eq}} = W''\, S''\, V''^{\mathrm{T}} \tag{22}$$

where S'' is diagonal and may have at most one negative diagonal element and W'' and V'' are both *proper* or both *improper* orthogonal. Markley [11] has shown that the optimal estimate of the attitude is given by[17]

$$A^{*\prime} = W''\, V''^{\mathrm{T}} \tag{23}$$

Hence, $A^{*\prime}$ must be the given attitude estimate. Comparing equations (21) and (22) yields

$$W'' = [\hat{\mathbf{W}}_1^{\mathrm{eq\ true}}, \hat{\mathbf{W}}_2^{\mathrm{eq\ true}}, \hat{\mathbf{W}}_3^{\mathrm{eq\ true}}] \quad \text{and} \quad V'' = [\hat{\mathbf{V}}_1^{\mathrm{eq}}, \hat{\mathbf{V}}_2^{\mathrm{eq}}, \hat{\mathbf{V}}_3^{\mathrm{eq}}] \tag{24ab}$$

where the matrices in each right member are labeled by their column vectors, and

$$S'' = \mathrm{diag}\left[1/(\sigma_1^2)^{\mathrm{eq}},\, 1/(\sigma_2^2)^{\mathrm{eq}},\, 1/(\sigma_3^2)^{\mathrm{eq}}\right] \tag{24c}$$

where $\mathrm{diag}\,[a, b, \ldots, z]$ denotes a diagonal matrix given by its diagonal elements. Thus,

$$A^{*\prime} = \sum_{i=1}^{3} \hat{\mathbf{W}}_i^{\mathrm{eq\ true}}\, \hat{\mathbf{V}}_i^{\mathrm{eq}\,\mathrm{T}} \tag{25}$$

which follows equally from equation (14c).[18] Equation (25) is reminiscent of a similar expression for the TRIAD attitude solution [3, 12–15].[19]

[15]Simple because the singular-value decomposition is normally included in Linear Algebra software packages. The SVD algorithm itself is not so simple.
[16]Again, note that B need not have been computed from QUEST-like measurements or even from 3×1 vector measurements or from Gaussian measurements.
[17]The usual mechanization of the singular-value decomposition applied to a 3×3 matrix leads to singular values $s_1 \geq s_2 \geq s_3 \geq 0$. For use in Markley's SVD attitude estimation algorithm and for the equivalent-direction representation, we must make the substitutions: $S''_{11} = s_1 \equiv s''_1$, $S''_{22} = s_2 \equiv s''_2$, $S''_{33} = \det(WV)s_3 \equiv s''_3$.
[18]We might equally well speak of an SVD representation with elements W'', V'', and s''_i, $i = 1, 2, 3$, from which $D_{\tilde{\epsilon}\tilde{\epsilon}} = W'' S'' W''^{\mathrm{T}}$ and $P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} = (\mathrm{tr}\, D_{\tilde{\epsilon}\tilde{\epsilon}}) I_{3\times 3} - D_{\tilde{\epsilon}\tilde{\epsilon}}$. $A^{*\prime}$ is then given by equation (23).
[19]The TRIAD attitude estimate can be written as $A^{*\mathrm{TRIAD}\prime} = \Sigma_{i=1}^{3}\, \hat{\mathbf{s}}'_i\, \hat{\mathbf{r}}_i'^{\mathrm{T}}$, which is similar in form to equation (25). Thus, the TRIAD algorithm constructs a set of three (non-QUEST-like) "equivalent" measurements. Obviously, one cannot create an equivalent-direction representation based on the TRIAD algorithm, because it would be characterized by only *eight* independent parameters, insufficient, in general, to reconstruct the attitude estimate and attitude covariance matrix.

