The Journal of the Astronautical Sciences, Vol. 52, No. 3, July–September 2004, pp. 405–419

Deterministic Three-Axis Attitude Determination

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Abstract

The general problem of determining the attitude deterministically, that is, directly without the optimization of a cost function, from measurements of arc lengths and directions is examined. While there is no continuous degeneracy for the solutions to this problem, because effectively three data are given, nonetheless, the attitude solution still has generally a discrete degeneracy which can be removed only by the addition of further data. The only case escaping the discrete degeneracy has an over-determined solution. Specific algorithms are developed for all cases, and the nature of the degeneracy is explored in detail.

Introduction

It is customary to divide algorithms for estimating three-axis attitude into two classes. The first class uses a minimal set of data and then solves three possibly nonlinear equations to obtain the attitude. This class is generally referred to as "deterministic," a name which has been popularized by Wertz [1]. The other class of algorithms, generally referred to as "optimal," determines the attitude by minimizing an appropriate cost function and using more than a minimal set of measurements. Such algorithms are called for when more than three scalar measurements are processed to obtain a more accurate estimate of the attitude. Perhaps the best known deterministic algorithm in current use is the TRIAD algorithm [2, 3], in use since at least 1964 [2], while the best known optimal algorithm nowadays is certainly the QUEST algorithm [3], in frequent use for computing spacecraft attitude for Earth-bound spacecraft since 1979 and throughout the solar system for more than a decade [4].

The overwhelming prevalence of complete vector data for determining spacecraft attitude and the availability of the QUEST algorithm for more than two decades has largely obviated the need to calculate spacecraft attitude from anything but complete vector data. In fact, even the TRIAD algorithm is used infrequently nowadays, since the computational burden of the QUEST algorithm and many of its competitors is hardly much greater. An excellent review of all batch optional

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algorithms for computing quaternions using vector data has been given by Markley and Mortari [5].

Deterministic attitude-determination algorithms, although in infrequent use, still find application. An example was the need to determine three-axis attitude for the Oscar-30 spacecraft from the observed geomagnetic field vector from a three-axis magnetometer (TAM) and the observed angle between the geomagnetic field vector and the Sun line from a spinning digital solar aspect detector (SDSAD). One of the methods developed in this paper is, in fact, the one employed for the processing of Oscar-30 attitude data (see below).

With the exception of studies of the TRIAD algorithm [3], no systematic studies have been taken of deterministic three-axis attitude determination methods. The TRIAD algorithm is not truly deterministic in that it uses more than the minimal number of measurements for the construction of the direction-cosine matrix [6]. It turns out that truly deterministic algorithms do not, in general, lead to unambiguous results for the attitude. The attitude solution may admit two-fold, four-fold, and eight-fold discrete degeneracies in the attitude solution. This is the subject of the present work.

Nature of the Measurements

Static attitude measurements (as opposed to "dynamic" gyro measurements) are generally of arc lengths or dihedral angles. Thus, for example, the z-component of a magnetic field vector measurement, when divided by the magnitude of the magnetic field yields the direction-cosine of the z-axis of the magnetometer with the magnetic field vector, which is the cosine of an arc length [7, 8]. Attitude sensors may also measure a dihedral angle, that is, the angle between two planes. This is the case for a slit Sun sensor, which measures the angle between the sensor reticule plane and the plane determined by the Sun position and the sensor slit. In data processing, however, this information is generally combined with that from a perpendicular slit and transformed into a direction vector [7, 8]. Conical horizon scanners provide an arc length (the nadir angle) and a dihedral angle (between the scanner axis-Sun direction and the scanner axis-nadir direction planes). This information is used, however, to determine the nadir direction as an effective measurement [9] rather than directly in the estimation of the attitude. Thus, for practical purposes, static sensors, no matter what their type, generally furnish as effective measurements either arc lengths or entire directions (i.e., three arc lengths from a set of orthogonal axes). In the present work, we treat only static sensors, whose data, we assume, is simultaneous.

