The Journal of the Astronautical Sciences, Vol. 53, No. 3, July–September 2005, pp. 337–351

QUEST and The Anti-QUEST: Good and Evil Attitude Estimation

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Abstract

An attitude determination algorithm, proposed 25 years ago as a back-up algorithm in case the QUEST algorithm failed to function properly on its maiden mission, is finally compared with the QUEST algorithm. A comparison of the attitude determination accuracies of both algorithms is carried out using the QUEST measurement model. The back-up algorithm, which we have chosen to call "The Anti-QUEST," while it lacks the special features that have made QUEST so popular today and is rather clumsy, slow and deficient in many ways, works well under the almost ideal conditions of the Magsat mission, but not generally. Further comparisons of QUEST and The Anti-QUEST provide useful insights into the different behaviors of optimal and deterministic attitude estimators. This work also presents useful practical techniques for the covariance analysis of nonoptimal algorithms.

Introduction: QUEST

When the QUEST algorithm [1–3] was first developed between August 1977 and October 1978, no one, especially not QUEST's author, anticipated that it would gain the popularity it enjoys today. QUEST came at a critical time in NASA attitude mission support, when accuracy requirements and attitude computation frequency requirements for the proposed Magsat mission threatened to overwhelm the computational capacity of Mission & Data Operations at NASA Goddard Space Flight Center. No one had a good idea what to do, and the expectation was that NASA would spend a lot of time computing spacecraft attitude (on ground-based main-frame computers) and the project scientists would all have long white beards by the time they received precisely referenced magnetic field data.

Fortunately, early in the development of the Magsat attitude software an algorithm, QUEST (for QUaternion ESTimator), was developed which was lightning fast and had other good properties which greatly streamlined attitude mission

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operations. The story of the development of QUEST is an interesting story in itself, which the reader can find in reference [4].

QUEST used somewhat unfamiliar mathematics to achieve its speed, and because of this there was some trepidation at NASA[3] that it might not perform as well with real data as in simulation. For that reason, about six months before the Magsat launch on October 30, 1979, it was decided to have a back-up algorithm just in case. The back-up algorithm was clumsy, slow, ugly, obviously less accurate than QUEST, and it could not do half of what QUEST could do, but no one doubted that it would work. As it turned out, QUEST performed remarkably well from the first frame of data, and the back-up algorithm, which never had a name, was never exercised with real mission data and quickly passed into well-deserved oblivion. But there is something to be learned from the alternative Magsat algorithm. Therefore, for the first time, we disclose this forgotten algorithm, orphaned at birth, abandoned by NASA-kind. We have dubbed this forlorn algorithm "The Anti-QUEST" and give it now a proper evaluation, so that the world may know what might have been.[4]

Despite a firm desire to be entertaining, this work also provides practical techniques for the covariance analysis of nonoptimal attitude estimation algorithms, i.e., algorithms for which the attitude-error covariance matrix cannot be calculated as the inverse of the Fisher information matrix. It also provides new results for the attitude-error covariance matrix of the TRIAD algorithm, which are much superior to those presented two decades ago. In addition, one gains important insights into some of the undesirable properties of nonoptimal attitude estimators. In some cases, The Anti-QUEST performs badly, because it makes use of *more* (!) data than the TRIAD algorithm, on which it is based, although it was expected naively to be better than the TRIAD algorithm, because it used all of the data. What is most surprising is that for the Magsat sensor configuration, for which The Anti-QUEST was proposed, it performs nearly as well as QUEST.

The TRIAD Algorithm

Although we will not repeat the derivation of the QUEST algorithm, which has been presented adequately elsewhere in the literature [2, 3], we will repeat the derivation of the TRIAD algorithm,[5] if only to introduce notation and definitions.

In the TRIAD algorithm, one is given two unit-vector measurements (i.e., 3×1 arrays of numbers) $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{W}}_2$, generally the observed directions of the Sun, a star, the magnetic field, or the nadir, all represented with respect to the spacecraft body frame, and their corresponding representations in the primary reference frame (frequently inertial), $\hat{\mathbf{V}}_1$ and $\hat{\mathbf{V}}_2$, which are the directions of these same objects resolved along primary reference axes. These must satisfy

$$\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1 + \Delta\hat{\mathbf{W}}_1 \quad \text{and} \quad \hat{\mathbf{W}}_2 = A\hat{\mathbf{V}}_2 + \Delta\hat{\mathbf{W}}_2 \tag{1ab}$$

[3]There was, in fact, more than just "some" trepidation at NASA Goddard Space Flight Center.

[4]The prefix of *The Anti-QUEST* is the Classical Greek preposition ’άντι, meaning *against.* Thus, *The Anti-QUEST* means *the rival of QUEST* or, more strongly, *the enemy of QUEST.*

[5]The TRIAD algorithm, despite the fact that it constructs the attitude matrix by first constructing righthand orthonormal triads, does not derive its name from that fact, but from the name of an attitude ground support system (TRI-axial Attitude Determination System, hence, TRIAD, in upper-case letters, because it was an acronym), in which that algorithm was implemented in the early seventies, and the earliest occurrence of the algorithm known to the second author when reference [2] was published. (He might, of course, have named it the "triad" algorithm because of its use of triads, but then the name would be in lower-case letters.) Harold Black [5], who first published the algorithm, gave it no name. Gerald Lerner [6] gave it the not very descriptive name, the "algebraic method."

with A the attitude matrix and $\Delta\hat{\mathbf{W}}_1$ and $\Delta\hat{\mathbf{W}}_2$ the measurement noise. These two equations are generally not solvable, owing to the presence of the noise terms. The TRIAD prescription for generating a (proper orthogonal) attitude matrix is as follows:

Define two right-hand orthonormal triads of vectors, $\{\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3\}$ and $\{\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_3\}$ according to

$$\hat{\mathbf{r}}_1 = \hat{\mathbf{V}}_1, \qquad \hat{\mathbf{r}}_2 = \frac{\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2}{|\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2|}, \qquad \hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \tag{2abc}$$

$$\hat{\mathbf{s}}_1 = \hat{\mathbf{W}}_1, \qquad \hat{\mathbf{s}}_2 = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|}, \qquad \hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \tag{3abc}$$

and then set

$$A^{\mathrm{TRIAD}} = [\hat{\mathbf{s}}_1 \quad \hat{\mathbf{s}}_2 \quad \hat{\mathbf{s}}_3][\hat{\mathbf{r}}_1 \quad \hat{\mathbf{r}}_2 \quad \hat{\mathbf{r}}_3]^{\mathrm{T}} \equiv SR^{\mathrm{T}} \tag{4}$$

In equation (4) the brackets denote two matrices labeled by their column vectors, and the T denotes the matrix transpose.

