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Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

General Formula for Extracting the Euler Angles

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Introduction

RECENTLY, the authors completed a study[1] of the Davenport angles, which are a generalization of the Euler angles for which the initial and final Euler axes need not be either mutually parallel or mutually perpendicular or even along the coordinate axes. During the conduct of that study, those authors discovered a relationship that can be used to compute straightforwardly the Euler angles characterizing a proper-orthogonal direction-cosine matrix for an arbitrary Euler-axis set satisfying

$$\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_2 = 0, \qquad \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{n}}_3 = 0 \tag{1a}$$

which is also satisfied by the more usual Euler angles that we encounter commonly in the practice of astronautics. When we have also

$$\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_3 = 0 \qquad \text{or} \qquad \pm 1 \tag{1b}$$

then the more general Davenport angles become identical to the Euler angles, although for the latter the Euler axes are usually taken to be the coordinate axes and not an arbitrary orthonormal set.

Rather than leave that relationship hidden in an article with very different focus from the present Engineering Note, we present it and the general algorithm derived from it for extracting the Euler angles from the direction-cosine matrix here. We also offer literal "code" for performing the operations, numerical examples, and general considerations about the extraction of Euler angles, which are not universally known.

Development of the Formula

Let $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3$ be a sequence of three (unit) Euler axes, and φ, ϑ, ψ be the associated Euler angles. Then the direction-cosine matrix D corresponding to these Euler axes and Euler angles is given by

$$D = R(\hat{\boldsymbol{n}}_3, \psi) R(\hat{\boldsymbol{n}}_2, \vartheta) R(\hat{\boldsymbol{n}}_1, \varphi) \equiv R(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3; \varphi, \vartheta, \psi) \tag{2}$$

where $R(\hat{\boldsymbol{n}}, \theta)$ denotes the direction-cosine matrix[2] of a rotation about an axis $\hat{\boldsymbol{n}}$ through an angle θ. Davenport[3] and Ref. 1 showed that such Euler angles exist for any proper-orthogonal D and any set of Euler axes satisfying Eq. (1a). Thus, in this work our interest is not limited to Euler axes chosen from the set $\mathcal{E} \equiv \{\pm\hat{\mathbf{1}}, \pm\hat{\mathbf{2}}, \pm\hat{\mathbf{3}}\}$, where

$$\hat{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{2}} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{3}} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

although in practice this is the case that occurs with greatest frequency. We do, however, assume in the present work that the axes satisfy also Eq. (1b).

The relationship discovered in Ref. 1 is

$$\begin{aligned} D &= C^T R(\hat{\mathbf{1}}, \lambda) R(\hat{\mathbf{3}}, \hat{\mathbf{1}}, \hat{\mathbf{3}}; \varphi, \vartheta - \lambda, \psi) C \\ &\equiv C^T R(\hat{\mathbf{1}}, \lambda) R_{313}(\varphi, \vartheta - \lambda, \psi) C \end{aligned} \tag{4}$$

with

$$\lambda = \arctan_2[(\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2) \cdot \hat{\boldsymbol{n}}_3, \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{n}}_3] \tag{5}$$

$$C = [\hat{\boldsymbol{n}}_2 \quad (\hat{\boldsymbol{n}}_1 \times \hat{\boldsymbol{n}}_2) \quad \hat{\boldsymbol{n}}_1]^T \tag{6}$$

where the matrix in Eq. (6) has been indicated by its column vectors, and $\arctan_2(y, x)$ returns the value of $\tan^{-1}(y/x)$ in the correct quadrant.

Writing

$$O \equiv R_{313}(\varphi, \vartheta - \lambda, \psi) \tag{7}$$

we can solve Eq. (4) as

$$O = R^T(\hat{\mathbf{1}}, \lambda) C D C^T \tag{8}$$

It is considerably easier to extract the values of $(\varphi, \vartheta, \psi)$ from O than from D directly.

From the familiar formula

$$R_{313}(\varphi, \vartheta', \psi) = \begin{bmatrix} c\psi c\varphi - s\psi c\vartheta' s\varphi & c\psi s\varphi + s\psi c\vartheta' c\varphi & s\psi s\vartheta' \\ -s\psi c\varphi - c\psi c\vartheta' s\varphi & -s\psi s\varphi + c\psi c\vartheta' c\varphi & c\psi s\vartheta' \\ s\vartheta' s\varphi & -s\vartheta' c\varphi & c\vartheta' \end{bmatrix} \tag{9}$$

with $c\varphi \equiv \cos\varphi$, $s\varphi \equiv \sin\varphi$, etc., and $\vartheta' = \vartheta - \lambda$, we have immediately

$$\vartheta = \lambda + \cos^{-1} O_{33} \tag{10a}$$

and, for $\lambda < \vartheta < \lambda + \pi$,

$$\varphi = \arctan_2(O_{31}, -O_{32}) \tag{10b}$$

$$\psi = \arctan_2(O_{13}, O_{23}) \tag{10c}$$

For $\vartheta = \lambda$ or $\vartheta = \lambda + \pi$ the arguments of Eqs. (10b) and (10c) all vanish, and the two equations have no unique solution for φ and