Our analysis of the equivalent-vector representation in this article, has been carried out in the absence of measurement noise,[20] while Markley's SVD algorithm operates with realizations of those measurements corrupted by measurement noise. Nonetheless, the equivalent-direction representation and Markley's SVD attitude estimation algorithm provide insights into one another. We know that the inverse equivalent variance smallest in magnitude corresponds to $s_3'' \equiv (\det W)(\det V)s_3$. Hence, a negative value of s_3'' corresponds to a negative equivalent variance in the equivalent-direction representation. Therefore, from the results for the equivalent-direction representation, we may conclude: (1) that if the measurements are all QUEST-like, then (from equation (17)) at truth, s_3 will always be non-negative and a negative s_3'' can derive only from the measurement noise; (2) (also from equation (17)) that s_3 will vanish if and only if the measurement vectors are coplanar; (3) that (from equation (20)) at truth s_1 and s_2 must always be positive; and (4) that a negative value of s_3'' for the *generalized* Wahba problem need not necessarily correspond to an unphysical or poorly-determined attitude (see the TRIAD example below). From the results for Markley's SVD algorithm, we know: (1) that at truth a negative equivalent inverse variance will always be smaller in magnitude than the other two (and likely so for corrupted measurements); and (2) that there can be no more than one negative equivalent inverse variance. Some of these results are known already from studies within the development of each of the two algorithms.

The Equivalent-Direction Representation and the TRIAD Algorithm

The TRIAD algorithm [3, 12–15] coupled with the QUEST measurement model, which leads to the commonly used formulas for the TRIAD covariance matrix, can lead to unphysical equivalent inverse variances. Here, because the second (generally, less accurate) measurement has been tampered with in order to create an orthonormal triad of unit-vector measurements, the attitude information matrix is given by

$$(P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{TRIAD}})^{-1} = \frac{1}{\sigma_1^2}(\hat{\mathbf{s}}_2^{\text{true}}\hat{\mathbf{s}}_2^{\text{true}T} + \hat{\mathbf{s}}_3^{\text{true}}\hat{\mathbf{s}}_3^{\text{true}T}) + \frac{1}{\sigma_2^2}\hat{\mathbf{s}}_4^{\text{true}}\hat{\mathbf{s}}_4^{\text{true}T} \tag{26}$$

where generation of an orthonormal triad has led to the loss of one term of the attitude information matrix compared to that of the QUEST algorithm. The attitude co-information matrix for the TRIAD algorithm becomes

$$D_{\text{TRIAD}} = \frac{1}{\sigma_1^2}\hat{\mathbf{s}}_1^{\text{true}}\hat{\mathbf{s}}_1^{\text{true}T} + \frac{1}{\sigma_2^2}\left(\frac{1}{2}I_{3\times3} - \hat{\mathbf{s}}_4^{\text{true}}\hat{\mathbf{s}}_4^{\text{true}T}\right) \tag{27}$$

where, in equations (26) and (27), following the usual TRIAD notation [3, 13],

$$\hat{\mathbf{s}}_1 = \hat{\mathbf{W}}_1, \qquad \hat{\mathbf{s}}_2 = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|}, \qquad \hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \quad \text{and} \quad \hat{\mathbf{s}}_4 = \hat{\mathbf{W}}_2 \times \hat{\mathbf{s}}_2 \tag{28}$$

Here, $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{W}}_2$ are the two QUEST-like direction measurements which are inputs to the TRIAD algorithm. It is usually assumed but not required that $\sigma_1^2 \leq \sigma_2^2$. The second term of the right member of equation (27) is clearly not positive-semidefinite, which hints that D_{TRIAD} may not be positive-semidefinite.

[20]By this we mean that the particular realization of the measurements contains no measurement noise, not that the measurement noise variances vanish. We call this condition "at truth."

The three characteristic values of D_{TRIAD} are readily calculated to yield

$$\frac{1}{(\sigma_a^2)^{\text{eq}}} = \frac{1}{2\sigma_2^2} \tag{29a}$$

$$\frac{1}{(\sigma_b^2)^{\text{eq}}} = \frac{1}{2\sigma_1^2} + \frac{1}{2}\sqrt{\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^2 - \frac{4\beta^2}{\sigma_1^2\sigma_2^2}} \tag{29b}$$

$$\frac{1}{(\sigma_c^2)^{\text{eq}}} = \frac{1}{2\sigma_1^2} - \frac{1}{2}\sqrt{\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^2 - \frac{4\beta^2}{\sigma_1^2\sigma_2^2}} \tag{29c}$$

with $\beta = |\hat{\mathbf{W}}_1^{\text{true}} \times \hat{\mathbf{W}}_2^{\text{true}}|$ as in equation (19). The argument of the square root can never be negative. However, the third equivalent inverse variance $1/(\sigma_c^2)^{\text{eq}}$ will become negative for