The matrix A^{TRIAD} perforce satisfies

$$A^{\mathrm{TRIAD}}\hat{\mathbf{r}}_k = \hat{\mathbf{s}}_k, \quad k = 1, 2, 3 \tag{5}$$

is always proper orthogonal, and satisfies $\hat{\mathbf{W}}_1 = A^{\mathrm{TRIAD}}\hat{\mathbf{V}}_1$ exactly. If there is no measurement noise, then $\hat{\mathbf{W}}_2 = A^{\mathrm{TRIAD}}\hat{\mathbf{V}}_2$ is also satisfied. Otherwise, the equation $\hat{\mathbf{s}}_3 = A^{\mathrm{TRIAD}}\hat{\mathbf{r}}_3$ plainly shows that A^{TRIAD} exactly aligns the component of $\hat{\mathbf{V}}_2$ perpendicular to $\hat{\mathbf{V}}_1$ along the component of $\hat{\mathbf{W}}_2$ perpendicular to $\hat{\mathbf{W}}_1$.

The TRIAD Covariance Matrix[6]

We assume for simplicity that $\hat{\mathbf{V}}_1$ and $\hat{\mathbf{V}}_2$ are free of error. When this is not the case it creates only a minor complication [2]. The error in the attitude matrix estimate is defined as

$$\Delta A \equiv A^* - A^{\mathrm{true}} = (\Delta S)R^{\mathrm{T}} \tag{6}$$

where A^* here denotes the estimate of the attitude matrix (in this case the TRIAD attitude matrix), and A^{true} is the true value of the attitude matrix to which, supposedly, A^* is infinitesimally close. We define the attitude-error vector $\Delta\boldsymbol{\theta}$ by

$$A^* = \delta A(\Delta\boldsymbol{\theta})A^{\mathrm{true}} \tag{7}$$

where δA is the direction-cosine matrix of an infinitesimal rotation (with argument $\Delta\boldsymbol{\theta}$), which we may write to linear order in $\Delta\boldsymbol{\theta}$ as

$$\delta A = I + [[\Delta\boldsymbol{\theta}]] \tag{8}$$

with $[[\mathbf{u}]]$ the 3×3 antisymmetric matrix given by [7]

$$[[\mathbf{u}]] = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{9}$$

[6]Although treated at length in reference [2], we repeat much of that earlier material here, because we require that machinery in the present work and also, because after more than two decades we are able to produce nicer results. This repetition has also given us the opportunity to correct some errors that appeared in reference [2].

Thus, to first order in $\Delta\boldsymbol{\theta}$

$$\Delta A = [[\Delta\boldsymbol{\theta}]]A^{\text{true}} \tag{10}$$

The *attitude-error covariance matrix* or simply the *attitude covariance matrix* is defined as

$$P_{\theta\theta} \equiv E\{\Delta\boldsymbol{\theta}\Delta\boldsymbol{\theta}^{\text{T}}\} \tag{11}$$

where $E\{\cdot\}$ denotes the expectation. It will be useful to define the *attitude-matrix covariance matrix* as

$$P_{AA} \equiv E\{\Delta A \Delta A^{\text{T}}\} = (\text{tr}P_{\theta\theta})I - P_{\theta\theta} \tag{12}$$

and "tr" denotes the trace operation. Equation (12) may be solved to yield

$$P_{\theta\theta} = \frac{1}{2}(\text{tr}P_{AA})I - P_{AA} \tag{13}$$

From equation (6)

$$P_{AA} = E\{\Delta S \Delta S^{\text{T}}\} = E\left\{\sum_{k=1}^{3}\Delta\hat{\mathbf{s}}_k\Delta\hat{\mathbf{s}}_k^{\text{T}}\right\} = \sum_{k=1}^{3}E\{\Delta\hat{\mathbf{s}}_k\Delta\hat{\mathbf{s}}_k^{\text{T}}\} \tag{14}$$

Because of the simplicity of equation (14), it will be simpler to calculate P_{AA} first and then $P_{\theta\theta}$ using equation (13).

To compute the expectations in equation (14) we require a statistical model of the measurement noise, for which we propose the QUEST measurement model [2–3], namely

$$E\{\Delta\hat{\mathbf{W}}_i\} = \mathbf{0}, \quad i = 1, \ldots, n \tag{15a}$$

$$E\{\Delta\hat{\mathbf{W}}_i\Delta\hat{\mathbf{W}}_j^{\text{T}}\} = \delta_{ij}\,\sigma_i^2\,(I - \hat{\mathbf{W}}_i^{\text{true}}\hat{\mathbf{W}}_i^{\text{true T}}), \quad i, j = 1, \ldots, n \tag{15b}$$

with n the number of measurements (for TRIAD $n = 2$, for The Anti-QUEST $n = 3$) and δ_{ij} is the Kronecker symbol. The $\Delta\hat{\mathbf{W}}_i$ are further assumed to have a Gaussian distribution. This measurement model can be only approximately true [3], but it is certainly adequate for our purposes here.

We introduce also the non-unitized vectors

$$\mathbf{s}_1 \equiv \hat{\mathbf{W}}_1, \quad \mathbf{s}_2 \equiv \hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2, \quad \mathbf{s}_3 \equiv \mathbf{s}_1 \times \mathbf{s}_2, \quad \mathbf{s}_4 \equiv \hat{\mathbf{W}}_2 \times \mathbf{s}_2 \tag{16abcd}$$

and

$$\hat{\mathbf{s}}_i = \mathbf{s}_i/|\mathbf{s}_i|, \quad i = 1, \ldots, 4 \tag{17}$$

Note that $|\mathbf{s}_1| = 1$, $|\mathbf{s}_2| = |\mathbf{s}_3| = |\mathbf{s}_4| = |\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|$, and that $\{\hat{\mathbf{W}}_2, \hat{\mathbf{s}}_2, \hat{\mathbf{s}}_4\}$ is also a right-hand orthonormal triad. It follows that

$$\hat{\mathbf{s}}_4 = |\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|\hat{\mathbf{s}}_1 + (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{s}}_3 \tag{18}$$

The following relations will also be useful:

$$\hat{\mathbf{s}}_3 = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|}[(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{W}}_1 - \hat{\mathbf{W}}_2] \tag{19a}$$

$$\hat{\mathbf{s}}_4 = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|}[\hat{\mathbf{W}}_1 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{W}}_2] \tag{19b}$$

To compute the errors in the unit vectors of the observation triad, we note now that[7]

$$\Delta\hat{\mathbf{s}}_i = \frac{1}{|\mathbf{s}_i|}(I - \hat{\mathbf{s}}_i\hat{\mathbf{s}}_i^{\mathrm{T}})\Delta\mathbf{s}_i, \quad i = 1,\ldots,4 \tag{20a}$$

$$\hat{\mathbf{s}}_i^{\mathrm{T}}\Delta\hat{\mathbf{s}}_i = 0, \quad i = 1,\ldots,4 \tag{20b}$$

and

$$\Delta\mathbf{s}_1 = \Delta\hat{\mathbf{W}}_1 \tag{21a}$$

$$\Delta\mathbf{s}_2 = [[\hat{\mathbf{W}}_2]]\Delta\hat{\mathbf{W}}_1 - [[\hat{\mathbf{W}}_1]]\Delta\hat{\mathbf{W}}_2 \tag{21b}$$

$$\Delta\mathbf{s}_3 = ([[\mathbf{s}_2]] + (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)I)\,\Delta\hat{\mathbf{W}}_1 + [[\hat{\mathbf{W}}_1]]^2\,\Delta\hat{\mathbf{W}}_2 \tag{21c}$$

The evaluation of equation (14) is now lengthy but straightforward.