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ψ. In those two special cases, O depends only on $\varphi - \psi$ or $\varphi + \psi$, respectively. Thus, for $\vartheta = \lambda$ one can write at best

$$\varphi - \psi = \arctan_2(O_{12} - O_{21}, O_{11} + O_{22}) \tag{11a}$$

and for $\vartheta = \lambda + \pi$

$$\varphi + \psi = \arctan_2(O_{12} + O_{21}, O_{11} - O_{22}) \tag{11b}$$

Typically, in these cases, one sets $\psi = 0$. Equations (11) are much better behaved numerically than the usual formulas[2] with slightly simpler arguments in the $\arctan_2$ functions. One or the other of these equations is also better behaved numerically than Eqs. (10b) and (10c) near $\theta = \lambda$ or $\theta = \lambda + \pi$, respectively. Unfortunately, there are no available numerically well-behaved equations for both φ and ψ in these regions.

If the Euler axes are chosen from $\mathcal{E}$, or from any orthonormal set of 3×1 matrices, then λ can take on the value $-\pi/2$, 0, $\pi/2$, or π (mod 2π). When this makes the range of ϑ inconvenient, the angles can be replaced by their equivalents according to[1]

$$(\varphi, \vartheta, \psi) \longleftrightarrow (\varphi + \pi, 2\lambda - \vartheta, \psi - \pi) \mod 2\pi \tag{12}$$

Frequently one desires that ϑ be in the range $0 \le \vartheta \le \pi$.

Thus, given $D, \hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3$, the algorithm for extracting the Euler angles from the (proper-orthogonal) direction-cosine matrix is as follows:

Given $D, \hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3$:

1) Set observability flag to "poor."
2) Compute λ and C from Eqs. (5) and (6).
3) Compute O from Eq. (8).
4) Compute ϑ from Eq. (10a).
5) If $|\vartheta - \lambda| \ge \epsilon$ and $|\vartheta - \lambda - \pi| \ge \epsilon$ (ϵ is machine and problem dependent),
   a) Set observability flag to "good."
   b) Compute φ and ψ from Eqs. (10b) and (10c).
6) Else
   a) Set $\psi = 0$.
   b) If $|\vartheta - \lambda| < \epsilon$, compute φ from Eq. (11a).
   c) If $|\vartheta - \lambda - \pi| < \epsilon$, compute φ from Eq. (11b).
7) Adjust angles according to Eq. (12) if necessary.
8) The outputs are φ, ϑ, ψ, and the observability flag.

Note that the preceding tests refer to the value of the argument (mod 2π), which is smallest.

Our result should be compared to that of Kolve,[4] who, instead of performing analytical operations on the direction-cosine matrix, does a special accounting of the indices. Kolve's method is applicable only to Euler axes that are parallel to the coordinate axes. Thus, Kolve's method cannot be applied to the second of the numerical examples below.

We note that φ and ψ cannot be calculated unambiguously using only Eqs. (11a) and (11b). The solution of each of the two equations yields numerical results for

$$\varphi - \psi + 2m\pi, \qquad \varphi + \psi + 2n\pi$$

respectively, where m and n are integers. By taking linear combinations of these two quantities, we can obtain numerical results for

$$\varphi + (m+n)\pi, \qquad \psi - (m-n)\pi$$

and neither $m + n$ nor $m - n$ need be even integers. Thus, both φ and $\varphi + \pi$ and both ψ and $\psi + \pi$ are possible solutions, which is unacceptable. It follows that we cannot use Eqs. (11a) and (11b) alone to calculate φ and ψ.

Instead of the program we have given following Eq. (12), we can use Eq. (11a) when $O_{33} \ge 0$ to solve for $\varphi - \psi$ or Eq. (11b) when $O_{33} < 0$ to solve for $\varphi + \psi$ and to supplement either of these solutions with that for φ or ψ from either Eq. (10a) or Eq. (10b), respectively. The resulting φ and ψ will not suffer from the ambiguity of multiples of π, but only from the usual ambiguity of multiples of 2π, which causes us no distress.