If $\hat{\mathbf{S}}$ is some direction fixed in the spacecraft, for example, the axis of a focal-plane sensor, and if $\hat{\mathbf{W}}$ is a direction of some object in space (or of a measurable vector field at the spacecraft, such as the geomagnetic field) coordinated in the spacecraft body frame, then the most basic scalar measurement is simply

$$d = \hat{\mathbf{S}} \cdot \hat{\mathbf{W}} + \Delta d \tag{1}$$

where d is the measurement, the cosine of an arc length, and Δd is the measurement noise. In general, we also know the representation of the partially observed direction in our reference coordinate system (typically geocentric inertial, which we will refer to in this work as the *space* frame). We write the space-referenced vector as $\hat{\mathbf{V}}$, and this is connected to the body-referenced vector $\hat{\mathbf{W}}$ by the direction-cosine matrix A [6]

$$\hat{\mathbf{W}} = A\hat{\mathbf{V}} \tag{2}$$

Thus, our basic measurement model is

$$d = \hat{\mathbf{S}}^{\mathrm{T}} A\hat{\mathbf{V}} + \Delta d \tag{3}$$

In general, we assume that Δd is normally distributed with mean zero and variance σ_d^2

$$\Delta d \sim \mathcal{N}(0, \sigma_d^2) \tag{4}$$

The measurement d is of a direction-cosine, from which we may unambiguously obtain the angle (i.e., arc length) between $\hat{\mathbf{W}}$ and $\hat{\mathbf{S}}$, if we restrict its value to lie in the interval $[0, \pi]$. For this reason we will refer to d henceforth as an arc length measurement.

For a vector sensor we are led to the effective measurement model for the direction measurement

$$\hat{\mathbf{W}} = A\hat{\mathbf{V}} + \Delta\hat{\mathbf{W}} \tag{5}$$

with (approximately)

$$E\{\Delta\hat{\mathbf{W}}\} = \mathbf{0} \tag{6a}$$

$$E\{\Delta\hat{\mathbf{W}}\Delta\hat{\mathbf{W}}^{\mathrm{T}}\} = R_{\hat{\mathbf{W}}} \tag{6b}$$

where $E\{\cdot\}$ denotes an expectation. $R_{\hat{\mathbf{W}}}$, the covariance matrix, must be singular because of the norm constraint on $\hat{\mathbf{W}}$ and satisfy[2]

$$\mathrm{R}_{\hat{\mathbf{W}}}\hat{\mathbf{W}} = \mathbf{0} \tag{7}$$

A useful approximate covariance matrix for direction measurements is the QUEST model [3, 10], which has been used for the covariance analysis of the QUEST and TRIAD algorithms [3]

$$R_{\hat{\mathbf{W}}} = \sigma_{\hat{\mathbf{W}}}^2 (I_{3\times 3} - \hat{\mathbf{W}}\hat{\mathbf{W}}^{\mathrm{T}}) \tag{8}$$

We expect this algorithm to be a particularly faithful representation of the effective direction-measurement error for a focal-plane sensor with a limited field of view. Henceforth, we will refer to $\hat{\mathbf{W}}$ as a direction measurement.

Since we are interested only in deterministic solutions, we will attempt to construct the direction-cosine matrix from the measurements under the assumption that we may neglect the measurement noise. We are led, therefore, to consider three measurement scenarios:

Scenario 1: Two Directions

We wish to determine the three-axis attitude given the two measurements

$$\hat{\mathbf{W}}_k = A\hat{\mathbf{V}}_k \qquad k = 1, 2 \tag{9}$$

Scenario 2: One Direction and One Arc Length

We wish to determine the three-axis attitude given the two measurements

$$\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1, \qquad d_2 = \hat{\mathbf{S}}_2^{\mathrm{T}} A\hat{\mathbf{V}}_2 \tag{10ab}$$

[2]This is true, of course, only in the limit that sensor field of view is infinitesimal. It is, however, a good approximation generally.

Scenario 3: Three Arc Lengths

We wish to determine the three-axis attitude given the three measurements

$$d_k = \hat{\mathbf{S}}_k^{\mathrm{T}} A \hat{\mathbf{V}}_k, \qquad k = 1, 2, 3 \tag{11}$$

These represent the minimum number of measurements of each type (directions only, directions and arc lengths, arc lengths only) for which the attitude solution will have at most a discrete degeneracy. We know already that Secnario 1 will not generally have solution because A is overdetermined, and a deterministic solution will require that some data be discarded, as we shall see below. For the other two scenarios a solution will indeed be possible if the d_k have values which are possible physically. However, the solution may not be unique. Note that when we compute the attitude covariance matrix, the noise terms must be added to the definition of the measurement vector and taken into account explicitly.