$$\beta < \frac{1}{2}\sqrt{2 + \sigma_1^2/\sigma_2^2} \equiv \beta_0 \tag{30}$$

Equations (29) should be compared with equations (18). As $\sigma_1^2/\sigma_2^2 \to 0$, a negative characteristic value will develop for $\beta < \beta_0 = 1/\sqrt{2} = \sin \pi/4$. For $\sigma_1^2 = \sigma_2^2$, $\beta_0 = \sqrt{3}/2 = \sin \pi/3$. Thus, for at least half of the values of β when $\sigma_1^2 \leq \sigma_2^2$, the TRIAD attitude covariance matrix cannot be derived from QUEST-like measurements with non-negative variance parameters.[21] This does not mean that the TRIAD attitude covariance matrix is an unacceptable attitude covariance matrix, only that it cannot be derived in maximum-likelihood estimation from three mutually-orthogonal *physically meaningful* equivalent direction measurements conforming to the QUEST measurement model.[22] A TRIAD estimate of the attitude can always be incorporated into the Wahba problem either by using the possibly unphysical equivalent directions and equivalent inverse variances or by using the special forms of the matrices B or K developed in reference [1]. It is certainly more efficient to do so as well.

The possible unphysicality of the equivalent-direction representation cannot be overcome by relaxing the constraint that the equivalent direction measurements be mutually orthogonal. We saw from equations (18) that given two original direction measurements separated by an angle other than 0 or π, one can always find mutually perpendicular equivalent direction measurements with positive equivalent variances which produce the same attitude estimate and attitude covariance matrix. Thus, the possible lack of physicality of the equivalent directions arising from the application of the equivalent-direction representation to the results of the TRIAD algorithm is insurmountable.

Further Generalization of the Wahba Problem

Using the insights gained from the application of the equivalent-direction representation to original direction measurements consistent with the QUEST measurement model, can we incorporate other measurement types in our generalized Wahba problem? An obvious case of interest would be unit-vector sensors whose measured

[21] The ratio $-1:2:3$ will be obtained for the equivalent variances when $\sigma_1^2 = 2\sigma_2^2$ and $\beta = \sqrt{5/8}$. Note that a negative equivalent inverse variance will occur whenever $\sigma_1^2 > 2\sigma_2^2$. Normally, one orders the two direction measurements in the TRIAD algorithm so that $\sigma_1^2 \leq \sigma_2^2$, but this is not a requirement.

[22] For a more microscopic discussion of the structure of the TRIAD covariance matrix, see reference [14].

directions have an ellipse of error in the measurement tangent plane rather than the simplified assumption of a circle of error in the QUEST measurement model.

One would imagine for such a measurement a (non-invertible) attitude information matrix of the form[23]

$$\text{“}R^{-1}\text{”} = \frac{1}{\sigma_1^2}\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^{\mathrm{T}} + \frac{1}{\sigma_2^2}\hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^{\mathrm{T}}, \qquad \sigma_1^2 < \sigma_2^2 \tag{31}$$

where here $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ are the directions of the minor and major axes of the error ellipse in the tangent plane, σ_1 and σ_2 the corresponding standard deviations, and $\hat{\mathbf{u}}_3 = \hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2$ the measured direction. Can we find equivalent directions in this case?

The corresponding attitude co-information matrix and attitude profile matrix are

$$D_{\text{true}} = \frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^{\mathrm{T}} + \frac{1}{2}\left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^{\mathrm{T}} + \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\hat{\mathbf{u}}_3\hat{\mathbf{u}}_3^{\mathrm{T}} \tag{32}$$

$$B_{\text{true}} = \frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\hat{\mathbf{u}}_1\hat{\mathbf{v}}_1^{\mathrm{T}} + \frac{1}{2}\left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\hat{\mathbf{u}}_2\hat{\mathbf{v}}_2^{\mathrm{T}} + \frac{1}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\hat{\mathbf{u}}_3\hat{\mathbf{v}}_3^{\mathrm{T}} \tag{33}$$

