AAS 07-102

ROBUSTNESS AND ACCURACY OF THE QUEST ALGORITHM

Yang Cheng[1] and Malcolm D. Shuster[2]

. . . the merit of service is seldom attributed
to the true and exact performer.

William Shakespeare (1564-1616)
All's Well That Ends Well, Act III, scene vi

Abstract

A detailed study is undertaken of the accuracy and robustness of solutions to the Wahba problem, the starting point for most modern estimators of three-axis attitude. The importance of the partial factorization of the Davenport characteristic equation is examined, and the expanded form of that equation is shown to lead to a complete loss of numerical significance in certain extreme cases. Recent criticisms of the QUEST algorithm are shown to be without practical significance and, with a trivial rearrangement of terms, without substance. Improvements for future versions of QUEST, unchanged for two decades, are presented.

Introduction

One of the active and enduring fields of research in attitude estimation has been the development of fast optimal batch attitude estimators. The first of these may have been the symmetric TRIAD algorithm [1],[3] a modification of Black's TRIAD algorithm [1–3], which can been shown to optimize

[1]Research Assistant Professor, Department of Mechanical and Aerospace Engineering, University at Buffalo, State University of New York, Amherst, New York, 14260. email: cheng3@buffalo.edu. website: www.buffalo.edu/∼cheng3.
[2]Director of Research, Acme Spacecraft Company, 13017 Wisteria Drive, Box 328, Germantown, Maryland 20874. email: mdshuster@comcast.net. website; http://home.comcast.net/∼mdshuster.
[3]While the first publication of the symmetric TRIAD algorithm is quite late (1978), its earliest proposal may have been very early. The origins of this algorithms are shrouded in darkness, and it is not even certain that it has ever been implemented in a real mission.

the least-square cost function

$$J_A(A) = \frac{1}{2} \sum_{k=1}^{n} a_k \, |\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k|^2 \tag{1}$$

for the special case $n = 2$ and $a_1 = a_2$. Here A is the attitude matrix, $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$, are the measured directions as observed in the spacecraft body frame, and $\hat{\mathbf{V}}_k$, $k = 1, \ldots, N$, are the corresponding reference directions in the primary reference frame, assumed to be noise-free.[4] Equation (1) is the famous cost function of Wahba [4], proposed in 1965, which has been the cornerstone of almost all later work on batch optimal three-axis attitude estimation.

The symmetric TRIAD algorithm was of very limited usefulness, because it was restricted to only two measurements and could lead to larger root-mean-square (rms) errors than the TRIAD algorithm when the accuracies of the original input vectors were very unequal (the inaccurate measurement would now corrupt both the "symmetrized" and the "antisymmetrized" inputs) [31]. Even for equally accurate direction measurements, the improvement in accuracy (expressed as the rms value of the magnitude of the attitude error increment [5]), of the symmetric TRIAD over the TRIAD algorithm was less than ten percent. A truly useful fast optimal batch attitude estimation algorithm would not come until the q-algorithm of Davenport [1, 3, 6] in 1977.

The Wahba problem could be written as

$$\begin{aligned} J_A(A) &= \sum_{k=1}^{n} a_k - \mathrm{tr}\Big[\Big(\sum_{k=1}^{n} a_k \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k \Big)^{\mathrm{T}} A\Big] \\ &\equiv \lambda_{\mathrm{o}} - \mathrm{tr}[B^{\mathrm{T}} A] \\ &\equiv \lambda_{\mathrm{o}} - g_A(A) \end{aligned} \tag{2}$$

The matrix B is often called the *attitude profile matrix*, and from it one can construct both the optimal attitude estimate and the attitude covariance matrix [5]. The function $g_A(A)$ is called the Wahba *gain function* and is a maximum when $J_A(A)$ is a minimum. Equation (2) has been known to many investigators of the Wahba problem from the time of its proposal [7]. A concise account of early (and late) work on the Wahba problem can be found in the review of Markley and Mortari [7].

Davenport's special and enormous contribution to the Wahba problem was to show that the Wahba gain function could be written in terms of the attitude quaternion $\bar{q}$ [5] as

$$g_{\bar{q}}(\bar{q}) \equiv g_A(A(\bar{q})) = \bar{q}^{\mathrm{T}} K \bar{q} \tag{3}$$

where

$$K = \begin{bmatrix} S - sI & \mathbf{Z} \\ \mathbf{Z}^{\mathrm{T}} & s \end{bmatrix} \tag{4}$$

with

$$S \equiv B + B^{\mathrm{T}}, \quad s \equiv \mathrm{tr}\, B \quad \text{and} \quad \mathbf{Z} \equiv [B_{23} - B_{32},\ B_{31} - B_{13},\ B_{12} - B_{21}]^{\mathrm{T}} \tag{5abc}$$

[4] We have been lax in labeling variables as random variables or sampled values and trust the reader to make the proper identification from the context.

As a result, the minimization of $J_A(A)$ or, equivalently, the maximization of $g_A(A)$, could be accomplished by finding the solution of the eigenvalue problem

$$K\bar{q}^* = \lambda_{\max}\bar{q}^* \tag{6}$$

where $\lambda_{\max}$ is the largest eigenvalue of the 4×4 real-symmetric matrix K. The attitude estimation problem has now been transformed into a standard problem of Numerical Linear Algebra, which could be solved by the application to K of Householder's method [8] for finding the eigenvectors and eigenvalues (characteristic or proper vectors and values) of a real-symmetric matrix. In this form Davenport's q-algorithm was first applied to the HEAO missions [9]. At its birth, Davenport's q-method was certainly the fastest optimal batch algorithm in existence for three-axis attitude estimation. Almost all faster algorithms which have succeeded it are simply faster methods of solving the eigenvalue equation (6).[5] However, the succession was rapid and, within a year of its first publication [6], the seed was planted for a much faster successor.

As a result of Davenport's discovery, almost all further attention to the Wahba problem was focused on developing faster methods for evaluating the maximum overlap eigenvalue $\lambda_{\max}$, after which the optimization problem became a simple algebraic problem. The first of these faster algorithms was QUEST [3, 7, 10], developed for the Magsat mission [11], whose speed requirements far exceeded those of the HEAO missions. To accelerate the computation of $\lambda_{\max}$, it was noted that

$$\lambda_{\max} = \lambda_{\mathrm{o}} - J_A(A^*) \tag{7}$$

For sensor accuracies of around 13 arc-seconds, (as in the Magsat mission), it was expected that the second term would be smaller than the first by ten orders of magnitude. Thus, for Magsat, λ_{o} is an excellent approximation for $\lambda_{\max}$, and further refinement of the maximum overlap eigenvalue, if it were desired, would be possible iteratively by means of Newton-Raphson method [8].

In the QUEST algorithm the characteristic polynomial[6] for K

$$\psi(\lambda) \equiv \det\left[\lambda\, I_{4\times 4} - K\right] \tag{8}$$

has the form [3, 7, 10]

$$\psi_{\mathrm{QUEST}}(\lambda) = \lambda^4 - (a+b)\lambda^2 - c\lambda + (ab + cs - d) \tag{9}$$

with[7]

$$a = s^2 - \mathrm{tr}\big(\mathrm{adj}\, S\big)\,, \qquad b = s^2 + \mathbf{Z}^{\mathrm{T}}\mathbf{Z} \tag{10ab}$$

$$c = \det S + \mathbf{Z}^{\mathrm{T}} S \mathbf{Z}\,, \qquad d = \mathbf{Z}^{\mathrm{T}} S^2 \mathbf{Z} \tag{10cd}$$

The function "adj" denotes the matrix adjoint, and "det" the determinant. The quantities a, b, c, and d enter also into the construction of the QUEST quaternion, further increasing the efficiency of that algorithm. For the computer resources of 1978, the year in which it was completed, QUEST was about 1000 times faster than Davenport's original implementation of the q-algorithm using

---

[5]To distinguish this implementation of Davenport's q-method from later implementations (such as QUEST), we will refer to the original implementation as q-Davenport

[6]We call $\psi(\lambda) = 0$ the characteristic equation.