We have next

$$E\{\Delta\mathbf{s}_1\Delta\mathbf{s}_1^{\mathrm{T}}\} = \sigma_1^2(I - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}) = \sigma_1^2(I - \mathbf{s}_1\mathbf{s}_1^{\mathrm{T}}) \tag{22a}$$

$$E\{\Delta\mathbf{s}_2\Delta\mathbf{s}_2^{\mathrm{T}}\} = \sigma_1^2[I - \hat{\mathbf{W}}_2\hat{\mathbf{W}}_2^{\mathrm{T}} - \mathbf{s}_2\mathbf{s}_2^{\mathrm{T}}] + \sigma_2^2[I - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}} - \mathbf{s}_2\mathbf{s}_2^{\mathrm{T}}] \tag{22b}$$

$$\begin{aligned} E\{\Delta\mathbf{s}_3\Delta\mathbf{s}_3^{\mathrm{T}}\} = {} & \sigma_1^2[|\mathbf{s}_2|^2\hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}} - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)(\hat{\mathbf{W}}_1\mathbf{s}_3^{\mathrm{T}} + \mathbf{s}_3\hat{\mathbf{W}}_1^{\mathrm{T}}) \\ & + (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)^2(I - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}})] + \sigma_2^2[I - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}} - \mathbf{s}_3\mathbf{s}_3^{\mathrm{T}}] \end{aligned} \tag{22c}$$

and, consequently, using equation (20a)

$$E\{\Delta\hat{\mathbf{s}}_1\Delta\hat{\mathbf{s}}_1^{\mathrm{T}}\} = \sigma_1^2(I - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}) = \sigma_1^2(I - \hat{\mathbf{s}}_1\hat{\mathbf{s}}_1^{\mathrm{T}}) \tag{23a}$$

$$E\{\Delta\hat{\mathbf{s}}_2\Delta\hat{\mathbf{s}}_2^{\mathrm{T}}\} = \frac{1}{|\mathbf{s}_2|^2}(\sigma_1^2\hat{\mathbf{s}}_4\hat{\mathbf{s}}_4^{\mathrm{T}} + \sigma_2^2\hat{\mathbf{s}}_3\hat{\mathbf{s}}_3^{\mathrm{T}}) \tag{23b}$$

$$E\{\Delta\hat{\mathbf{s}}_3\Delta\hat{\mathbf{s}}_3^{\mathrm{T}}\} = \frac{1}{|\mathbf{s}_2|^2}\{\sigma_1^2[|\mathbf{s}_2|^2\hat{\mathbf{s}}_1\hat{\mathbf{s}}_1^{\mathrm{T}} + (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)^2\,\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}}] + \sigma_2^2\,\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}}\} \tag{23c}$$

The three equations (23) are easily summed to yield

$$P_{AA} = \sigma_1^2\left\{I + \frac{1}{|\mathbf{s}_2|^2}[\hat{\mathbf{s}}_4\hat{\mathbf{s}}_4^{\mathrm{T}} + (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)^2\,\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}}]\right\} + \sigma_2^2\frac{1}{|\mathbf{s}_2|^2}(I - \hat{\mathbf{s}}_1\hat{\mathbf{s}}_1^{\mathrm{T}}) \tag{24}$$

Straightforward substitution of equation (24) into equation (13) leads to

$$P_{\theta\theta}^{\mathrm{TRIAD}} = \sigma_1^2\,\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}} + \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|^2}(\sigma_1^2\,\hat{\mathbf{W}}_2\hat{\mathbf{W}}_2^{\mathrm{T}} + \sigma_2^2\,\hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}) \tag{25a}$$

$$= \frac{1}{|\mathbf{s}_2|^2}[\sigma_1^2(\hat{\mathbf{W}}_2\hat{\mathbf{W}}_2^{\mathrm{T}} + \mathbf{s}_2\mathbf{s}_2^{\mathrm{T}}) + \sigma_2^2\,\hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}] \tag{25b}$$

[7]In equations (20), and subsequently in expressions containing noise terms or expectations, whole quantities, as opposed to their errors, should bear the superscript *true,* since the errors are measured from the true values. Such a practice, however, would overburden our notation, and we trust the reader to make the correct interpretation. Note also that all expressions for errors are, as usual, correct only to first order in the measurement noise errors. It is only in first order in $\Delta\hat{\mathbf{s}}_i$ that equations (20) are true.

We note in passing the equivalent expressions[8] from reference [2]

$$P_{\theta\theta}^{\mathrm{TRIAD}} = \left[(\sigma_1^2 + \sigma_2^2) \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|^2} - \sigma_1^2 \right] \hat{\mathbf{s}}_1 \hat{\mathbf{s}}_1^{\mathrm{T}} + \sigma_1^2 (\hat{\mathbf{s}}_2 \hat{\mathbf{s}}_2^{\mathrm{T}} + \hat{\mathbf{s}}_3 \hat{\mathbf{s}}_3^{\mathrm{T}}) - \sigma_1^2 \frac{(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|} (\hat{\mathbf{s}}_1 \hat{\mathbf{s}}_3^{\mathrm{T}} + \hat{\mathbf{s}}_3 \hat{\mathbf{s}}_1^{\mathrm{T}}) \tag{26a}$$

$$= \sigma_1^2 I + \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|^2} [(\sigma_2^2 - \sigma_1^2) \hat{\mathbf{W}}_1 \hat{\mathbf{W}}_1^{\mathrm{T}} + \sigma_1^2 (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)(\hat{\mathbf{W}}_1 \hat{\mathbf{W}}_1^{\mathrm{T}} + \hat{\mathbf{W}}_2 \hat{\mathbf{W}}_1^{\mathrm{T}})] \tag{26b}$$

Equation (26a) may be simplified to

$$P_{\theta\theta}^{\mathrm{TRIAD}} = \sigma_1^2 I + (\sigma_1^2 + \sigma_2^2) \frac{1}{|\mathbf{s}_2|^2} \hat{\mathbf{s}}_1 \hat{\mathbf{s}}_1^{\mathrm{T}} - \sigma_1^2 \frac{1}{|\mathbf{s}_2|} (\hat{\mathbf{s}}_1 \hat{\mathbf{s}}_4^{\mathrm{T}} + \hat{\mathbf{s}}_4 \hat{\mathbf{s}}_1^{\mathrm{T}}) \tag{27}$$

Clearly equation (25b) is the most efficient form.