with $\hat{\mathbf{v}}_i = A^{\mathrm{T}}\hat{\mathbf{u}}_i$, $i = 1, 2, 3$, the corresponding reference direction. Here lies the rub. $\hat{\mathbf{u}}_3$ is the actual measured direction and $\hat{\mathbf{v}}_3$ is the reference direction as given by a Sun ephemeris, geomagnetic field model, spacecraft orbit, etc. However, there is no way to model $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ without knowing the attitude. To estimate the attitude by solving a smaller estimation problem involving this sensor and one other (in order to produce an equivalent B) makes little sense because of the large computational burden. Thus, in one way or another one is simply back to general iterative batch least-square attitude estimation. There is no escape from this situation. The attractiveness of the Wahba problem is that the computational burden is small if one uses one of the fast algorithms, but that savings is lost if one must execute a burdensome least-square calculation in order to obtain the parameters for the Wahba problem. In addition, the equivalent inverse variance for $\hat{\mathbf{u}}_2$ is negative. Thus, it would seem that the generalized Wahba problem and the equivalent-direction representation are limited to direction sensors with circles of error and complete-attitude sensors. The difficulty here is similar to that of the closely related problem of constructing equivalent matrices B and K for scalar measurements explored in reference [1].

Equivalent Directions as Random Variables

The conceit of the equivalent-direction representation is that we are given a realization of the attitude and an *exact true value* of the *body-referenced* attitude covariance matrix. Thus, from the attitude covariance matrix we know $\hat{\mathbf{W}}_i^{\text{eq true}}$ and $1/(\sigma_i^2)^{\text{eq}}$, $i = 1, 2, 3$, exactly but know $\hat{\mathbf{V}}_i^{\text{eq}}$ only as $A^{*\prime\mathrm{T}}\, \hat{\mathbf{W}}_i^{\text{eq true}}$, $i = 1, 2, 3$. For a given $P_{\tilde{\epsilon}\tilde{\epsilon}}$, we obtain a different set of equivalent *reference* directions for every value of realization $A^{*\prime}$. This is the opposite of how we expect a random measurement to behave. The randomness of our equivalent vectors, however, is that it is $\hat{\mathbf{W}}_i^{\text{eq}\prime}$, $i = 1, 2, 3$, which changes for every realization of $\Delta\hat{\mathbf{W}}_i^{\text{eq}}$. This is not a contradiction, however. There is no need for the random errors in $\Delta\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, to be related to those of A^*. We are interested in only one realization of the $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$,

[23]Perhaps, it would be better to write $R^{\#}$, the Moore-Penrose pseudo-inverse of R, because R is not always invertible, or to speak instead of the Fisher information matrix associated with this measurement.

in any event, namely, that corresponding to $\Delta\hat{\mathbf{W}}_i^{\text{eq}\prime} = \mathbf{0}$, $i = 1, 2, 3$.

Suppose a different problem were posed, and we were given instead the realization $A^{*\prime}$ again but now the exact true value of the *space-referenced* attitude covariance matrix. In that case we would determine $1/(\sigma_i^2)^{\text{eq}}$ and $\hat{\mathbf{V}}_i^{\text{eq}}$, $i = 1, 2, 3$, from the space-referenced attitude covariance matrix while determining $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, from the inverse of equation (14c). In this case, $\hat{\mathbf{V}}_i^{\text{eq}}$, $i = 1, 2, 3$, are fixed for all values of $A^{*\prime}$, and

$$\hat{\mathbf{W}}_i^{\text{eq}\prime} = A^{*\prime}\hat{\mathbf{V}}_i^{\text{eq}}, \qquad i = 1, 2, 3 \tag{34}$$

varies with different values of the realization $A^{*\prime}$, which sits better with our intuitive notions.[24]

From equation (34) it is tempting to assert for the *random* equivalent direction that

$$\hat{\mathbf{W}}_i^{\text{eq}} = A^{*}\hat{\mathbf{V}}_i^{\text{eq}}, \qquad i = 1, 2, 3 \tag{35}$$

This, however, cannot be true if $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, are QUEST-like measurements. The three QUEST-like random equivalent direction measurements are characterized by six independent Gaussian zero-mean error sources, namely, the six active components of the combined three $\Delta\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, while the attitude estimate A^* is characterized by only three independent error sources, namely, the three statistically independent components of $\tilde{\boldsymbol{\epsilon}}^*$, which need not be Gaussian. Thus, the random distribution of the equivalent directions cannot be derived from the distribution of A^*, which may not even be Gaussian. Equation (34) can hold only for a single realization $A^{*\prime}$ and cannot be understood as the realization of an equation in random variables. Equation (35) is simply inconsistent with $\hat{\mathbf{W}}_i^{\text{eq}}$, $i = 1, 2, 3$, being QUEST-like for any A^*.