[7]It is more accurate numerically to evaluate $c = 8 \det B$, which is identical to equation (10c) for infinitely precise arithmetic.

Householder's method. Even more than a quarter century after its first implementation in a real attitude ground support system in 1979, the QUEST algorithm remains the most popular batch algorithm for estimating three-axis attitude. The human story of the development of QUEST has also been published [12].

It has been shown that for constant λ_o and assuming the QUEST measurement model [3, 13], the optimized cost function would be smallest if the weights were chosen to satisfy

$$a_k = \mu / \sigma_k^2 \tag{11}$$

[3, 13] for some positive μ with σ_k the angle-equivalent error level of the k-th direction measurement in the QUEST measurement model [2]. For the Magsat mission, μ was chosen so that $\lambda_\mathrm{o} = 1$. In the present work, for the most part, we choose $\mu = 1$, in which case $J_A(A)$ becomes the data-dependent part of the negative-log-likelihood function [13, 14] of the attitude given the QUEST measurement model [3, 13], which is

$$\hat{\mathbf{W}}_k = A^{\mathrm{true}} \hat{\mathbf{V}}_k + \Delta \hat{\mathbf{W}}_k \tag{12}$$

with

$$E\{\Delta \hat{\mathbf{W}}_k\} = \mathbf{0}\,, \qquad k = 1, \ldots, n \tag{13a}$$

$$E\{\Delta \hat{\mathbf{W}}_k \Delta \hat{\mathbf{W}}_l^{\mathrm{T}}\} = \delta_{kl}\, \sigma_k^2 \left(I_{3\times 3} = \hat{\mathbf{W}}_k^{\mathrm{true}}\, \hat{\mathbf{W}}_k^{\mathrm{trueT}} \right), \quad k, l = 1, \ldots, n \tag{13b}$$

After a decade, other fast algorithms began to appear. The first was Markley's SVD algorithm [7, 15], which calculates the optimal attitude estimate by applying the singular value decomposition (SVD) [8] of Numerical Linear Algebra to the attitude profile matrix B. Like Davenport's original implementation of the q-method [3, 6, 7] Markley's SVD algorithm[8] is highly stable and robust, because, like Davenport's use of Householder's method, it too utilizes a mathematical method that has benefitted from more than a half-century of careful scrutiny and improvement. The price paid for this reliability is in speed. Both the q-Davenport and the M-SVD algorithms are relatively slow compared with QUEST and later optimal algorithms, although the difference has been decreasing over the past two decades.

The next algorithm to appear was Markley's highly imaginative FOAM algorithm [7, 16], which, like QUEST, begins with the approximation $\lambda_{\max} \approx \lambda_\mathrm{o}$ and then uses Newton-Raphson iteration to determine a more refined value of the maximum overlap eigenvalue. The characteristic polynomial for the FOAM problem (identical to that of QUEST for infinitely precise arithmetic) has the form

$$\psi_{\mathrm{FOAM}}(\lambda) = \left(\lambda^2 - \|B\|_F^2\right)^2 - 8\lambda \det B - 4\,\|\mathrm{adj}\, B\|_F^2 \tag{14}$$

where $\|\cdot\|_F$ denotes the Frobenius norm

$$\|M\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |M_{ij}|^2 \tag{15}$$

and M is any $n \times n$ matrix.

[8]When "SVD" is not qualified by Markley's name, we will refer to his SVD algorithm as M-SVD to avoid confusion with the general algorithm of Numerical Linear Algebra.

It has been known for some time that for extremely unbalanced measurement accuracies (equivalently, weights) the FOAM form of the characteristic polynomial is better behaved numerically than the corresponding QUEST form for extremely unbalanced weights, although the reason for this has not been understood before the present work. We will investigate the origin and elimination of this difference in numerical performance in a later section.

Lastly, we have Mortari's Euler [17, 18] and ESOQ [7, 19–21] families of batch attitude estimation algorithms, of which only the ESOQ algorithms will concern us here.[9] The important characteristic (for the present work) of the ESOQ1[10] [7, 19] and late ESOQ2 [7, 20, 21][11] algorithms is that they calculate $\lambda_{\max}$ by applying the Newton-Raphson method to the FOAM form of the characteristic equation.[12] Recently, Bruccoleri, Lee and Mortari [22] have developed a Wahba estimator MRAD based on the modified Rodrigues vector [5], whose execution is very similar to that of ESOQ2. We do not examine this new algorithm here. Mortari [21] has claimed that early ESOQ2 is faster and more accurate than QUEST.

In the above algorithm descriptions we have avoided all discussion of the manner of construction of the optimal quaternion from $\lambda_{\max}$ and K. Excellent descriptions can be found in Reference 7, and it is unlikely that we could improve on them. Rather, our focus in the present study has been on the characteristic polynomials and their effect on attitude estimation accuracy.

Part I of the present work has two main subparts: (A) the discussion of the results of Reference 7; and (B) an analysis of the difference in numerical behavior of the QUEST and FOAM characteristic polynomials in the Newton-Raphson search for the largest root and the elimination of those differences. Part II of this work [23] will examine the topic of computational speed.

The Review of Markley and Mortari

Summary of the Review

The review of Markley and Mortari [7] is in many ways a superb document, which gives concise excellent descriptions of all of the solutions to the Wahba problem up to the year 2000. Despite its title, it describes also the solutions to the Wahba problem which employ the direction-cosine matrix as the attitude representation. It is, without question, an important resource on the many solutions to the Wahba problem from 1965 until 2000 and a key paper on attitude estimation.

[9]Like many Italian families, these have many members. The Euler family consists of Euler, Euler-2, Euler-n, and Euler-q, while the ESOQ family proudly boasts ESOQ, ESOQ1, early and late ESOQ2, ESOQ1.1, and ESOQ2.1.

[10]In Reference 7, ESOQ1 is frequently referred to as ESOQ, a very different algorithm, which leads to some confusion in the nomenclature, since Mortari's ESOQ1.1 did not become ESOQ.1 in Reference 7. We use the unambiguous designation ESOQ1 in this work. The ESOQ1.1 and ESOQ2.1 algorithms each use a special procedure for computing $\lambda_{\max}$ to first order only and do not use the characteristic polynomial. Their similarity to ESOQ1 and late ESOQ2, respectively, is in the manner in which the quaternion estimate is constructed given the first-order approximation for $\lambda_{\max}$. ESOQ2 underwent a similar metamorphosis to that of ESOQ but the name wasn't changed. Hence, we use the designations early and late ESOQ2.

[11]References 20 and [21] really refer to early ESOQ2.

[12]This was not always true. The earliest report of ESOQ2 [20] determined $\lambda_{\max}$ from an exact analytic expression in terms of surds. Later the method was changed to applying Newton-Raphson iteration to the FOAM form of the characteristic equation without changing the algorithm name or numerical designation. For clarity we denote the version of ESOQ2 using the expression for $\lambda_{\max}$ in terms of surds as "early ESOQ2" and the version using the Newton-Raphson method as "late ESOQ2."

Unfortunately, while the algorithm descriptions are excellent, the performance comparisons are marred by incomplete information about the performance of the QUEST algorithm, which may give a false impression of the performance of QUEST, perhaps even make some readers think that the use of QUEST may be dangerous, which is certainly not the case.

The assertions of Reference 7 about QUEST are :

- QUEST is less accurate and less robust than the algorithms developed by the authors of Reference 7 and does not necessarily converge.
- QUEST requires more floating-point operations than some of the algorithms of the authors of Reference 7.

The quick response to the first assertion is that QUEST appears to be less accurate and less robust only if (1) the algorithm is tested for an extremely unphysical case (scenario 2 of Reference 7) which cannot occur in the real world, and (2) the QUEST algorithm is executed in an unnecessary and undesirable manner. Only if both of these conditions are met, will the performance of the QUEST algorithm be degraded. In addition, if a minor rearrangement of terms is made in the QUEST characteristic polynomial,[13] there will be no degradation in the performance of QUEST, not even in the far-fetched scenario 2 of Reference 7. The detailed examination of the accuracy and robustness of the various fast optimal batch algorithms is the principal subject of the present work (Part I).