Of interest also is the inverse attitude-error covariance matrix

$$(P_{\theta\theta}^{\mathrm{TRIAD}})^{-1} = \frac{1}{\sigma_1^2} (I - \hat{\mathbf{W}}_1 \hat{\mathbf{W}}_1^{\mathrm{T}}) + \frac{1}{\sigma_2^2} \hat{\mathbf{s}}_4 \hat{\mathbf{s}}_4^{\mathrm{T}} \tag{28}$$

which can be verified most easily by multiplying this expression with that in equation (25a), (25b), (26b), or (27).

The Anti-QUEST

The Magsat spacecraft, launched on October 30, 1979, was equipped with three fine-attitude sensors: two Ball Brothers CT-401 fixed-head star trackers (FHST1 and FHST2) and an Adcole fine Sun senor (FSS). We shall label these sensors with the indices 1, 2, and 3, respectively. The accuracies (1σ) of the three sensors were determined inflight [8] to be 9.2, 8.0, and 11.2 arc seconds per axis, respectively, with statistical error levels (in the estimated accuracies) on the order of 1 arc second. These error levels were consistent with those specified by the vendors before launch.[9] The respective boresight vectors, with respect to Magsat body coordinates, were approximately

$$\hat{\mathbf{U}}_1 = \begin{bmatrix} \sqrt{3/8} \\ \sqrt{3/8} \\ 1/2 \end{bmatrix}, \qquad \hat{\mathbf{U}}_2 = \begin{bmatrix} -\sqrt{3/8} \\ \sqrt{3/8} \\ 1/2 \end{bmatrix}, \qquad \hat{\mathbf{U}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{29}$$

Thus, the boresights of any two fine-attitude sensors are separated in angle by at least 60 deg.

The QUEST algorithm [1–3] was able to handle all three measurements simultaneously. The TRIAD algorithm, which had been NASA's tried-and-true algorithm up to the Magsat launch, could handle only two vectors at a time. Thus, when data

[8]Note that equation (23) in reference [2] contains some typographical errors. All $\mathbf{s}_i$ in that equation should be $\hat{\mathbf{s}}_i$.

[9]The inflight estimation of the sensor error levels was made because errors in the earliest estimated attitudes were outside specification. After the study of reference [8] was reported, the problem was traced to a software coding error.

from all three fine-attitude sensors were available simultaneously, it was proposed as a back-up algorithm with the following steps:

- The TRIAD algorithm would be used to compute the attitude matrices $A(a)$, $A(b)$ and $A(c)$ for each of the three possible pairs of sensors, denoted by a, b and c. (Note: a, b, c denote measurement pairs, not individual direction measurements.)
- Each of the attitude matrices would be converted to an appropriate set of Euler angles $\{\varphi(a), \vartheta(a), \psi(a)\}$, $\{\varphi(b), \vartheta(b), \psi(b)\}$, and $\{\varphi(c), \vartheta(c), \psi(c)\}$.
- The Euler angles of the estimated attitude would be the average of the Euler angles sets for each measurement pair.

$$\chi^{\text{average}} = (\chi(a) + \chi(b) + \chi(c))/3 \tag{30}$$

  where χ is either φ, ϑ or ψ.
- These averaged Euler angles would be used, finally, to generate the Anti-QUEST attitude matrix

$$A^{\text{Anti-QUEST}} = \mathrm{A}(\varphi^{\text{average}}, \vartheta^{\text{average}}, \psi^{\text{average}}) \tag{31}$$

This is the algorithm we call here The Anti-QUEST.[10]

It is certainly not the intent of the authors to offer this algorithm as a candidate replacement for QUEST. Too much would be lost by that substitution, however enticing it might be to return to the "Bronze Age" of NASA Attitude Support. Nonetheless, it would be interesting to know how it would have performed compared to QUEST. Thus, the main part of this work will be taken up in developing expressions from which the Anti-QUEST attitude-error covariance matrix can be calculated.

Let us note first that there is no need in our forthcoming analysis of the Anti-QUEST attitude-error covariance matrix to duplicate exactly the steps of The Anti-QUEST. Since our attitude-error covariance matrix is defined with respect to true body axes, we may refer the (asymmetric set of) Euler angles also to these, so that the true value of each of the Euler angles will be zero.[11] (Equivalently, we may choose $A^{\text{true}} = I$.) Then for each pair of vectors, the errors in the corresponding 1-2-3 Euler angles will be simply the three components of the attitude-error vector for that pair, and their average will be simply the Anti-QUEST attitude-error vector. All the same, for the development of this work we will retain A in our mathematical expressions for clarity. However, because the attitude-error covariance matrix, as we have defined it, will not depend on the attitude, any "virtual" dependence on the attitude cannot be explicit in the final expressions, which will depend only on body-referenced direction measurements.

Given that the measurements are all of about the same accuracy, we do not expect a great difference in the performance of The Anti-QUEST for different choices of which is the first measurement vector in each measurement pair. Therefore, we will choose our three *ordered* pairs of measurements to be

$$a \equiv (1,2), \qquad b \equiv (2,3), \qquad c \equiv (3,1) \tag{32abc}$$

[10]Officially, the attitude output was the three Anti-QUEST Euler angles (and, perhaps, the three sets of Euler angles for each pair). Later in the mission, after people became comfortable with quaternions, the attitudes on the attitude output tapes were expressed as the four components of the quaternion in integer format.

[11]In actual mission support, of course, one does not know the directions of the true body axes and must calculate "whole" Euler angles.

again with literal rather than numerical labels to avoid confusion of the measurement pairs with the measurements. In each pair of equations (32abc) the first index is that of the measurement vector chosen to be $\hat{\mathbf{s}}_1$. In each pair the measurements have been chosen to be in cyclic order, and the three pairs themselves are also cyclically ordered, which will lead to further simplification of our task in the sequel. We could equally well have chosen the anticyclic pairs

$$a' \equiv (2,1), \qquad b' \equiv (3,2), \qquad c' \equiv (1,3) \tag{32def}$$

or any combination of cyclic and anticyclic measurement pairs can be used provided they correspond to the same set of three *unordered* pairs. We will have cause to make use of this greater generality in a later section.