There is a set of effective measurements which does satisfy an equation like equation (35). These are the predicted directions, which are examined in the succeeding article [9].

Some Comments on Implementation in Mission Support

Although it is not anticipated that the equivalent-direction representation will be implemented in day-to-day mission support, it is nonetheless worthwhile to consider some "practical" questions.

Computational Considerations for Sampled Data

The application of the equivalent-direction representation to mission support immediately encounters an ambiguity. There are two incompatible attitude covariance matrices when one deals with real data, for example, with star trackers. There, one has a body-referenced attitude covariance matrix calculated from the measured star directions, and one has also the space-referenced attitude covariance matrix calculated from the directions in the star catalog. Generally, there is no proper orthogonal matrix connecting the two.

Since the directions in the star catalog are generally known with far greater accuracy than the directions measured by the star tracker, it makes sense to compute the space-referenced attitude information matrix and by spectral decomposition the

[24]Of course, the fact that our intuition may not be satisfied with the result of the previous case says more about our intuition than about the correctness of the equivalent-direction representation.

equivalent reference directions and the equivalent inverse variances. The equivalent direction measurements are then obtained from the inverse of equation (14c). However, this more accurate procedure will make little practical difference.

Markley's SVD Algorithm

Markley's SVD algorithm works in the same way. One may still call the equivalent observation directions $\hat{\mathbf{W}}_i^{\text{eq true}}$, $i = 1, 2, 3$, because they are the truth model within the equivalent-direction representation. As said earlier, the mechanization of the equivalent-direction representation is independent of the provenance of the given attitude and attitude covariance matrix, although the values are not.

The Tastelessness of Equivalent Directions

In mission support operations, we have no choice but to determine the equivalent direction representation from the sampled covariance matrix $P'_{\tilde{\epsilon}\tilde{\epsilon}}$, since $P_{\tilde{\epsilon}\tilde{\epsilon}}$ is unavailable. Because the sampled equivalent inverse variances are computed from the sampled $D'_{\tilde{\epsilon}\tilde{\epsilon}}$ rather than from $D_{\tilde{\epsilon}\tilde{\epsilon}}$, we must have [1]

$$\sum_{i=1}^{3} \frac{1}{(\sigma'^2_i)^{\text{eq}}} = \text{tr}\, D''_{\tilde{\epsilon}\tilde{\epsilon}} = \lambda'_{\max} \tag{36}$$

and not $\lambda_0 = \Sigma_{k=1}^{N} 1/\sigma_k^2$. Therefore, the equivalent-direction representation disables the TASTE test [1, 16].

Equivalent Directions in Data Fusion

Let us suppose that we have estimated the attitude and computed the attitude covariance matrix from a set of data and from these determined the equivalent-direction representation. Then, if one is given additional measurements $\mathbf{z}'_i$, $i = 1, \ldots, n$, we may combine them with the old data for a better estimate of the attitude by finding the attitude increment vector $\boldsymbol{\xi}$ which minimizes the cost function[25]

$$J(A) = \frac{1}{2}\sum_{i=1}^{3} \frac{1}{(\sigma_i^2)^{\text{eq}}} |\hat{\mathbf{W}}_i^{\text{eq true}} - A(\boldsymbol{\xi})\hat{\mathbf{V}}_i^{\text{eq}}|^2 + \frac{1}{2}\sum_{i=1}^{n} [\mathbf{z}'_i - \mathbf{f}_i(A(\boldsymbol{\xi}))]^{\text{T}} R_i^{-1} [\mathbf{z}'_i - \mathbf{f}_i(A(\boldsymbol{\xi}))] \tag{37}$$

where[26]

$$A(\boldsymbol{\xi}) \equiv e^{[[\boldsymbol{\xi}]]} A^{\text{eq true}} \tag{38}$$

$\mathbf{f}_i(A)$, $i = 1, \ldots, n$, are known functions of the attitude of unspecified dimension, and where the equivalent directions represent the earlier data. We have assumed that the $\mathbf{z}_i$, $i = 1, \ldots, n$, are characterized by additive zero-mean Gaussian measurement noise with respective measurement covariance matrices R_i, $i = 1, \ldots, n$. Such a method is less practical than the general method embodied in equations (66) through (70) of reference [1], but in theoretical analyses the equivalent directions may offer insights, because the error levels of the input attitude estimate are so manifest in the equivalent variances. Equation (37) is particularly attractive when the $\mathbf{z}'_i$, $i = 1, \ldots, n$, conform to the QUEST measurement model.