The quick response to the second assertions is that the examples reported in Reference 7 are indeed true, but that these represent only a minority of the cases. For most cases, QUEST requires fewer floating-point operations than the other algorithms and in terms of execution time, the only true measure of algorithm speed, QUEST is even slightly faster within MATLAB than any of the other other fast algorithms in all cases. Reference 7 divides the algorithms into robust or non-robust according to their behavior under iteration. Because QUEST showed problems in the far-fetched scenaro 2, Reference 7 classifies it as non-robust, and compares it only with no iterations of ESOQ and ESOQ2 and with ESOQ1.1 and ESOQ2.2 and finds it require the greatest number of Matlab floating-point operations of the zeroth or first order algorithms for the cases considered. As Part II of this work [23] will show, it should be classified as the robust algorithm requiring the *fewest* number of MATLAB® floating-point operations. A more general and more careful treatment of floating-point operations and as well as a general treatment of execution time are the subject of Part II of this work [23].

Performance of the Fast Attitude Estimators in the Review of Markley and Mortari

Reference 7 considered among others the following test scenario (scenario 2). A star tracker of accuracy $\sigma_1 = 1$ arcsec measures a single star direction $\hat{\mathbf{W}}_1$ along its boresight (the body x-axis), while two coarse sensors, each of accuracy $\sigma_2 = 1$ deg, measure directions, $\hat{\mathbf{W}}_2$ and $\hat{\mathbf{W}}_3$, in the xy-plane offset from the negative boresight by an angle $\pm\alpha$ with $\alpha \approx 4.35$ deg. The exact value of α was chosen so that $\sin\alpha$ and $\cos\alpha$ would have finite decimal representations. Thus,

$$\hat{\mathbf{W}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{W}}_2 = \begin{bmatrix} -0.99712 \\ 0.07584 \\ 0 \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{W}}_3 = \begin{bmatrix} -0.99712 \\ -0.07584 \\ 0 \end{bmatrix} \qquad \text{(16abc)}$$

[13] This rearrangement is that of simply replacing $\lambda^4 - (a+b)\lambda^2 + ab$ by $(\lambda^2 - a)(\lambda^2 - b)$.

and all three vectors have unit norm exactly. The values of the reference directions will depend on the attitude.[14] This particular symmetric choice of measurements insures also that the attitude estimate-error covariance matrix will be diagonal with respect to the star-tracker axes. The errors were assumed to be modeled by the QUEST measurement model [3, 10, 13]. Thus,

$$P_{\theta\theta}^{-1} = \mathrm{diag}\left[\frac{2\sin^2\alpha}{\sigma_2^2}, \left(\frac{1}{\sigma_1^2} + \frac{2\cos^2\alpha}{\sigma_2^2}\right), \left(\frac{1}{\sigma_1^2} + \frac{2}{\sigma_2^2}\right)\right] \tag{17}$$

In the present study one thousand samples were computed for the q-Davenport, M-SVD, FOAM, QUEST, ESOQ1, ESOQ1.1, late ESOQ2 and ESOQ2.1 algorithms, which led to the results of Table 1.[15] The weights have been chosen (as in Reference 7) to be $a_k = 1/\sigma_k^2$. "Iterations" in Table 1 refers to the number of Newton-Raphson iterations in the calculation of $\lambda_{\max}$; "Δ cost function" denotes the root-mean-square[16] value of the difference of the computed cost function from the value computed by the q-Davenport algorithm;[17] "x" denotes the root-mean-square (rms) value of the incremental error angle (in degrees) of the attitude estimate for the given algorithm about the star-tracker x-axis; "yz" denotes the rms value (over the one thousand samples of the rss value (over the two axes) of the incremental angles about the y- and z-axes). We have written QUEST$^{\mathrm{MM}}$ in Table 1, because Reference 7 has implemented a slightly different version of QUEST than that developed in References [1] and [2]. QUEST$^{\mathrm{MM}}$ is actually slightly faster than QUEST [23].[18] In the remainder of this report QUEST without a superscript will always denote the version of QUEST as created in 1978 by the second author of the present work. The distinction will be important in the sequel work [23], since it affects the execution time. The results in Table 1 should be compared with the attitude estimation accuracies about these axes computed from equation (17)[19]

$$\sigma_x \approx 9.32 \text{ deg} \qquad \text{and} \qquad \sigma_{yz} \approx 1.41 \text{ arcsec} \tag{18ab}$$

[14]Since one wishes to maintain the same values of the observations in every sample, one must add Gaussian random noise to the reference directions, which was what was done in Reference 7.

[15]For greater readability we use the notation 3 e-5 rather than the 3×10^{-5} of Reference 7, and we seldom present more than three significant digits. Unlike Reference 7 we present the results of the first five iterations of the iterative algorithms. Reference 7 also presents results for the maximum error, a figure of merit of questionable value [24].

[16]Mislabeled root-sum-square (rss) in Reference 7

[17]Barring inadequacies in the algorithms, the expected value of the Wahba cost function (with $\mu = 1$) should be 3/2, as shown in Reference 25. Clearly, except for QUEST, all iterative algorithms in Table 1 are seen to converge effectively to the value 3/2 in no more two steps. The difference in performance for the convergence of the cost function (equivalently, for $\lambda_{\max}$) are due to differences in numerical precision. Note that many entries for the cost function are identical for many algorithms since they all calculate a Newton-Raphson sequence for the largest root of the same characteristic polynomial. If the (optimal) weights are chosen instead to have unit sum, the expected value of the cost function (with infinite-precision arithmetic) is $(3/2)\sigma_{\mathrm{tot}}^2$ with $\sigma_{\mathrm{tot}}^2 = (1/\mathrm{arcsec}^2 + 2/\mathrm{deg}^2)^{-1} \approx 2.350 \times 10^{-11}$, so that the cost function for this choice of the weights (with exact arithmetic) is 3.525×10^{-11}. The sampled mean and variance of the cost function for one-thousand sampled values (with $\mu = 1$) gave $J(\bar{q}^*)^{\mathrm{sampled}} = (3/2)(1.027 \pm 0.857)$, consistent with the statistical model. (Note equations (7), (19) and (20).)

[18]In fact, while the FORTRAN mission code of 1979 used an expanded expression for the determinant, the second author's MATLAB® implementations have always used a function call for the determinant, the sole difference between QUEST and QUEST$^{\mathrm{MM}}$. Thus, QUEST$^{\mathrm{MM}}$ is not an innovation of Reference 7.

[19]The differences between the results in equations (18) and the last two columns of Table 1 are consistent with the expected variation of the rms sampled errors.

The numerical results in Table 1 are very similar to those of Table 2 of Reference 7, except, alas, those for QUEST, which are somewhat worse even than those reported by Reference 7. However, only the attitude estimate errors about the star-tracker boresight for *iterated* values of $\lambda_{\rm max}$ in QUEST$^{\rm MM}$ show a significantly greater error level. Without iteration the increase in error level is less than two percent, hardly a significant difference when the best one can do is 9 deg and for a scenario which is not realizable physically (see below). Thus, the QUEST estimate without iteration of $\lambda_{\rm max}$ has all the accuracy one could desire.

The Very Unrealistic Test Scenario 2 of Reference 7

We make the following remarks concerning scenario 2 of Reference 7 :

(1) **Scenario 2 of Reference 7 is impossible with existing sensors**. The variance ratio of fine and coarse attitude sensors in scenario 2 of Reference 7 is 12,960,000. For a typical star tracker (accuracy = 3 arcsec) and for typical coarse attitude sensors (accuracy = 0.5 deg), the variance ratio is only 360,000. The variance ratio of scenario 2 for 3-arcsec star trackers is realizable only if the coarse sensors are vector magnetometers operating near a magnetic pole. The Earth has only one geomagnetic field, not two geomagnetic fields separated in angle by $2\alpha \approx 8.7$ deg. Scenario 2 of Reference 7 is *not* of this Earth.