The expressions in equations (22) through (28) have a hybrid appearance given the simultaneous use of s-vectors and W-vectors. We can impose a more uniform style by defining $\mathbf{s}_5 \equiv \hat{\mathbf{W}}_2$, i.e., $\hat{\mathbf{s}}_5$ is the second direction in an (ordered) measurement pair. Then we can always make the substitutions in these equations: $\hat{\mathbf{W}}_1 \to \hat{\mathbf{s}}_1$, $\hat{\mathbf{W}}_2 \to \hat{\mathbf{s}}_5$, $\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2 \to \mathbf{s}_2$, and $\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2 \to \hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_5$. Such a formula will be particularly advantageous when a measurement pair is not the canonical $(\hat{\mathbf{W}}_1, \hat{\mathbf{W}}_2)$, even if it is simplest to derive an expression originally with these specific numerical values for the indices.

The Anti-QUEST Covariance Matrix

The relationship connecting P_{AA} and $P_{\theta\theta}$ that was useful for the analysis of the TRIAD algorithm also holds with minor modification for The Anti-QUEST. We write for the error in the Anti-QUEST attitude matrix

$$\Delta A = \frac{1}{3}[\Delta A(a) + \Delta A(b) + \Delta A(c)] \tag{33}$$

and

$$P_{AA} = E\{\Delta A \Delta A^{\mathrm{T}}\} = \frac{1}{9}\sum_{\alpha,\beta} E\{\Delta A(\alpha)\Delta A^{\mathrm{T}}(\beta)\} = \frac{1}{9}\sum_{\alpha,\beta} P_{AA}(\alpha,\beta) \tag{34}$$

The Greek indices α and β range over the three pairs a, b and c. Clearly, $P_{AA}(\alpha,\alpha)$ is just the TRIAD result from a previous section with the proper choice of indices for the measurement vectors, and

$$P_{AA}(\alpha,\beta) \equiv E\{\Delta A(\alpha)\Delta A^{\mathrm{T}}(\beta)\} = P_{AA}^{\mathrm{T}}(\beta,\alpha) \tag{35}$$

Note that

$$P_{AA}(\alpha,\beta) = (\mathrm{tr} P_{\theta\theta}(\beta,\alpha))I - P_{\theta\theta}(\beta,\alpha) \tag{36}$$

with a transposition of arguments, and likewise

$$P_{\theta\theta}(\alpha,\beta) = \frac{1}{2}(\mathrm{tr} P_{AA}(\beta,\alpha))I - P_{AA}(\beta,\alpha) \tag{37}$$

We are led, therefore, to examine

$$P_{AA}(\alpha,\beta) = E\{\Delta A(\alpha)\Delta A^{\mathrm{T}}(\beta)\} = E\{\Delta S(\alpha)R^{\mathrm{T}}(\alpha)R(\beta)\Delta S^{\mathrm{T}}(\beta)\} \tag{38}$$

Evaluation of the expectation is hampered by the presence of $R^{\mathrm{T}}(\alpha)R(\beta)$ in the center of the expression. For the TRIAD case, this expression was $R^{\mathrm{T}}(a)R(a)$ which was equal to the identity matrix and, therefore, disappeared.

To eliminate the presence of the reference orthogonal matrices, we note from the orthogonality of the attitude matrix that

$$\Delta A = -A^{\text{true}}(\Delta A)^{\text{T}}A^{\text{true}} \tag{39a}$$

$$\Delta A(\alpha) = -A^{\text{true}}(\alpha)(\Delta A(\alpha))^{\text{T}}A^{\text{true}}(\alpha) \tag{39b}$$

Substituting these expressions into equations (12) and (34), respectively, yields

$$P_{AA} = A^{\text{true}}E\{\Delta A^{\text{T}}\Delta A\}A^{\text{true}\,\text{T}} \tag{40a}$$

$$P_{AA}(\alpha,\beta) = A^{\text{true}}(\alpha)E\{\Delta A^{\text{T}}(\alpha)\Delta A(\beta)\}A^{\text{true}\,\text{T}}(\beta) \tag{40b}$$

and further substituting equations (4) and (6) leads to

$$P_{AA} = S^{\text{true}}E\{\Delta S^{\text{T}}\Delta S\}S^{\text{true}\,\text{T}} \tag{41a}$$

$$P_{AA}(\alpha,\beta) = S^{\text{true}}(\alpha)E\{\Delta S^{\text{T}}(\alpha)\Delta S(\beta)\}S^{\text{true}\,\text{T}}(\beta) \tag{41b}$$

where we have used[12]

$$A^{\text{true}} = S^{\text{true}}(a)R^{\text{true}\,\text{T}}(a) = S^{\text{true}}(b)R^{\text{true}\,\text{T}}(b) \tag{42}$$

Our task now becomes the evaluation of

$$M(\alpha,\beta) \equiv E\{\Delta S^{\text{T}}(\alpha)\Delta S(\beta)\} \tag{43}$$

whence

$$P_{AA}(\alpha,\beta) = S(\alpha)M(\alpha,\beta)S^{\text{T}}(\beta) \tag{44}$$

(Henceforth, when no confusion can arise, we no longer write the superscript "true" on whole quantities.) Clearly

$$M(\beta,\alpha) = M^{\text{T}}(\alpha,\beta) \tag{45}$$

Since we have conveniently chosen the order of vector indices in each measurement pair to be in cyclic order and have defined the pairs a, b and c to be in cyclic order as well, we need develop expressions only for $M(a,a)$ and $M(a,b)$. The expressions for the remaining seven terms can then be obtained from these by cyclic permutation of the pairs and indices and matrix transposition. In fact, it is sufficient to develop expressions for the corresponding $P_{\theta\theta}(a,a)$ and $P_{\theta\theta}(b,a)$ in terms of the $\hat{\mathbf{W}}_i$, $i = 1,2,3$, and then apply the appropriate transformations to obtain the remaining seven components of $P_{\theta\theta}$. This makes our task simpler still, because $P_{\theta\theta}(a,a)$ is just the expression for attitude-error covariance matrix of the TRIAD algorithm presented in equation (25). We are left, therefore, with only one further expression to develop in order to obtain an expression for $P_{\theta\theta}$, namely, that for $M(a,b)$.