[25]The treatment of such cost functions in the form of equation (37) receives considerable attention in reference [17].
[26]Recall that $A^{\text{eq true}}$ is the given $A^{*\prime}$.

Discussion and Conclusions

A general method has been developed for including estimates of the spacecraft attitude and the associated attitude covariance matrices in the Wahba problem by means of three equivalent (observation) directions, three equivalent reference directions and equivalent inverse variances. Equivalent directions with three non-negative equivalent inverse variances do not exist for every given attitude covariance matrix, so that this method of representing the attitude and attitude covariance cannot be applied universally. A physically meaningful equivalent-direction representation will always occur if the original measurements were QUEST-like, as shown by equation (17), and the attitude estimates were calculated within maximum-likelihood estimation. When the equivalent inverse variances are all non-negative, then the equivalent observations can, in fact, be used in any estimation problem, whether or not the original measurements leading to the attitude estimate and attitude covariance matrix conformed to the QUEST measurement model or were even vectors. Thus, the equivalent directions may be used in data fusion, but they are impractical because of the need to compute the spectral decomposition of the covariance matrix of the attitude estimate from the original data. They do, however, provide insight into such problems when the equivalent inverse variances are non-negative.

Within the Wahba problem, the equivalent-direction representation, even when one equivalent inverse variance is negative, is identical to the complete Wahba problem using all data and with no loss of attitude information. As we have seen, the attitude covariance matrix of the TRIAD algorithm does not always admit a representation in terms of equivalent directions with non-negative inverse variances.

Using an equivalent direction with negative variance in the attitude Kalman filter [18] could be disastrous, as we have seen.[27] Only in a least-square batch cost function might it prove trouble-free, because the substitution is equivalent to using the original cost function based on the positive-definite original covariance matrix. However, elimination of one of the equivalent direction measurements, say the one with negative variance (the "ghost" measurement in this case) could introduce significant error and effectively overweight some of the data. It is probably best to leave the equivalent-direction representation to the Wahba problem. A phasmaphobic approach would seem to be advisable. *Cave phantasmata!* Beware of ghosts! While the equivalent directions will likely never find a place in day-to-day mission support, they may be a useful tool for analysis and for design.

It is important to note that the introduction of the equivalent direction measurements in the data-fusion problem causes the earlier attitude estimator to be approximated as Gaussian, whatever its true statistics. This is true also for equations (66) through (70) of reference [1]. We may speak of maximum-likelihood estimation, but in practice we are almost always forced to perform least-square estimation.

As we shall show in a succeeding article [9], the equivalent directions are not the only possible choice for a set of effective direction measurements which satisfy $\hat{\mathbf{W}}_i' = A^{*\prime}\hat{\mathbf{V}}_i$ and which in maximum-likelihood estimation lead to a given attitude estimate and attitude covariance matrix. There is at least one other kind of effective

[27] There are other reasons not to use the equivalent vectors in simulation, as shown in reference [9].

direction measurement, the *predicted directions* [9], which lack the one bad property of the equivalent directions (they are always physically meaningful), but lack their most important property (they cannot be used in the Wahba problem). The fact there there are more than one fundamental kind of effective direction measurement is certainly of considerable import.

An application of the equivalent-direction representation (and of the predicted directions) may be found in reference [19].

Acknowledgments

The author is grateful to F. Landis Markley and, especially, Yang Cheng for many interesting and helpful discussions.