(2) **Scenario 2 of Reference 7 uses the coarse attitude sensors unrealistically**. Even if scenario 2 of Reference 7 used only a single vector magnetometer, and, therefore, were physically realizable, it would still be using magnetometer data at magnetic latitudes in which such data are generally discarded as useless for attitude determination. Generally, one discards vector magnetometer data at magnetic latitudes greater than 70 deg in magnitude.

(3) **Scenario 2 of Reference 7 relies on a very unrealistic star tracker.** It is extraordinarily unlikely nowadays (and when Reference 7 was being written) that a star tracker will observe only a single star. Consider, for example, the star tracker of the Wilkinson Microwave Anisotropy Probe (WMAP, launched June 2001) [26]. The Lockheed AST-201 autonomous star tracker of the WMAP spacecraft can track up to 50 stars simultaneously. During the first few years of the WMAP mission it never observed fewer than 15 stars at a time. The average number of simultaneous star observations per frame in that mission is about 25. In normal operation the Lockheed star tracker even lacks the capability to output individual star directions but only the attitude quaternion, because the observation of only a single star is so impossibly unlikely, and the amount of output data would be uncomfortably large. The Sun sensor on WMAP, the "coarser" sensor, has an accuracy not of 1.0 deg, the accuracy of the coarse sensors in scenario 2 of Reference 7, but 20 arcsec (1 sigma), a more reasonable complement to a star tracker with a single-star standard deviation of about 10 arcsec (1 sigma). The variance ratio for the WMAP sensors is not 12,960,000 but 4.[20]

[20]During the late 1970s, when QUEST was being developed, star trackers, specifically the NASA standard star tracker (the Ball Brothers Research Corporation's CT-401) were capable of observing only a single star at one time. If two simultaneous star-direction measurements were needed, then two star trackers were included in the attitude determination system, as was the case of Magsat. In 1990, star trackers existed which could observe five stars simultaneously, so that only a single star tracker was needed in the attitude determination system, but the likelihood that only a single star (or no star) would be observed in a given data frame was still not negligible. In 2000, as illustrated by the Lockheed AST-201 for WMAP, this was obviously no longer the case.

TABLE 1. Estimation Results for Scenario 2 of Reference 7 (after Table 2 of Reference 7)

Algorithm	Iterations	Δ Cost Function	x (deg)	yz (arcsec)
q-Davenport (q)	—	—	9.30	1.43
M-SVD	—	1.40 e-5	9.30	1.43
FOAM	0	—	9.42	1.41
	1	0.102	9.42	1.41
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
QUEST^{MM}	0	—	9.47	1.43
	1	749	106.	1.43
	2	3900	101.	1.43
	3	2100	99.1	1.43
	4	2890	99.3	1.43
	5	2500	98.4	1.43
ESOQ1	0	—	9.25	1.43
	1	0.102	9.29	1.43
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
ESOQ1.1	1	334	49.0	1.43
ESOQ2*	0	—	9.42	1.43
	1	0.102	9.30	1.43
	2	0.952 e-3	9.30	1.43
	3	1.40 e-5	9.30	1.43
	4	1.37 e-5	9.30	1.43
	5	1.38 e-5	9.30	1.43
ESOQ2.1	1	64.1	37.4	1.43

*In the tables ESOQ2 will always mean late ESOQ2

(4) **QUEST^{MM}, in fact, works well in scenario 2 of Reference 7**. There is no requirement that QUEST perform iterations of the Newton-Raphson method to calculate λ_{max}. It has been shown [25] that

$$\lambda_{max}^{r.v.} = \lambda_o \left[1 - (1/2)\, \sigma_{tot}^2\, \chi^2(2N-3)\right] \tag{19a}$$

with

$$\frac{1}{\sigma_{\text{tot}}^2} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{19b}$$

and where $\lambda_{\max}^{\text{r.v.}}$ denotes the random variable (a function of the random $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$) whose sampled value is $\lambda_{\max}$. Thus, for scenario 2 of Reference 7 we have roughly[21]

$$\begin{aligned}\lambda_{\max}/\lambda_o &\approx 1 - (3/2)\left((1.0\text{ arcsec})^2 \pm 2.45\,(1.0\text{ arcsec})^2\right)\\ &\approx 1 - 3.53 \times 10^{-11} \pm 8.64 \times 10^{-11}\end{aligned} \tag{20}$$

The maximum overlap eigenvalue $\lambda_{\max}$ differs from λ_o only by terms of (relative) order 10^{-10}; hence, without Newton-Raphson iteration of the characteristic equation should be an accurate approximation of $\lambda_{\max}$ *to ten significant digits*. From the beginning (1978) QUEST possessed an input NEWT, which controled the number of Newton-Raphson iterations. Thus, when there is only one star direction—easily detected in flight—one simply sets NEWT = 0, and QUEST (and QUEST$^{\text{MM}}$) will yield an accurate and robust result.[22]

The statement in Reference 7 that "the only useful results of QUEST are obtained by not performing any iterations for $\lambda_{\max}$" is thus misleading. It would have been more justified to say that *all* of the useful attitude information can be obtained from QUEST (i.e., QUEST$^{\text{MM}}$) *without* performing any iterations, rather than the disparaging wording of Reference 7. $\lambda_{\max}$ is generally computed to higher numerical accuracy than simply as λ_o not for attitude estimation but for the TASTE test, which uses the value $\lambda_o - \lambda_{\max}$. However, $\lambda_o - \lambda_{\max}$ can be obtained almost as efficiently by other means[23] as demonstrated in the previous footnote.

(5) **Scenario 2 of Reference 7 presents a very unwise system design**. From the point of view of system design the choice of sensor accuracies simply does not make sense. What reasonable attitude-determination-system designer would couple a one-arc-second sensor with one-degree sensors if it were anticipated that all sensors would be needed for attitude determination? What system designer would spend millions of dollars for a star tracker in an attitude determination system which will provide attitude accuracy no better than 9 deg? The WMAP attitude determination system (sic) [26] exhibits a sensible design choice. Duh?

(6) **In real life one would never use a batch algorithm for scenario 2 of Reference 7**. For an attitude determination system, when the highest possible accuracy is a priority, when a CCD star tracker is part of the attitude sensor suite, so is a three-axis gyro assembly. Such systems with CCD star trackers and three-axis gyro assemblies, invariably compute the attitude by means of a Kalman filter.[24] Thus, if rare situations, similar to that of scenario 2 of Reference 7 are encountered,

---

[21] In fact, equation (19) has been known since 1980 and noted in Reference 7. The first proof published in the open literature, however, was only in 2005 [25].

[22] The only loss in this situation is that one can no longer calculate the figure of merit TASTE by computing the first iteration of $\lambda_{\max}$. But TASTE can be computed just as well from $\lambda_{\max} \approx s + \mathbf{Z}^{\mathrm{T}}\mathbf{Y}^*(0)$ with $\mathbf{Y}^*(0)$ the estimate of the Rodrigues vector based on λ_o.

[23] This provision was not in the original QUEST FORTRAN code of 1978, because its first application was to the Magsat mission, for which the three fine attitude sensors were nearly equal in accuracy and the angular separation between the sensors was large. The version tested in Reference 7 was based on a modification of that QUEST code made in 1987 by Markley to improve numerical significance.

[24] It is typically as a preprocessor for the Kalman filter that QUEST now has its most frequent application [27, 28].

it is better simply to use only the single star-direction measurement, gyro outputs and the previous attitude estimate to estimate the attitude.[25]

Thus, the statements of Reference 7 that QUEST$^{\mathrm{MM}}$ performs poorly are simply not true as far as the attitude-estimation accuracy is concerned and create an unnecessary distraction. Nor would one wish to exercise either QUEST or late ESOQ2 in a situation like scenario 2 of Reference 7. QUEST performs poorly only if implemented in an unwise manner in this far-fetched scenario.