To compute $M(a,b)$ we note that the elements of this matrix are given by

$$[M(a,b)]_{ij} = E\{\Delta\hat{\mathbf{s}}_i^{\text{T}}(a)\Delta\hat{\mathbf{s}}_j(b)\} = \text{tr}(E\{\Delta\hat{\mathbf{s}}_j(b)\Delta\hat{\mathbf{s}}_i^{\text{T}}(a)\}) \tag{46}$$

The second expression will be more useful, because it places the measurement errors in immediate proximity. It follows immediately that

$$M_{11}(a,b) = M_{12}(a,b) = M_{13}(a,b) = M_{21}(a,b) = 0 \tag{47}$$

[12]Note that in general $S^{\text{true}}(\alpha) \neq S^{\text{true}}(\beta)$ for $\alpha \neq \beta$.

so that there are at most five nontrivial elements for which we must develop expressions. These are:

$$M_{22}(a,b) = -\frac{\sigma_2^2}{|\mathbf{s}_2(a)|\,|\mathbf{s}_2(b)|}(\hat{\mathbf{s}}_2(a)\cdot\hat{\mathbf{s}}_2(b))\,(\hat{\mathbf{s}}_3(a)\cdot\hat{\mathbf{s}}_4(b)) \tag{48a}$$

$$M_{23}(a,b) = -\frac{\sigma_2^2}{|\mathbf{s}_2(a)|\,|\mathbf{s}_2(b)|}(\hat{\mathbf{s}}_2(a)\cdot\hat{\mathbf{s}}_3(b))\,(\hat{\mathbf{s}}_3(a)\cdot\hat{\mathbf{s}}_4(b)) \tag{48b}$$

$$M_{31}(a,b) = -\frac{\sigma_2^2}{|\mathbf{s}_2(a)|} \tag{48c}$$

$$M_{32}(a,b) = \frac{\sigma_2^2}{|\mathbf{s}_2(a)|\,|\mathbf{s}_2(b)|}(\hat{\mathbf{s}}_2(a)\cdot\hat{\mathbf{s}}_2(b))\,(\hat{\mathbf{s}}_2(a)\cdot\hat{\mathbf{s}}_4(b)) \tag{48d}$$

$$M_{33}(a,b) = -\frac{\sigma_2^2}{|\mathbf{s}_2(a)|\,|\mathbf{s}_2(b)|}(\hat{\mathbf{s}}_5(a)\cdot\hat{\mathbf{s}}_5(b))|\hat{\mathbf{s}}_2(a)\times\hat{\mathbf{s}}_3(b)|^2 \tag{48e}$$

The reader must be cautioned against applying equation (45) naively. The specific form of the matrix elements of equations (47) and (48) depends on the fact that b is the cyclic follower of a. If one attempted to calculate $M_{ij}(b,a)$ simply by interchanging a and b, one would not obtain, necessarily, a correct result.

On the other hand, if one were to make the substitutions (b,c) or (c,a) for (a,b), the form of equations (48) would be maintained with the appropriate cyclic substitutions.

Had we chosen our measurement set to be the anticyclic pairs c', b' and a', then the elements of $M(\alpha,\beta)$ would have expressions very similar in form to those of equations (48) provided that the pairs are also in anticyclic order, that is, we calculate the values $M(b',a')$, $M(a',c')$ and $M(c',b')$.

From equations (47) and (48) we can calculate $P_{AA}(a,b)$ and by transposition and cyclic permutation of the measurement indices and by using the expression for the TRIAD attitude-error covariance matrix developed earlier the entire P_{AA} and thence, $P_{\theta\theta}$. It will be generally easier, however, to calculate first the portion of P_{AA} arising from the six cross terms alone, convert this to the corresponding portion of $P_{\theta\theta}$, and then add the TRIAD terms using equation (25b). A simple closed-form expression is likely an unattainable goal and certainly of limited applicability.

This complicated procedure should be compared with the simplicity of computing the attitude-error covariance matrix for the QUEST algorithm

$$P_{\theta\theta}^{\text{QUEST}} = \left[\sum_{i=1}^{3}\frac{1}{\sigma_i^2}(I - \hat{\mathbf{W}}_i\hat{\mathbf{W}}_i^{\mathrm{T}})\right]^{-1} \tag{49}$$

QUEST Versus The Anti-QUEST: Magsat, the First and Final Contest

We have computed the attitude-error covariance matrix for the QUEST and the Anti-QUEST algorithms for the Magsat mission assuming that the three measurement directions are along the sensor boresights and the measurement accuracies are as mentioned earlier with the results

$$P_{\theta\theta}^{\text{QUEST}} = \begin{bmatrix} 40.18 & -3.53 & -3.72 \\ -3.53 & 46.41 & 19.14 \\ -3.72 & 19.14 & 56.61 \end{bmatrix} \text{arcsec}^2 \tag{50a}$$

$$P_{\theta\theta}^{\text{Anti-QUEST}} = \begin{bmatrix} 50.70 & -12.58 & -4.86 \\ -12.58 & 54.46 & 19.42 \\ -4.86 & 19.42 & 65.21 \end{bmatrix} \text{arcsec}^2 \tag{50b}$$

The results are remarkably similar. The QUEST algorithm must do better, of course, because of the Cramér-Rao theorem [9]. Not a great deal of accuracy was gained for Magsat by using QUEST rather than The Anti-QUEST, only 20 percent in variance or 10 percent in standard deviation, but QUEST was certainly a far faster algorithm and easier to implement, as well as providing other extremely important benefits, such as the readily calculable figure of merit TASTE.

The Two Faces of the Anti-QUEST

In the above study of The Anti-QUEST, we arranged the data in pairs *a*, *b*, and *c*. However, if the variances satisfy $\sigma_1^2 \le \sigma_2^2 \le \sigma_3^2$, then a more accurate estimate would have been obtained using the three pairs *a*, *b* and *c′*. We will call The Anti-QUEST with this arrangement of the data *The Arch Anti-QUEST* (from the Classical Greek prefix ’άρχ-, which means *chief,* and the prototype version presented earlier in this paper *The Proto-Anti-QUEST* (from Classical Greek πρῶτος, -η, -ον, which means *first*).[13]

Because the accuracies of the Magsat fine attitude sensors were so close in value, there will be little difference in the value of the attitude-error covariance matrix for the two forms of The Anti-QUEST. We examine these two variations of The Anti-QUEST and compare them with QUEST for several general scenarios in the next section.

Some Simple Examples: The Attitude-Axis Measurement Model

Previous sections, while developing useful methods and providing a result of historical interest, have not provided much in the way of physical insight. We hope in the present section to remedy this deficiency.

Consider again a set of three direction measurements $\hat{\mathbf{W}}_1$, $\hat{\mathbf{W}}_2$ and $\hat{\mathbf{W}}_3$ with measurement errors described by the QUEST measurement model with respective variances σ_1^2, σ_2^2 and σ_3^2. Without loss of generality, we will assume that

$$\sigma_1^2 \le \sigma_2^2 \le \sigma_3^2 \tag{51}$$

To simplify the calculation of the attitude-error covariance matrices we will assume further that

$$\hat{\mathbf{W}}_1^{\text{true}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \equiv \hat{\mathbf{1}}, \quad \hat{\mathbf{W}}_2^{\text{true}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \equiv \hat{\mathbf{2}}, \quad \hat{\mathbf{W}}_3^{\text{true}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \equiv \hat{\mathbf{3}} \tag{52abc}$$

For this very simple configuration we can obviate all of the machinery of the previous section developed for the general case. We call the above model the *Attitude-Axis Measurement Model.* Its usefulness extends, obviously, beyond the present examples.