When $|\vartheta - \lambda|$ is close to 0 or π, the alternate method will yield an accurate value for $\varphi \pm \psi$ for one choice of the sign, but the value for the solution from Eq. (10a) or Eq. (10b) (and its accuracy) will be the same as for the preceding program. When this is combined with the $\varphi \pm \psi$ to obtain the remaining angle, that angle will suffer then from the same lack of significance as that calculated from this work's proposed program. Thus, whether one uses the program just proposed for the individual angles or the alternate method is purely a matter of esthetic taste.

Numerical Examples

Example 1

As a simple example, consider the computation of φ, ϑ, and ψ for a 3-1-2 set of Euler axes with true values $\varphi = 45$ deg, $\vartheta = 30$ deg, and $\psi = 20$ deg. The resulting direction-cosine matrix is

$$D = R(\hat{\mathbf{2}}, 20 \text{ deg})\, R(\hat{\mathbf{1}}, 30 \text{ deg})\, R(\hat{\mathbf{3}}, 45 \text{ deg}) = \begin{bmatrix} 0.5435 & 0.7854 & -0.2962 \\ -0.6124 & 0.6124 & 0.5000 \\ 0.5741 & -0.0904 & 0.8138 \end{bmatrix} \tag{13}$$

One finds straightforwardly that

$$C = I_{3\times 3}, \qquad \lambda = \arctan_2(1, 0) = \pi/2 \tag{14}$$

Performing the multiplications of Eq. (8) yields

$$O = \begin{bmatrix} 0.5435 & 0.7854 & -0.2962 \\ -0.5741 & 0.0904 & -0.8138 \\ -0.6124 & 0.6124 & 0.5000 \end{bmatrix} \tag{15}$$

and applying Eqs. (10) yields

$$\varphi = 45.0000, \qquad \vartheta = 30.0000, \qquad \psi = 20.0000 \tag{16}$$

as expected. Deviations occur only in the 14th decimal place.

Example 2

To appreciate the power of this algorithm, consider the following more complex example:

$$\hat{\boldsymbol{n}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{n}}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \qquad \hat{\boldsymbol{n}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{17}$$

The Euler-axis set is orthonormal but not proper orthonormal, and two of the axes are certainly not along body coordinate axes. Let the direction-cosine matrix be

$$D = R(\hat{\boldsymbol{n}}_3, 20 \text{ deg}) R(\hat{\boldsymbol{n}}_2, 30 \text{ deg}) R(\hat{\boldsymbol{n}}_1, 45 \text{ deg}) = \begin{bmatrix} 0.9929 & 0.1171 & 0.0216 \\ -0.0887 & 0.6063 & 0.7903 \\ 0.0795 & -0.7866 & 0.6124 \end{bmatrix} \tag{18}$$

Then

$$C = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \tag{19a}$$

$$\lambda = \arctan_2(-1, 0) = -\pi/2 \tag{19b}$$

Performing the multiplications of Eq. (8) yields

$$O = \begin{bmatrix} 0.7854 & 0.5435 & 0.2962 \\ 0.0904 & -0.5741 & 0.8138 \\ 0.6124 & -0.6124 & -0.5000 \end{bmatrix} \tag{20}$$

and applying Eqs. (10) yields

$$\varphi = 45.0000, \qquad \vartheta = 30.0000, \qquad \psi = 20.0000 \tag{21}$$

as expected. Again, deviations from the input values occur only in the 14th decimal place.

Note that although the Euler angles have the same values in the two examples, the direction-cosine matrices O are not identical. This is because the middle angle for O is not ϑ, but $\vartheta - \lambda$ and λ has different values in the two examples. Also, ϑ from the first example had a value outside the interval $0 \leq \vartheta \leq \pi$ and required adjustment according to Eq. (12). Note the similarities (if not equality) of the matrix entries, although they might differ by a sign and not always be in the same place. These similarities are caused by the fact that in our preceding examples λ has the value $\pi/2$ or $-\pi/2$ so that the transformation of Eq. (12) is of a rather trivial sort. Had the two λ had very different values from multiples of $\pi/2$, then the similarity of the matrix elements might not be present. That situation will occur, however, only when the three rotation axes are chosen from a nonorthonormal set [but satisfying Eq. (1a), in which case we are dealing not with Euler angles but with the Davenport angles[1,3]].

Summary

A very simple parameterization of the direction-cosine matrix has been developed in terms of Euler angles about axes drawn from an orthonormal triad that need not be the coordinate axes. Very efficient algorithms have been presented for constructing a direction-cosine matrix using these more general Euler axes and for extracting the corresponding Euler angles from a direction-cosine matrix by constructing a related direction-cosine matrix generated by the same angles but for which the Euler axes are the familiar 3-1-3 set. Two illustrative examples were presented.

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