The QUEST and FOAM Characteristic Polynomials

The Two-Vector Case

Although we have seen that the poor or non-convergence of the $\lambda_{\max}$ iteration in the QUEST algorithm for certain extreme, unrealistic, and unrealizable cases is not of practical concern, it is, nonetheless, true that the QUEST form of the characteristic polynomial is less accurate numerically than the FOAM form in these physically unrealizable situations. The original QUEST algorithm developed for Magsat included the method of sequential rotations [2] simply to avoid a problem that might occur once every 20,000 years for that mission [10, 12]. It will repay us to pay the same attention to the convergence issue. It will turn out that the numerical problem of the QUEST form of the characteristic polynomial in the ultra-bizarre scenario 2 of Reference 7 can be eliminated entirely by a very simple rearrangement of terms. In this section, we examine a very simplified case of two measured directions, which will provide simple insights into the cause of the problems of the QUEST form in the Newton-Raphson iteration for the maximum overlap eigenvalue. We will treat scenario 2 of Reference 7 specifically and in detail in the next section.

The Overlap Eigenvalues

We have said that the QUEST characteristic polynomial is not as well-behaved numerically as that for FOAM for extremely unbalanced measurements. We will investigate here the reason for this difference in performance. Let us consider the simplest case of only two measured directions, one with accuracy $\sigma_1 = 1$ arcsec and the other with accuracy $\sigma_2 = 1$ deg, the error levels of scenario 2 or Reference 7. Let the angle between the observed directions be θ_W and that between the reference directions θ_V. Then the analytic solutions for the four eigenvalues of K are[26]

$$\lambda_4 = \sqrt{a_1^2 + 2a_1 a_2 \cos(\theta_W - \theta_V) + a_2^2} = -\lambda_1 \tag{21a}$$

$$\lambda_3 = \sqrt{a_1^2 + 2a_1 a_2 \cos(\theta_W + \theta_V) + a_2^2} = -\lambda_2 \tag{21b}$$

with

$$\cos(\theta_W \pm \theta_V) = (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)(\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2) \mp |\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|\,|\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2| \tag{22}$$

[25]It is worth noting that scenario 2 of Reference 7 is the perfect candidate for the application of the SCAD algorithm [29] (finally, an application for the SCAD algorithm!), which effectively in this case finds the rotation that satisfies $\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1$ exactly and optimizes the overlap for the two coarse vectors. If there were only a single coarse measurement, then scenario 2 of Reference 7 would be the perfect candidate for the TRIAD algorithm [3].

[26]Equation (21a) can be found in Reference 3.

Thus, for θ_V not too small[27]

$$\lambda_0 \geq \lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1 \geq -\lambda_o \tag{23}$$

. Likewise,

$$\theta_W = \arctan_2\left(|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|,\ \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2\right) \tag{24a}$$

$$\theta_V = \arctan_2\left(|\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2|,\ \hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2\right) \tag{24b}$$

Here, $\arctan_2(y,\ x)$ returns the arc tangent of y/x in the proper quadrant. In the present case, θ_W and θ_V will have values in the interval $[0,\ \pi]$. We assume, as usual, that the reference directions are noise-free, in which case

$$\hat{\mathbf{W}}_k^{\text{true}} \equiv A^{\text{true}}\hat{\mathbf{V}}_k\,, \qquad k = 1,\ 2 \tag{25}$$

whence,

$$\theta_V = \theta_W^{\text{true}}\,, \qquad \text{and} \qquad \theta_W = \theta_W^{\text{true}} + \Delta\theta_W \tag{26ab}$$

and we can write equivalently

$$\cos(\theta_W \pm \theta_V) = \cos(\theta_W \pm \theta_W^{\text{true}}) \tag{27}$$

A more revealing form of equations (21) is

$$\lambda_4 = \sqrt{\lambda_o^2 - 4a_1a_2\sin^2[(\theta_W - \theta_W^{\text{true}})/2]} = -\lambda_1 \tag{28a}$$

$$\lambda_3 = \sqrt{\lambda_o^2 - 4a_1a_2\sin^2[(\theta_W + \theta_W^{\text{true}})/2]} = -\lambda_2 \tag{28b}$$

For simplicity we choose $\mu = \sigma_{\text{tot}}^2$ in equation (11) so that λ_o is unity. Approximating the sine function above for small angles θ_W and θ_V and making the usual approximation for the square root when the argument is close to unity, we obtain then

$$\lambda_4 = 1 - (1/2)\,a_1a_2\,|\Delta\theta_W|^2 \qquad\qquad = -\lambda_1 \tag{29a}$$

$$\lambda_3 = 1 - (1/2)\,a_1a_2\,|2\theta_W^{\text{true}} + \Delta\theta_W|^2 = -\lambda_2 \tag{29b}$$

Comparing equation (29a) with equation (19) we obtain

$$|\Delta\theta_W^{\text{r.v.}}|^2 = (\sigma_1^2 + \sigma_2^2)\,\chi^2(1) \tag{30}$$

where $\Delta\theta_W^{\text{r.v.}}$ is the random variable which has the sampled value $\Delta\theta_W$. Thus, within terms of order 10^{-8} we have

$$(\Delta\theta_W)_{\text{rms}} = 1\ \text{deg} \tag{31}$$

[27]For small θ_V it may not be true that $\lambda_4 \geq \lambda_3$. We will assume it to be true, however. Whether λ_4 or λ_3 is largest is not important to our discussion in this section.

The Characteristic Polynomial and the Newton-Raphson Method

The characteristic polynomial for the Davenport matrix K when there are only two measurements is simply

$$\psi(\lambda) = (\lambda - \lambda_4)(\lambda - \lambda_3)(\lambda - \lambda_2)(\lambda - \lambda_1) \tag{32a}$$

$$= (\lambda^2 - \lambda_4^2)(\lambda^2 - \lambda_3^2) \tag{32b}$$

$$= \lambda^4 - (\lambda_4^2 + \lambda_3^2)\,\lambda^2 + \lambda_4^2\lambda_3^2 \tag{32c}$$

For obvious reasons, we will call the form of equation (32b) the *factored* form, $\psi_{\text{fac}}(\lambda)$, and the form of equation (32c) the *expanded* form, $\psi_{\text{exp}}(\lambda)$. In infinitely precise arithmetic, of course, $\psi_{\text{fac}}(\lambda) = \psi_{\text{exp}}(\lambda)$. The expanded form of the characteristic polynomial is analogous to that for QUEST, and the factored form to that for FOAM. The two forms of equations (32) are much simpler than those for QUEST and FOAM, as presented in equations (9) and (14), respectively because of the restriction to only two measurements. The Newton-Raphson iteration for λ_{max} will take the form[28]

$$\lambda_{\text{max}}^{(\text{o})} = \lambda_{\text{o}} \tag{33a}$$

$$\lambda_{\text{max}}^{(i+1)} = \lambda_{\text{max}}^{(i)} - \psi(\lambda_{\text{max}}^{(i)})/\psi'(\lambda_{\text{max}}^{(i)}) \tag{33b}$$

with $\psi'(\lambda)$ the derivative of $\psi(\lambda)$.

Examine first the expanded form. Following equations (29) we may write

$$\lambda_4 = 1 - \epsilon_4\,, \qquad \text{and} \qquad \lambda_3 = 1 - \epsilon_3 \tag{34ab}$$

with ϵ_4 and ϵ_3 very small quantities. If we chose[29]

$$\Delta\theta_W = \Delta\theta_W^{\text{rms}} = 1.0 \text{ deg} \qquad \text{and} \qquad \theta_W^{\text{true}} = 2.0 \text{ deg} \tag{35ab}$$

then

$$\epsilon_4 \approx 1.175 \times 10^{-11} \qquad \text{and} \qquad \epsilon_3 \approx 2.938 \times 10^{-10} \tag{36ab}$$

Given these values, we have to second order in ϵ_4 and ϵ_3

$$\psi_{\text{exp}}(\lambda) = \lambda^4 - (2 - \delta_1)\,\lambda^2 + (1 - \delta_{\text{o}}) \tag{37}$$

with

$$\delta_1 = 2(\epsilon_4 + \epsilon_3) - (\epsilon_4^2 + \epsilon_3^2) \qquad \text{and} \qquad \delta_{\text{o}} = \delta_1 - 4\epsilon_3\epsilon_4 \tag{38ab}$$

---

[28]We use k as a data index and i as an iteration index.