[13]It is interesting to note that the two Classical Greek roots ’άρχ- and πρῶτο- can each have the meaning first in time, as we see from the English words *Archaeology* and *prototype.* Note also Latin *principium* “beginning” and *princeps* “chief, prince.”

Since the attitude-error covariance matrix has been defined in such a way that it does not depend on the attitude, we may assume that the true attitude is null, i.e.

$$A^{\text{true}} = I \tag{53}$$

Thus, we have that

$$\hat{\mathbf{V}}_k = \hat{\mathbf{W}}_k^{\text{true}}, \quad \hat{\mathbf{r}}_k = \hat{\mathbf{s}}_k^{\text{true}}, \quad k = 1, 2, 3 \tag{54}$$

This and the special choice of the $\hat{\mathbf{W}}_k^{\text{true}}$ make it easy to calculate the error in the attitude matrices $\Delta A(\alpha)$ directly from

$$\Delta A(\alpha) = (\Delta\hat{\mathbf{s}}_1(\alpha))\hat{\mathbf{r}}_1^{\mathrm{T}}(\alpha) + (\Delta\hat{\mathbf{s}}_2(\alpha))\hat{\mathbf{r}}_2^{\mathrm{T}}(\alpha) + (\Delta\hat{\mathbf{s}}_3(\alpha))\hat{\mathbf{r}}_3^{\mathrm{T}}(\alpha) \tag{55}$$

One establishes readily that

$$\Delta\hat{\mathbf{s}}_1(a) = (\Delta\hat{\mathbf{W}}_1)_2\,\hat{\mathbf{2}} + (\Delta\hat{\mathbf{W}}_1)_3\,\hat{\mathbf{3}} \tag{56a}$$

$$\Delta\hat{\mathbf{s}}_2(a) = -(\Delta\hat{\mathbf{W}}_1)_3\,\hat{\mathbf{1}} - (\Delta\hat{\mathbf{W}}_2)_3\,\hat{\mathbf{2}} \tag{56b}$$

$$\Delta\hat{\mathbf{s}}_3(a) = -(\Delta\hat{\mathbf{W}}_1)_2\,\hat{\mathbf{1}} + (\Delta\hat{\mathbf{W}}_2)_3\,\hat{\mathbf{3}} \tag{56c}$$

where $(\Delta\hat{\mathbf{W}}_1)_2$ denotes the second component of $\Delta\hat{\mathbf{W}}_1$, and similarly for the other terms. Evaluating equation (55) yields

$$\begin{aligned}\Delta A(a) &= -(\Delta\hat{\mathbf{W}}_2)_3\,[[\hat{\mathbf{1}}]] + (\Delta\hat{\mathbf{W}}_1)_3\,[[\hat{\mathbf{2}}]] - (\Delta\hat{\mathbf{W}}_1)_2\,[[\hat{\mathbf{3}}]] \\ &= \begin{bmatrix} 0 & -(\Delta\hat{\mathbf{W}}_1)_2 & -(\Delta\hat{\mathbf{W}}_1)_3 \\ (\Delta\hat{\mathbf{W}}_1)_2 & 0 & -(\Delta\hat{\mathbf{W}}_2)_3 \\ (\Delta\hat{\mathbf{W}}_1)_3 & (\Delta\hat{\mathbf{W}}_2)_3 & 0 \end{bmatrix}\end{aligned} \tag{57}$$

from which we can read the rotation angles to obtain

$$\Delta\boldsymbol{\theta}(a) = [-(\Delta\hat{\mathbf{W}}_2)_3, (\Delta\hat{\mathbf{W}}_1)_3, -(\Delta\hat{\mathbf{W}}_1)_2]^{\mathrm{T}} \tag{58}$$

Proceeding in similar fashion for $\Delta A(b)$ and $\Delta A(c)$ leads to

$$\Delta\boldsymbol{\theta}^{\text{Proto-Anti-QUEST}} = \frac{1}{3}\begin{bmatrix} (\Delta\hat{\mathbf{W}}_3)_2 - 2(\Delta\hat{\mathbf{W}}_2)_3 \\ (\Delta\hat{\mathbf{W}}_1)_3 - 2(\Delta\hat{\mathbf{W}}_3)_1 \\ (\Delta\hat{\mathbf{W}}_2)_1 - 2(\Delta\hat{\mathbf{W}}_1)_2 \end{bmatrix} \tag{59}$$

and

$$P_{\theta\theta}^{\text{Proto-Anti-QUEST}} = \frac{1}{9}\,\mathrm{diag}[\sigma_3^2 + 4\sigma_2^2,\ \sigma_1^2 + 4\sigma_3^2,\ \sigma_2^2 + 4\sigma_1^2] \tag{60}$$

where

$$\mathrm{diag}[a_1, a_2, a_3] \equiv \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{61}$$

By a similar path (after calculating $\Delta A(c')$) we find

$$P_{\theta\theta}^{\text{Arch-Anti-QUEST}} = \frac{1}{9}\,\mathrm{diag}[\sigma_3^2 + 4\sigma_2^2,\ 4\sigma_1^2 + \sigma_3^2,\ \sigma_2^2 + 4\sigma_1^2] \tag{62}$$

Note that $P_{\theta\theta}^{\text{Arch-Anti-QUEST}}$ differs from $P_{\theta\theta}^{\text{Proto-Anti-QUEST}}$ only in the $(2, 2)$ element. From equation (49)

$$P_{\theta\theta}^{\text{QUEST}} = \mathrm{diag}\left[\left(\frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}\right)^{-1}, \left(\frac{1}{\sigma_3^2} + \frac{1}{\sigma_1^2}\right)^{-1}, \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}\right] \tag{63}$$

Case 1: Equally Accurate Measurements

Consider now the special case that $\sigma_1^2 = \sigma_2^2 = \sigma_2^2 = \sigma^2$. We have immediately that

$$P_{\theta\theta}^{\text{Proto-Anti-QUEST}} = P_{\theta\theta}^{\text{Arch-Anti-QUEST}} = \frac{5}{9}\sigma^2 I \tag{64a}$$

$$P_{\theta\theta}^{\text{QUEST}} = \frac{1}{2}\sigma^2 I \tag{64b}$$

Both versions of The Anti-QUEST are only ten percent less accurate in variance, which is similar to our more realistic example above. The Anti-QUEST does so well in this case because the perpendicularity of the measurements attenuates the ill-effects of the correlations.