References

[1] SHUSTER, M. D. "The Generalized Wahba Problem," *The Journal of the Astronautical Sciences,* Vol. 54, No. 2, April–June 2006, pp. 245–259.
[2] WAHBA, G. "Problem 65-1: A Least Squares Estimate of Spacecraft Attitude," *SIAM Review,* Vol. 7, No. 3, July 1965, p. 409.
[3] SHUSTER, M. D. and OH, S. D. "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control,* Vol. 4, No. 1, January–February. 1981, pp. 70–77.
[4] MARKLEY, F. L. and MORTARI, M. "Quaternion Attitude Estimation Using Vector Measurements," *The Journal of the Astronautical Sciences,* Vol. 48, Nos. 2 and 3, April–September 2000, pp. 359–380.
[5] CHENG, Y. and SHUSTER, M. D. "Robustness and Accuracy of the QUEST Algorithm," presented as paper AAS-07-102 at the 17th Space Flight Mechanics Meeting, Sedona Arizona, January 28–February 2, 2007; Proceedings: *Advances in the Astronautical Sciences,* Vol. 127, 2007, pp. 41–61.
[6] CHENG, Y. and SHUSTER, M. D. "The Speed of Attitude Estimation," presented as paper AAS-07-105 at the 17th Space Flight Mechanics Meeting, Sedona Arizona, January 28–February 2, 2007; Proceedings: *Advances in the Astronautical Sciences,* Vol. 127, 2007. pp. 101–116.
[7] SHUSTER, M. D. "Maximum Likelihood Estimation of Spacecraft Attitude," *The Journal of the Astronautical Sciences,* Vol. 37, No. 1, January–March, 1989, pp. 79–88.
[8] SHUSTER, M. D. "A Survey of Attitude Representations," *The Journal of the Astronautical Sciences,* Vol. 41, No. 4, October–December 1993, pp. 439–517.
[9] SHUSTER, M. D. "Effective Direction Measurements for Spacecraft Attitude: II. Predicted Directions," *The Journal of the Astronautical Sciences,* Vol. 55, No. 4, October–December 2007, pp. 479–492.
[10] GOLUB, G. H. and VAN LOAN, C. F. *Matrix Computations,* The Johns Hopkins University Press, Baltimore, 1983.
[11] MARKLEY, F. L. "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *The Journal of the Astronautical Sciences,* Vol. 36, No. 3, July–September 1988, pp. 245–258.
[12] BLACK, H. D. "A Passive System for Determining the Attitude of a Satellite," *AIAA Journal,* Vol. 2, July 1964, pp. 1350–1351.
[13] CHENG, Y. and SHUSTER, M. D. "QUEST and the Anti-QUEST: Good and Evil Attitude Estimation," *The Journal of the Astronautical Sciences,* Vol. 53, No. 3, July–September 2005, pp. 337–351.
[14] SHUSTER, M. D. "The TRIAD Algorithm as Maximum-Likelihood Estimation," *The Journal of the Astronautical Sciences,* Vol. 54, No. 1, January–March 2006, pp. 113–123.
[15] TANYGIN, S. and SHUSTER, M. D. "The Many TRIAD Algorithms," presented as paper AAS-07-104 at the AAS/AIAA 17th Space Flight Mechanics Meeting, Sedona, Arizona, January 28–February 2, 2007; Proceedings: *Advances in the Astronautical Sciences,* Vol. 127, 2007, pp. 81–99.
[16] SHUSTER, M. D. and FREESLAND, D. C. "The Statistics of TASTE and the Inflight Estimation of Attitude Sensor Precision," presented as paper No. 56 at the NASA Goddard Space Flight Center Flight Mechanics Symposium, Greenbelt, Maryland, October 18–20, 2005.

[17] SHUSTER, M. D. "Constraint in Attitude Estimation Part I: Constrained Estimation," *The Journal of the Astronautical Sciences,* Vol. 51, No. 1, January–March, 2003, pp. 51–74.

[18] LEFFERTS, E. J., MARKLEY, F. L., and SHUSTER, M. D. "Kalman Filtering for Spacecraft Attitude Estimation," *Journal of Guidance, Control and Dynamics,* Vol. 5, No. 5, September–October 1982, pp. 417–429.

[19] SHUSTER, M. D. "Effective Direction Measurements for Spacecraft Attitude: III. Defective Directions and Data Fusion," *The Journal of the Astronautical Sciences,* Vol. 55, No. 4, October–December 2007, pp. 493–510.