Three-Axis Attitude from Two Directions

Scenario 1 above is simply that of the TRIAD algorithm, whose derivation we shall repeat here, because it is very short and will be of value in later discussion.

We begin by constructing two dextral (right-handed orthonormal) triads of unit vectors from the observations and the reference vectors, namely

$$\hat{\mathbf{r}}_1 = \hat{\mathbf{V}}_1, \qquad \hat{\mathbf{r}}_2 = \frac{\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2}{|\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2|}, \qquad \hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2 \tag{12abc}$$

$$\hat{\mathbf{s}}_1 = \hat{\mathbf{W}}_1, \qquad \hat{\mathbf{s}}_2 = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|}, \qquad \hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \tag{12def}$$

In the absence of measurement noise, these ancillary vectors would satisfy

$$\hat{\mathbf{s}}_k = A \hat{\mathbf{r}}_k, \qquad k = 1, 2, 3 \tag{13}$$

or, equivalently

$$M_s = A\, M_r \tag{14}$$

with

$$M_s \equiv [\hat{\mathbf{s}}_1 \quad \hat{\mathbf{s}}_2 \quad \hat{\mathbf{s}}_3], \qquad M_r \equiv [\hat{\mathbf{r}}_1 \quad \hat{\mathbf{r}}_2 \quad \hat{\mathbf{r}}_3] \tag{15ab}$$

and the right members of equations (15) denote 3×3 matrices labeled by their columns. The matrices M_r and M_s are both proper orthogonal, because the two triads of column vectors are each dextral. Hence, we may solve equation (14) for A to obtain

$$A = M_s M_r^{-1} = M_s M_r^{\mathrm{T}} \tag{16}$$

This is the TRIAD algorithm [2, 3].

Although the attitude is overdetermined by the data, the TRIAD algorithm is deterministic, as opposed to the QUEST algorithm, which finds an attitude solution optimally from the same data. To understand the nature of the TRIAD algorithm, note that it is sufficient to find A which satisfies equation (13) for $k = 1$ and $k = 3$. By construction, the TRIAD attitude satisfies

$$\hat{\mathbf{W}}_1 = A \hat{\mathbf{V}}_1 \tag{17}$$

exactly. The equation for $k = 3$, however is equivalent to

$$\left(\frac{\hat{\mathbf{W}}_2 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{W}}_1}{|\hat{\mathbf{W}}_2 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{W}}_1|}\right) = A\left(\frac{\hat{\mathbf{V}}_2 - (\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)\hat{\mathbf{V}}_1}{|\hat{\mathbf{V}}_2 - (\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)\hat{\mathbf{V}}_1|}\right) \tag{18}$$

Thus, equivalently, one of the four equivalent scalar data is discarded by removing the component of $\hat{\mathbf{W}}_2$ which is in the direction of $\hat{\mathbf{W}}_1$, and similarly for $\hat{\mathbf{V}}_2$ and $\hat{\mathbf{V}}_1$, and readjusting the normalization of the two vectors to be unity. On the other hand, if we examine $\hat{\mathbf{r}}_2$ and $\hat{\mathbf{s}}_2$, which are also perpendicular to $\hat{\mathbf{V}}_1$ and $\hat{\mathbf{W}}_1$, respectfully, and regard the attitude construction as first rotating $\hat{\mathbf{V}}_1$ into alignment with $\hat{\mathbf{W}}_1$ and then rotating $\hat{\mathbf{r}}_2$ about $\hat{\mathbf{W}}_1$ until it is aligned with $\hat{\mathbf{s}}_2$, we see immediately that the only information needed for the second step is the angle between the $(\hat{\mathbf{W}}_1, \hat{\mathbf{r}}_2')$ plane and the $(\hat{\mathbf{W}}_1, \hat{\mathbf{s}}_2)$ plane. Here $\hat{\mathbf{r}}_2'$ is the resultant of the first rotation acting on $\hat{\mathbf{r}}_2$. This angle, clearly, is a dihedral angle. $\hat{\mathbf{W}}_1$ can be expressed unambiguously by three arc lengths or one arc length (the azimuthal angle, commonly denoted by θ) and one dihedral angle (the polar angle, commonly denoted by ϕ). Thus, of the six components of the two unit-vector measurements, the effective data set consists of two dihedral angles and one arc length. As we have said, however, the dihedral angle data is never processed as such.