Case 2: Two Levels of Accuracy

Sensors today fall into two broad classes: (a) coarse sensors with accuracies around 0.5 deg; and (b) fine sensors with accuracies around a few arc seconds. One can find poorer sensors, such as the Earth albedo sensor on the Thrusted-Vector Mission, which was made to do double duty as an attitude sensor [10] and had an effective error level as an attitude sensor of about 7 deg, and more accurate sensors, like the fine guidance sensors of the Hubble Space Telescope which have a precision of about one thousandth of an arc second. The accuracy of a fine guidance sensor as an attitude sensor, however, is limited to about 0.3 arc second, because that is the accuracy of star directions in star catalogs. For this reason, our examples treat only the cases of sensors with a common level of accuracy (Case 1 above) and sensors with one of two levels of accuracy. We do not consider the case that the levels of accuracy of all three sensors may be qualitatively different.

Thus, the more realistic cases are Case 2-A with

$$\sigma_1^2 = \sigma_2^2 = \sigma^2 \quad \text{and} \quad \sigma_3^2 = \lambda\sigma^2 \tag{65a}$$

and Case 2-B with

$$\sigma_1^2 = \sigma^2 \quad \text{and} \quad \sigma_2^2 = \sigma_3^2 = \lambda\sigma^2 \tag{65b}$$

Given the current sensor technology, $\lambda \approx 300{,}000$.

Case 2-A: Two Fine, One Coarse Sensor

When there are two accurate direction measurements, the results for the three attitude-error covariance matrices are

$$P_{\theta\theta}^{\text{Proto-Anti-QUEST}} = \frac{\sigma^2}{9}\text{diag}[4 + \lambda, 1 + 4\lambda, 5] \tag{66a}$$

$$P_{\theta\theta}^{\text{Arch-Anti-QUEST}} = \frac{\sigma^2}{9}\text{diag}[4 + \lambda, 4 + \lambda, 5] \tag{66b}$$

$$P_{\theta\theta}^{\text{QUEST}} = \sigma^2\text{diag}\left[\frac{\lambda}{1+\lambda}, \frac{\lambda}{1+\lambda}, \frac{1}{2}\right] \tag{66c}$$

In this case, we have once again that

$$P_{\theta\theta}^{\text{QUEST}} \leq P_{\theta\theta}^{\text{Arch-Anti-QUEST}} \leq P_{\theta\theta}^{\text{Proto-Anti-QUEST}} \tag{67}$$

consistent with our naive expectation and the Cremér-Rao lower bound [9]. Two of the attitude-error variances for the Anti-QUEST algorithms can become very large as λ increases.[14] Only the attitude error of the Anti-QUEST algorithms about one axis is bounded as λ increases without bound.

Case 2-B: One Fine, Two Coarse Sensors

When there is only one accurate direction measurement, the results are

$$P_{\theta\theta}^{\text{Proto-Anti-QUEST}} = \frac{\sigma^2}{9}\text{diag}[5\lambda, 1 + 4\lambda, 4 + \lambda] \tag{68a}$$

$$P_{\theta\theta}^{\text{Arch-Anti-QUEST}} = \frac{\sigma^2}{9}\text{diag}[5\lambda, 4 + \lambda, 4 + \lambda] \tag{68b}$$

$$P_{\theta\theta}^{\text{QUEST}} = \sigma^2\text{diag}\left[\frac{\lambda}{2}, \frac{\lambda}{1 + \lambda}, \frac{\lambda}{1 + \lambda}\right] \tag{68c}$$

In this case, all three attitude-error variance of the Anti-QUEST algorithms are unbounded and only one attitude-error variance of the QUEST algorithm is unbounded as λ increases without bound. The relative accuracy of the Arch-Anti-QUEST and Proto-Anti-QUEST algorithms is again what one would expect naively.

Discussion

As we have seen, in very auspicious cases when the three measurements are well separated in angle, the accuracies are all approximately at the same level, and there are only three measured directions, the estimate error levels of the Anti-QUEST attitude are not much greater than those of the QUEST algorithm. In fact, in our attitude-axis measurement model, the Anti-QUEST algorithm captures ninety percent of the accuracy improvement (in standard deviation) of the QUEST algorithm (with three measurements) above the TRIAD algorithm with only two measurements! However, when the attitude system provides measurements of widely differing accuracies, the deficiencies of The Anti-QUEST become obvious.

The case where there are three measurements only is common nowadays only for low-accuracy missions, in which the attitude sensor configuration may consist of a coarse vector Sun sensor, an infra-red Earth horizon scanner, and a three-axis magnetometer. When the spacecraft is equipped with a star tracker, then there may be as many as 25 measured directions. Would anyone in his right mind construct an Anti-QUEST-type algorithm with $n(n - 1)/2$ pairs for $n \approx 25$? Fortunately, if n is an even number, one really need consider only $n/2$ pairs to achieve attitude-error covariance matrices which will have approximately the same magnitude. But even three pairs are too many. Trade-offs between QUEST and The Anti-QUEST might have received serious consideration twenty-five years ago (they did not from the second author of this work), when attitude estimation was still a primitive art. We have come a long way since that time, and such trade-offs no longer merit our serious consideration.

It is too easy to forget that the great value of QUEST for Magsat lay not in its lightning speed but in its capacity to recognize outliers immediately using the TASTE test [4]. This led to a great streamlining of ground attitude operations for Magsat,

[14] An interesting occurrence in the case where there are at least two very different levels of accuracy of the three attitude sensors is that for λ sufficiently large, the Anti-QUEST variances can be larger (!) than the corresponding variances for the TRIAD algorithm with only two of the measurements.

for which data editing would otherwise have been the most time-consuming segment. In fact, NASA had half expected the data processing with "hands-on" data validation and editing to take as long as two days for each day of attitude data, which would have meant nearly a year-and-a-half of data processing for the eight months of Magsat data. As it turned out, with QUEST's automatic data validation and the consequent avoidance of frequent analyst intervention in the data validation and editing, only four hours were required for one day of attitude data. Once fine-attitude data processing was well underway [4], the definitive attitude tapes were ready for the scientists in nearly real time.

The computation time needed by QUEST for the attitude is of limited importance nowadays when every student has a computer which is more than 1000 times faster and more capacious than NASA mainframes of twenty-five years ago. Nonetheless, explorers of new solutions to the Wahba problem [11] still use speed as the chief figure of merit.[15]

Requiescat in pace Anti-QUEST.

Acknowledgment

The first author wishes to thank John L. Crassidis and the Department of Mechanical and Aerospace Engineering of the University at Buffalo, State University of New York, for their hospitality and financial support while this work was being performed. Both authors wish to thank F. Landis Markley of NASA Goddard Space Flight Center and one of the anonymous reviewers of this paper for helpful comments.

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[15]Even so, almost a quarter century after the publication of the QUEST algorithm, the fastest competitor of QUEST claims to be only ten percent faster [11]. However, we find in the majority of cases it is actually QUEST which is faster. It seems that there is no absolutely fastest solution to the Wahba problem.