[29]The value for θ_W^{true} has been chosen heuristically. Crudely, following the form of equation (30), we might anticipate for three vectors that

$$|\Delta\theta_W^{\text{r.v.}}|^2 \approx (\sigma_1^2 + 2\sigma_2^2)\,\chi^2(3)$$

so that $\theta_w^{\text{true}} \approx 4.35$ deg is nearly twice $\Delta\theta_W^{\text{rms}} \approx 2.45$. Therefore, we have chosen θ_W^{true} to be approximately twice $\Delta\theta_w^{\text{rms}}$ in our two-vector example. The same arguments apply if we choose $\theta_W = \pi - 2.0$ deg.

Numerically,

$$\delta_1 = 6.111150996221957 \times 10^{-10} \tag{38c}$$

$$\delta_o = 6.111150996083843 \times 10^{-10} \tag{38c}$$

The difference in these two numbers are much less than their accuracy. Thus,

$$\psi_{\exp}(1) = \bigcirc \times 10^{-20} \tag{39}$$

in IEEE double-precision arithmetic. We have used "$\bigcirc$" to denote a number of order unity but with no significant digits. Since the precision of an IEEE double-precision floating-point number is approximately $10^{-16.8}$, the cancellations in $\psi_{\exp}(1)$ cause all numerical significance to be lost. Similarly,[30]

$$\begin{aligned}\psi'_{\exp}(\lambda) &= 4\lambda^3 - 2(\lambda_4^2 + \lambda_3^2)\,\lambda \\ &= 4 - 2(2 - 6.111 \times 10^{-10} + 8.646 \times 10^{-20}) \\ &\approx 1.222 \times 10^{-9} \qquad \text{for } \lambda = 1\end{aligned} \tag{40}$$

so that the denominator in the Newton-Raphson iteration retains about seven significant digits, since we have achieved this numerical result by subtracting quantities of order unity.

Examining now the factored form we obtain to second order

$$\psi_{\text{fac}}(\lambda) = [\lambda - (1 - 2\epsilon_4 + \epsilon_4^2)]\,[\lambda - (1 - 2\epsilon_3 + \epsilon_3^2)] \tag{41}$$

Numerically,

$$\begin{aligned}\psi_{\text{fac}}(1) &= [1 - (1 - 2.350 \times 10^{-11} + 1.381 \times 10^{-22})]\,[1 - (1 - 5.876 \times 10^{-10} + 8.632 \times 10^{-20})] \\ &= 1.381 \times 10^{-20}\end{aligned} \tag{42}$$

and now one retains six significant digits (not all displayed) in the numerator of the first Newton-Raphson iteration, because we never subtract quantities which differ by more than 11 orders of magnitude. Likewise,

$$\begin{aligned}\psi'_{\text{fac}}(1) &= 2[1 - (1 - 2.350 \times 10^{-11} + 1.381 \times 10^{-22})] \\ &\quad + 2[1 - (1 - 5.876 \times 10^{-10} + 8.632 \times 10^{-20})] \\ &= 1.222 \times 10^{-9}\end{aligned} \tag{43}$$

and $\psi'_{\text{fac}}(1)$ retains, in fact, six significant digits (not all displayed).

Thus, all significance is lost in the Newton-Raphson iteration of the expanded form of the characteristic equation, but seven significant digits are retained in the Newton-Raphson iteration of the factored form of the characteristic equation. This is the origin of the differences in precision of the characteristic polynomials of the QUEST and FOAM algorithms, as we shall see in detail in the next section.

---

[30]When we are not subtracting numbers very close in value, we display no more than four significant figures.

The QUEST and FOAM Characteristic Polynomials— Scenario 2 of Reference 7

From equations (9) and (14) we see that the characteristic polynomial for the QUEST algorithm in common use is in expanded form, and that for the FOAM algorithm is in what we shall call *partially-factored* form. We shall label these characteristic polynomials henceforth as $\psi_{\text{QUEST-exp}}(\lambda)$ and $\psi_{\text{FOAM-fac}}(\lambda)$. By direct substitution, we find for scenario 2 of Reference 7 that

$$a = 0.9999999974609042 \qquad \text{and} \qquad b = 0.9999999971236696 \tag{44ab}$$

$$c = 0 \ \ \text{(exactly)} \qquad \text{and} \qquad d = 5.946137136732494 \times 10^{-18} \tag{44cd}$$

$$s = 1.285462824336719 \times 10^{-6} \tag{44e}$$

whence,

$$a + b = 1.999999994584574 \qquad \text{and} \qquad ab + cs - d = 0.9999999945845738 \tag{45ab}$$

and, finally,

$$\begin{aligned} \psi_{\text{QUEST-exp}}(1) &= 1 - 1.999999994584574 - 0 + 0.9999999945845738 \\ &= \bigcirc \times 10^{-20} \end{aligned} \tag{46a}$$

$$\psi'_{\text{QUEST-exp}}(1) = 4 - 3.999999989169 - 0 = 1.0830853 \times 10^{-8} \tag{46b}$$

The variable c vanishes exactly when the measurements are coplanar, because $\det \boldsymbol{B}$ vanishes identically in that case. To show this for the QUEST form requires a bit of work, but from the FOAM form we see that the coefficient of the term linear in λ is $-8 \det \boldsymbol{B}$. With infinite-precision arithmetic the QUEST and FOAM forms of the characteristic polynomial must be identically equal. The tremendous loss of significance in the evaluation of the expanded form of the QUEST characteristic polynomial is clear.

For the FOAM cost function, on the other hand,

$$\|\boldsymbol{B}\|_F = 0.999999986461434 \qquad \text{and} \qquad \det \boldsymbol{B} = 0 \text{ (exactly)} \tag{47ab}$$

$$\|\text{adj}\, \boldsymbol{B}\|_F = 1.222146568006809 \tag{47c}$$

so that

$$\begin{aligned} \psi_{\text{FOAM-fac}}(1) &= 7.3317106 \times 10^{-18} - 0 - 5.974568934763287 \times 10^{-18} \\ &= 1.3571416 \times 10^{-18} \end{aligned} \tag{48a}$$

$$\psi'_{\text{FOAM-fac}}(1) = 1.08308527 \times 10^{-8} - 0 = 1.08308527 \times 10^{-8} \tag{48b}$$

and eight significant digits remain in the calculation of λ_{max}.

We now note that we may factor the QUEST characteristic polynomial partially as

$$\psi_{\text{QUEST-fac}}(\lambda) = [\lambda^2 - a]\,[\lambda^2 - b] - c\lambda + (cs - d) \tag{49a}$$

$$\psi'_{\text{QUEST-fac}}(\lambda) = 2\lambda\,[2\lambda^2 - a - b] - c \tag{49b}$$

so that

$$\psi_{\text{QUEST-fac}}(1) = (2.5390958 \times 10^{-9})\,(2.8763304 \times 10^{-9}) - 0 \\ - 5.946137136732494 \times 10^{-18} \\ = 1.3571414 \times 10^{-18} \tag{50}$$

and retains eight significant digits.

In a similar manner we may write the FOAM characteristic polynomial in expanded form to obtain

$$\psi_{\text{FOAM-exp}}(\lambda) = \lambda^4 - 2\|B\|_F^2\,\lambda^2 - 8(\det B)\,\lambda + (\|B\|_F^4 - 4\,\|\text{adj}\,B\|_F^2) \tag{51}$$

and it is easy to demonstrate that all significance is lost in $\psi_{\text{FOAM-exp}}(1)$.