Covariance Analysis of the TRIAD Algorithm

The attitude covariance matrix is defined as the covariance matrix of the attitude error vector, which is defined as the rotation vector of the very small rotation taking the true attitude into the estimated attitude. Thus, if A_{true} denotes the true attitude and A^* denotes the estimated attitude, then

$$A^* = C(\Delta\boldsymbol{\theta}^*)A_{\text{true}} \tag{19}$$

where

$$C(\boldsymbol{\theta}) = I_{3\times3} + \frac{\sin(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|}[\![\boldsymbol{\theta}]\!] + \frac{1 - \cos(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|^2}[\![\boldsymbol{\theta}]\!]^2 \tag{20}$$

is the formula for a proper orthogonal matrix parameterize by the rotation vector [6] and

$$[\![\boldsymbol{\theta}]\!] \equiv \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \tag{21}$$

Note that for $|\Delta\boldsymbol{\theta}^*| << 1$ we have that

$$C(\Delta\boldsymbol{\theta}^*) = I_{3\times3} + [\![\Delta\boldsymbol{\theta}^*]\!] + O(|\Delta\boldsymbol{\theta}^*|^2) \tag{22}$$

The attitude covariance matrix is defined as

$$P_{\theta\theta} \equiv E\{\Delta\boldsymbol{\theta}^*\Delta\boldsymbol{\theta}^{*\mathrm{T}}\} \tag{23}$$

The covariance matrix of the TRIAD algorithm has been computed in reference [3]. The result, assuming the QUEST Model, is stated most simply as[3]

$$(P_{\theta\theta}^{\text{TRIAD}})^{-1} = \frac{1}{\sigma^2_{\hat{\mathbf{W}}_1}}(I_{3\times3} - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}) + \frac{1}{\sigma^2_{\hat{\mathbf{W}}_2}}\hat{\mathbf{s}}_4\hat{\mathbf{s}}_4^{\mathrm{T}} \tag{24}$$

[3]Equation (24) is an unpublished result of Markley.

where

$$\hat{\mathbf{s}}_4 \equiv \hat{\mathbf{W}}_2 \times \hat{\mathbf{s}}_2 \tag{25}$$

We can write equation (24) equivalently as

$$(P_{\theta\theta}^{\mathrm{TRIAD}})^{-1} = \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2}(\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}} + \hat{\mathbf{s}}_3\hat{\mathbf{s}}_3^{\mathrm{T}}) + \frac{1}{\sigma_{\hat{\mathbf{W}}_2}^2}\hat{\mathbf{s}}_4\hat{\mathbf{s}}_4^{\mathrm{T}} \tag{26}$$

while the inverse covariance matrix (information matrix) of the QUEST algorithm is given by

$$\begin{aligned}(P_{\theta\theta}^{\mathrm{QUEST}})^{-1} &= \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2}(I_{3\times 3} - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}) + \frac{1}{\sigma_{\hat{\mathbf{W}}_2}^2}(I_{3\times 3} - \hat{\mathbf{W}}_2\hat{\mathbf{W}}_2^{\mathrm{T}}) \\ &= \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2}(\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}} + \hat{\mathbf{s}}_3\hat{\mathbf{s}}_3^{\mathrm{T}}) + \frac{1}{\sigma_{\hat{\mathbf{W}}_2}^2}(\hat{\mathbf{s}}_2\hat{\mathbf{s}}_2^{\mathrm{T}} + \hat{\mathbf{s}}_4\hat{\mathbf{s}}_4^{\mathrm{T}})\end{aligned} \tag{27}$$

Clearly, one axis' worth of information has been lost in constructing the TRIAD attitude solution.

Three-Axis Attitude from One Direction and One Arc Length

Consider now the set of measurements posed by equations (10). To solve for the attitude in this case we begin by seeking all direction-cosine matrices A which satisfy $\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1$. These are given by

$$A = R(\hat{\mathbf{W}}_1, \theta)A_o \tag{28}$$

where A_o is any direction-cosine matrix satisfying $\hat{\mathbf{W}}_1 = A_o\hat{\mathbf{V}}_1$, $R(\hat{\mathbf{W}}_1, \theta)$ is the rotation matrix for a rotation about the axis $\hat{\mathbf{W}}_1$ through an angle θ, and θ is any angle satisfying $0 \le \theta < 2\pi$. $R(\hat{\mathbf{W}}_1, \theta)$ is given by Euler's formula

$$R(\hat{\mathbf{n}}, \theta) = \cos\theta\, I_{3\times 3} + (1 - \cos\theta)\,\hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} + \sin\theta [\![\hat{\mathbf{n}}]\!] \tag{29}$$

with $[\![\hat{\mathbf{n}}]\!]$ defined in equation (21).

To prove the assertion of equation (28), assume that there exist two distinct direction-cosine matrices, A and A_o, satisfying $\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1$ and $\hat{\mathbf{W}}_1 = A_o\hat{\mathbf{V}}_1$, respectively. Then

$$\hat{\mathbf{W}}_1 = A(A_o^{-1}A_o)\hat{\mathbf{V}}_1 = (AA_o^{-1})A_o\hat{\mathbf{V}}_1 = (AA_o^{-1})\hat{\mathbf{W}}_1 \tag{30}$$

Thus, $\hat{\mathbf{W}}_1$ must be the axis of rotation of the rotation matrix AA_o^{-1}. Since AA_o^{-1} must be different from the identity matrix, the axis of rotation is well-defined and unique (within a sign). Hence

$$AA_o^{-1} = R(\hat{\mathbf{W}}_1, \theta) \tag{31}$$

for some angle θ, equation (28) follows. Every direction-cosine matrix given by equation (28) satisfies $\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1$. Therefore, there is a continuum of solutions satisfying this equation.

Equation (28) is equivalent to

$$A = A_oR(\hat{\mathbf{V}}_1, \theta) \tag{32}$$

with identical A_o and θ. This follows from [6]

$$A_oR(\hat{\mathbf{V}}_1, \theta) = A_oR(\hat{\mathbf{V}}_1, \theta)A_o^{\mathrm{T}}A_o = R(A_o\hat{\mathbf{V}}_1, \theta)A_o = R(\hat{\mathbf{W}}_1, \theta)A_o \tag{33}$$

Having found a candidate matrix A_o which satisfies equation (10a), we then determine the values of θ for which equation (10b) is also satisfied.

We must first find a single A_o which satisfies $\hat{\mathbf{W}}_1 = A_o\hat{\mathbf{V}}_1$. Let us look for an A_o of the form

$$A_o = R(\hat{\mathbf{n}}_o, \theta_o) \tag{34}$$

For the special case that $\hat{\mathbf{W}}_1 = \hat{\mathbf{V}}_1$, the choice of $\hat{\mathbf{n}}_o$ is arbitrary provided we choose $\theta_o = 0$. Likewise, for the special case that $\hat{\mathbf{W}}_1 = -\hat{\mathbf{V}}_1$, we may choose $\hat{\mathbf{n}}_o$ to be any direction perpendicular to $\hat{\mathbf{V}}_1$ and $\theta_o = \pi$. In all other cases, we may chose

$$\hat{\mathbf{n}}_o \equiv \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|} \tag{35}$$

Thus, in every case, we can chose $\hat{\mathbf{n}}_o$ to satisfy

$$\hat{\mathbf{n}}_o \cdot \hat{\mathbf{V}}_1 = \hat{\mathbf{n}}_o \cdot \hat{\mathbf{W}}_1 = 0 \tag{36}$$

a fact which will be useful later. Assuming equation (35) we obtain[4]

$$R(\hat{\mathbf{n}}_o, \theta_o)\hat{\mathbf{V}}_1 = \left[\cos\theta_o - \frac{(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}\sin\theta_o\right]\hat{\mathbf{V}}_1 + \frac{\sin\theta_o}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}\hat{\mathbf{W}}_1 = \hat{\mathbf{W}}_1 \tag{37}$$

For $\hat{\mathbf{W}}_1 \neq \pm\hat{\mathbf{V}}_1$, $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{V}}_1$ are linearly independent, and a unique solution exists for θ_o, namely

$$\theta_o = \arctan_2(|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|, (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)) \tag{38}$$

where $\arctan_2(y, x)$ is the function which computes the arc tangent of y/x and in the correct quadrant. This is just the familiar FORTRAN function ATAN2.