As a further demonstration, we have repeated the calculations of Table 1 of the present work with the forms of the characteristic polynomials reversed. Thus, the QUEST characteristic polynomial is now in partially-factored form and the FOAM characteristic polynomial is now in expanded form. The results are shown in Table 2.

With this minor rearrangement of terms in the QUEST characteristic polynomial the alleged (but not real) robustness and inaccuracy problems of QUEST disappear. Poof! To demonstrate more clearly that the numerical precision of the characteristic polynomial is the cause of the poorer performance of QUEST in scenario 2 of Reference 7 we have recalculated the attitude estimates for scenario 2 with the QUEST characteristic polynomial in partially-factored form and the FOAM characteristic polynomial in expanded form. The results are shown in Table 2. We see that QUEST now performs as well as any of the other algorithms did in Table 1 but the other algorithms perform in a manner similar to that of QUEST in Table 1. Whee!

Discussion and Conclusions

The QUEST algorithm, virtually unchanged since its first implementation in 1978, has been shown, if executed properly, to be as accurate as any batch optimal attitude estimator. With a minor rearrangement of the terms of the characteristic polynomial, it will perform accurately and robustly, even for the unrealizable scenario 2 of Reference 7. As we shall see in a later work [23], assessment of the speed of QUEST and the many ESOQ algorithms is much more complex than presented in Reference 7. Overall, all algorithms (except ESOQ1.1 and ESOQ2.1, which are approximate by design) are equal in accuracy, and QUEST is slightly faster in MATLAB when it comes to speed.[31] Assertions that another algorithm is the most robust, the most accurate, the fastest, or the state of the art must be discredited. Given the smallness of the differences, the fairest statement would seem to be that QUEST, FOAM, ESOQ1, late ESOQ2, M-SVD and q-Davenport are equal in accuracy; that q-Davenport and M-SVD are more robust but slower; and that QUEST, FOAM, ESOQ1, and late ESOQ2 are faster. There seems to be little justification for an Olympic Games of the Wahba problem.

The poorer convergence of $\lambda_{\max}$ in QUEST for very imbalanced sensor accuracies, although nothing like the absurd behavior of an incorrectly configured QUEST for the unphysical scenario 2 of Reference 7 (i.e., INEWT $\neq$ 0 for scenario 2 of Reference 7), had, in fact, been known for more than a decade before the appearance of Reference 7, and had been observed even for the more

[31]As determined by execution time; the result for the number of MATLAB® floating-point operations is less clear-cut.

TABLE 2. Estimation Results for Scenario 2 of Reference 7 with Reversed Characteristic Polynomials

Algorithm	Iterations	Δ Cost Function	x (deg)	yz (arcsec)
q-Davenport	—	—	9.30	1.43
M-SVD	—	1.40 e-5	9.30	1.43
FOAM	0	—	9.42	1.43
	1	471	108.	1.43
	2	6620	109.	1.43
	3	3430	105.	1.43
	4	5950	104.	1.43
	5	3470	104.	1.43
QUEST^{MM}	0	—	9.47	1.43
	1	0.102	9.29	1.43
	2	0.953 e-3	9.30	1.43
	3	1.33 e-5	9.30	1.43
	4	1.34 e-5	9.30	1.43
	5	1.33 e-5	9.30	1.43
ESOQ1	0	—	9.25	1.43
	1	471	57.0	1.43
	2	6620	56.3	1.43
	3	3430	56.4	1.43
	4	5950	56.2	1.43
	5	3470	56.5	1.43
ESOQ1.1	1	334	49.0	1.43
ESOQ2	0	—	9.42	1.43
	1	471	108.	1.43
	2	6620	109.	1.43
	3	3430	105.	1.43
	4	5950	104.	1.43
	5	3470	103.	1.43
ESOQ2.1	1	64.1	37.4	1.43

physical accuracies of a 3.0 arcsec star tracker and a coarse sensor of accuracy 0.5 deg (variance ratio = 360,000). This situation for a more reasonable scenario (including also less extreme values for the angle α) was encountered in practice during the design of the on-board attitude determination software for the Mid-Course Space Experiment mission (MSX) (launched 1996), in which the second author of the present work played a large rôle. The MSX on-board computer (Harris 1750) followed the IEEE standard for double-precision word length. In cases when only a single star direction was observed, single-frame attitude was determined using the single star and the outputs of the Barnes Engineering infrared horizon scanner (not the decision of the second author of the present work). It was discovered that the values of $\lambda_{\max}$ in the Newton-Raphson iterations in those cases sometimes did not converge rapidly but exhibited a damped oscillation, converging sometimes only after five or six iterations to the correct $\lambda_{\max}$. Since there was excess CPU time available for onboard calculations, but not excess research time for investigations, the modification (in 1993 and due officially to the second author of the present work) was to replace the QUEST characteristic polynomial with that from FOAM, which under more normal circumstances would have made QUEST slower.[32] The QUEST algorithm so modified performed well in flight.[33]

The ESOQ1.1 and ESOQ2.1 also show poor performance in scenario 2 of Reference 7. These algorithms do not use the characteristic polynomials to calculate their single iteration of $\lambda_{\max}$. We have not sought a modification which will correct the poor performance of the ESOQ1.1 and ESOQ2.1 algorithms for scenario 2 of Reference 7. It is not clear what the need is for such algorithms, since they can be only approximate.

Most of the competing fast attitude estimators are not very different from QUEST. The more important features of QUEST as developed over the past three decades have been the recognition that $\lambda_{\max} = \lambda_{\text{o}}$ to extremely high accuracy [3, 10], the use of Newton-Raphson iteration to refine the calculation of $\lambda_{\max}$ [3, 10], the method of sequential rotations [3, 10], the QUEST measurement model [3, 13], the choice of optimal weights [3], the simple expression for the attitude covariance matrix [3], the exact expression of $\lambda_{\max}$ for two direction measurements [3], the TASTE test [25], the connection of QUEST and TRIAD [3], and the realization that QUEST was the maximum-likelihood estimator for the QUEST measurement model [13]. All of these aspects, when appropriate, have been adopted by the fast algorithms which came after QUEST. Thus, to a very large degree, these "competing" fast algorithms, especially ESOQ1 and to almost as great as extent ESOQ2, are mostly QUEST.

In two works currently in preparation [23, 30] we shall discuss the implementation of the method of sequential rotations in QUEST and how this can be made more efficient [30] and the complexity of determining the speed of execution of QUEST and the other Wahba algorithms [23]. We will present results not only for the algorithms presented here, but also for better optimized implementations of the ESOQ1 and late ESOQ2 algorithms.

With the exception of ESOQ1.1 and ESOQ2.1, which by design are only first-order approximation, all of the fast solutions to the Wahba problem are excellent candidates for mission support in general, including the relative slowpokes, q-Davenport and M-SVD, which, when computers become still faster, may someday become the only reasonable choices [12].

[32] The elimination of all iterations and the use of other means to calculate $\lambda_{\max}$ for the TASTE test, a preferable approach in the opinion of the second author of the present work, was also considered but not accepted.

[33] Of course, the frames of data containing only one star direction were sufficiently infrequent that an accurate attitude estimate could have been obtained simply by propagating the previous estimate forward using gyro data and performing a single-vector update with the Kalman filter. This was, in fact, the second author's original suggestion, but there was a strong desire to have a single-frame attitude estimate available in all possible cases for frame-by-frame data checking using the TASTE test [25]. It was this hybridization of the QUEST and FOAM algorithms which prompted the second author to pose the rhetorical question in his Dirk Brouwer lecture [12]: Is QUEST still QUEST if it uses the FOAM form of the characteristic polynomial? With the partially factored form of the QUEST characteristic polynomial, that question is no longer of even rhetorical interest.

Our work here has also suggested some improvements which should be made to QUEST: (1) the implementation of the partially-factored form of the QUEST characteristic polynomial equation (49a) rather than the expanded form of equation (9); and (2) to compute the coefficient c of equation (9) as

$$c = 8 \det B \tag{52}$$

rather than according to equation (10c). In addition, the present authors plan to include (3) a more efficient means for detecting the need for a sequential rotation during the execution of QUEST [30]. These improvements will make QUEST (possibly renamed QUEST2000 or QUEST2007) a much more formidable attitude estimation tool.