It is a simple matter to show that the corresponding quaternion is given by

$$\bar{q}_o = \sqrt{\frac{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}{2}}\begin{bmatrix}\left(\dfrac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}\right) \\ 1\end{bmatrix} \tag{39}$$

which can now be computed without the need to compute θ_o. The Rodrigues vector[5] $\boldsymbol{\rho}_o$ is given obviously by [6]

$$\boldsymbol{\rho}_o = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1} \tag{40}$$

and the matrix A_o is given equivalently by

$$A_o = (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)I_{3\times3} + \frac{(\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1)(\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1)^{\mathrm{T}}}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1} + [\![\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1]\!] \tag{41a}$$

$$= I_{3\times3} + [\![\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1]\!] + \frac{1}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}[\![\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1]\!]^2 \tag{41b}$$

Given A_o we must now compute θ. Define

$$\hat{\mathbf{W}}_3^* \equiv A_o\hat{\mathbf{V}}_2 \tag{42}$$

[4]Details of the derivation, if needed, can be found in reference [11].
[5]Also called the Gibbs vector.

Then θ is a solution of

$$\hat{\mathbf{S}}_2 \cdot R(\hat{\mathbf{W}}_1, \theta)\hat{\mathbf{W}}_3^* = d_2 \tag{43}$$

Substituting Euler's formula and rearranging terms leads to

$$B\cos(\theta - \beta) = (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3) - d_2 \tag{44}$$

with

$$B = |\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1|\,|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*| \tag{45a}$$

$$\beta \equiv \arctan_2(\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*), \hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*))) \tag{45b}$$

From equation (44) we see that a necessary condition that a solution exists is that

$$|(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^*) - d_2| \leq |\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1|\,|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*| \tag{46}$$

If this condition is satisfied, then θ has the solutions

$$\theta = \beta + \cos^{-1}\left[\frac{(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^*) - d_2}{|\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1|\,|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*|}\right] \tag{47}$$

and the inverse cosine is indeed two-valued. Given A_o and θ we can now construct the two direction-cosine matrix solutions according to equations (28) and (41). Other criteria must be brought to bear to choose the correct attitude solution. This is the algorithm that was developed in support of the Oscar-30 mission.

Covariance Analysis for the Case of One Direction and One Arc Length

The two direction-cosine matrices constructed by the above algorithm solve equations (10) exactly. Therefore, if attitude solutions exist, they each certainly minimize the negative-log-likelihood function

$$J(A) = \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2}|\hat{\mathbf{W}}_1 - A\hat{\mathbf{V}}_1|^2 + \frac{1}{\sigma_{d_2}^2}|\hat{\mathbf{S}}_2 \cdot A\hat{\mathbf{V}}_2 - d_2|^2 \tag{48}$$

where $\sigma_{\hat{\mathbf{W}}_1}$ and σ_{d_2} are standard deviations defined earlier.[6]

The calculation of the Fisher information is tedious but straightforward. The result for the attitude covariance matrix is

$$P_{\theta\theta}^{-1} = \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2}(I_{3\times3} - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^{\mathrm{T}}) + \frac{1}{\sigma_{d_2}^2}(\hat{\mathbf{W}}_3^* \times \hat{\mathbf{S}}_2)(\hat{\mathbf{W}}_3^* \times \hat{\mathbf{S}}_2)^{\mathrm{T}} \tag{49}$$

Note that generally

$$\hat{\mathbf{W}}_3^* \neq \hat{\mathbf{S}}_2 \tag{50}$$

even in the absence of measurement noise. Note also that $P_{\theta\theta}$ will not exist unless

$$\hat{\mathbf{W}}_1 \cdot (\hat{\mathbf{W}}_3^* \times \hat{\mathbf{S}}_2) \neq 0 \tag{51}$$

or, equivalently, unless

$$\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times (A\hat{\mathbf{V}}_2)) = (A\hat{\mathbf{V}}_2) \cdot (\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1) \neq 0 \tag{52}$$

[6]If a deterministic attitude solution constructed according to the above methodology does not exist (say, because equation (46) is not satisfied due to the effect of measurement noise) then one can at least find an optimal attitude solution which minimizes a cost function. This is one of the advantages of an optimal algorithm.