In summary, Markley and Mortari presented a genuine case in which the performance of QUEST was poorer than that for competing algorithms. However, that case is unphysical and unlikely to occur unless one has a very stupid attitude system design. With a very minor rearrangement of terms in one equation, however, that bad case disappears entirely. As a result, QUEST has been shown to be at least equal in accuracy and robustness to any other fast optimal batch estimator. In a succeeding work we shall discuss the question of algorithm speed in great detail. Again, QUEST will appear among the best performers. To modify a famous quote of the American author and humorist Mark Twain, the rumors of the death of QUEST are highly exaggerated.

Acknowledgment

The authors are very grateful to F. Landis Markley, creator of the M-SVD and FOAM algorithms, for generously supplying them with the MATLAB® m-files used in Reference 7 and for helpful comments. It is altogether fitting that Dr. Markley, who played an important rôle in the development of the first QUEST form of the characteristic polynomial [12], should have been such a strong motivating force for its improvement. We thank him and Daniele Mortari, creator of the many Euler and ESOQ algorithms and, like Dr. Markley, a noble, consummate and gallant wahbateer, for many interesting discussions. The QUEST algorithm and its continued improvement have been the greatest beneficiaries of their efforts. The first author wishes to thank John L. Crassidis of the University at Buffalo for his continued support.

References

[1] LERNER, G. M. “Three-Axis Attitude Determination,” in Wertz, J. R. (ed.), *Spacecraft Attitude Determination and Control*, Springer, Berlin and New York, 1978, pp. 420–428, especially the footnote on p. 425.

[2] BLACK, H. D. “A Passive System for Determining the Attitude of a Satellite,” *AIAA Journal*, Vol. 2, July 1964, pp. 1350–1351.

[3] SHUSTER, M. D. and OH, S. D. ‘Three-Axis Attitude Determination from Vector Observations,” *Journal of Guidance and Control*, Vol. 4, No. 1, Jan.-Feb. 1981, pp. 70–77.

[4] WAHBA, G. “Problem 65–1: A Least Squares Estimate of Spacecraft Attitude,” *Siam Review*, Vol. 7, No. 3, July 1965, p. 409.

[5] SHUSTER, M. D. “A Survey of Attitude Representations,” *The Journal of the Astronautical Sciences*, Vol. 41, No. 4, October–December 1993, pp. 439–517.

[6] KEAT, J. *Analysis of Least Squares Attitude Determination Routine DOAOP*, Computer Sciences Corporation, CSC/TM–77/6034, February 1977.

[7] MARKLEY, F. L. and MORTARI, M. "Quaternion Attitude Estimation Using Vector Measurements, *The Journal of the Astronautical Sciences*, Vol. 48, Nos. 2 and 3, April–September 2000, pp. 359–380.

[8] GOLUB, G. H. and VAN LOAN, C. F. *Matrix Computations*, The Johns Hopkins University Press, Baltimore, 1983.

[9] FALLON, L. III, HARROP, I. H., and STURCH, C. R. "Ground Attitude Determination and Gyro Calibration Procedures for the HEAO Missions," *Proceedings, AIAA 17th Aerospace Sciences Meeting*, New Orleans, Louisiana, January 1979.

[10] SHUSTER, M. D. "Approximate Algorithms for Fast Optimal Attitude Computation" Paper No. 78–1249, *Proceedings, AIAA Guidance and Control Conference*, Palo Alto, California, August 1978, pp. 88–95.

[11] ABSHIRE, G., McCUTCHEON, R., SUMMERS, G. and VanLANDINGHAM, F. G. "High Precision Attitude Determination for Magsat," *Proceedings, ESA International Symposium on Spacecraft Flight Dynamics,* Darmstadt, Germany, May 1981.

[12] SHUSTER, M. D. "In Quest of Better Attitudes" (Dirk Brouwer lecture), Paper No. AAS-01-250, *Advances in the Astronautical Sciences*, Vol. 108, 2001, pp. 2089–2117.

[13] SHUSTER, M. D. "Maximum Likelihood Estimation of Spacecraft Attitude," *The Journal of the Astronautical Sciences*, Vol. 37, No. 1, January–March, 1989, pp. 79–88.

[14] SORENSON, H. *Parameter Estimation*, Marcel Dekker, New York, 1980.

[15] MARKLEY, F. L. "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *The Journal of the Astronautical Sciences*, Vol. 36, No. 3, July-September 1988, pp. 245–258.

[16] MARKLEY, F. L. "Attitude Determination Using Vector Observations: a Fast Optimal Matrix Algorithm," *The Journal of the Astronautical Sciences,* Vol. 41, No. 2, April-June 1993, pp. 261–280.

[17] MORTARI, D. "Euler-2 and Euler-n Algorithms for Attitude Determination from Vector Observations," *Space Technology*, Vol. 16, Nos. 5–6, 1996, pp. 317–321.

[18] MORTARI, D. "Euler-q Algorithm for Attitude Determination from Vector Observations," *Journal of Guidance, Control and Dynamics*, Vol. 12, No. 2, March-April 1998, pp. 328–334.

[19] MORTARI, D. "ESOQ: A Closed-Form Solution to the Wahba Problem," *The Journal of the Astronautical Sciences*, Vol. 45, No. 2, April-June 1997, pp. 195–204.

[20] MORTARI, D. "ESOQ2 Single-Point Algorithm for Fast Optimal Attitude Determination," Paper AAS-97-167, *Advances in the Astronautical Sciences*, Vol. 97, Part II, 1997, pp. 803–816.

[21] MORTARI, D. "Second Estimator for the Optimal Quaternion," *Journal of Guidance, Control and Dynamics*, Vol. 23, No. 4, September–October 2000, pp. 885–888.

[22] BRUCCOLERI, C., LEE, D.-J. and MORTARI, D. "Single-Point Optimal Attitude Determination Using Modified Rodrigues Parameters," *Advances in the Astronautical Sciences*, Vol. 122, 2006, pp. 137–147.

[23] CHENG, Y. and SHUSTER, M. D. "Fast Optimal Attitude Estimators: II. Speed," (in preparation).

[24] SHUSTER, M. D. "The Maximum-Error Test," to appear in *The Journal of the Astronautical Sciences*.

[25] SHUSTER, M. D. and FREESLAND, D. C. “The Statistics of TASTE and the Inflight Estimation of Sensor Precision,” *Proceedings (CD), NASA Flight Mechanics Symposium*, NASA Goddard Space Flight Center, Greenbelt, Maryland, October 18–20, 2005.

[26] MARKLEY, F. L., ANDREWS, S. F., O’DONNELL, J. R., and WARD, D. K., “Attitude Control System of the Wilkinson Microwave Anisotropy Probe,” *Journal of Guidance, Control and Dynamics*, Vol. 28, No. 2, May–June 2005, pp. 385–397.

[27] VAROTTO, S. E. C., ORLANDO, V., and LOPES, R. V. F. “Um procedimento para determinação da atitude de satélites artificiais utilizando técnicas de estimação ótima estática e dinâmica,” *Atas, 6º Congresso Brasileiro de Automática,* Belo Horizonte, 1986, p. 946–951.

[28] SHUSTER, M. D. “Kalman Filtering of Spacecraft Attitude and the QUEST Model,” *The Journal of the Astronautical Sciences* Vol. 38, No. 3, July–September, 1990, pp. 377–393; Erratum: Vol. 51, No. 3, July-September 2003, p. 359.

[29] SHUSTER, M. D. “SCAD—A Fast Algorithm for Star-Camera Attitude Determination,” *The Journal of the Astronautical Sciences*, Vol. 52, No. 3, July–September 2004, pp. 391–404.

[30] CHENG, Y. and SHUSTER, M. D. “The Method of Sequential Rotations” (in preparation).

[31] TANYGIN, S. and SHUSTER, M. D. “The TRIAD Algorithms” (in preparation).