Even though the direction-cosine matrix may be defined in this case the geometry represents an extremum situation in which the sensitivity of the attitude to the measurements vanishes along one direction in parameter space.

Note that although the covariance matrix may be small, the actual uncertainty in the attitude can be large because of the finite degeneracy. Thus, given only a single direction and a single arc length as measurements, then with probability roughly 2/3 (the one-sigma probability) we know only that the attitude (in terms of, say, the rotation vector) will lie in the interior of either of two error ellipsoids whose axes are determined from the attitude covariances matrices calculated above and whose centers lie at the two possible estimates of the attitude.

A TRIAD-Like Algorithm from One Direction and One Arc Length

Instead of first calculating the direction-cosine matrix from the data and then determining a vector $\hat{\mathbf{W}}_3^*$ which satisfies equation (42), we might try instead to calculate this $\hat{\mathbf{W}}_3^*$ directly, without first determining the attitude, and, once this vector has been determined, calculate A using the TRIAD algorithm [3].

To compute $\hat{\mathbf{W}}_3^*$ we write

$$\hat{\mathbf{W}}_3^* = a\hat{\mathbf{W}}_1 + b\hat{\mathbf{S}}_2 + c\frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|} \tag{53}$$

which is possible provided that $\hat{\mathbf{W}}_1 \neq \pm\hat{\mathbf{S}}_2$. It then follows that

$$\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^* = a + b(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) = \hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2 \tag{54a}$$

$$\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_3^* = a(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b = d_2 \tag{54b}$$

$$\hat{\mathbf{W}}_3^* \cdot \hat{\mathbf{W}}_3^* = a^2 + 2ab(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b^2 + c^2 = 1 \tag{54c}$$

The solution for a and b is immediate and is given by

$$a = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|^2}((\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2) - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2)d_2) \tag{55a}$$

$$b = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|^2}(d_2 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)) \tag{55b}$$

The solution for c is then given by

$$c = \pm\sqrt{1 - (a^2 + 2ab(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b^2)} \tag{56}$$

This last calculation can be simplified by noting that

$$a^2 + 2ab(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b^2$$
$$= \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|^2}[d_2^2 - 2d_2(\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + (\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)^2] \tag{57}$$

The lack of a unique solution is now obvious from equation (56). Although the TRIAD algorithm [3] can now be used to calculate the attitude from the four vectors $\hat{\mathbf{V}}_1$, $\hat{\mathbf{V}}_2$, $\hat{\mathbf{W}}_1$, and $\hat{\mathbf{W}}_3^*$, the measured unit vectors are no longer uncorrelated and the attitude covariance matrix is still that computed earlier (equation (49)).

While the present algorithm is clearly more efficient than that developed above, it also suffers from some numerical problems. In particular, in extreme cases measurement error may cause the argument of the square root in equation (56) to be negative.

Three-Axis Attitude from Three Arc Lengths

We now seek a direction cosine matrix which satisfies equations (11). In general, there will be an eight-fold degeneracy for this problem. To see this, let us define

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{58}$$

and consider the special case

$$\hat{\mathbf{u}}_k^{\mathrm{T}} A \hat{\mathbf{u}}_k = d_k \qquad k = 1, 2, 3 \tag{59}$$

Substituting equation (29) into this equation leads to three equations for the axis and angle of rotation

$$\cos\theta + (1 - \cos\theta) n_k^2 = d_k \qquad k = 1, 2, 3 \tag{60}$$

where n_k denotes the n-th component of $\hat{\mathbf{n}}$. This leads immediately to the solution for θ

$$\theta = \cos^{-1}\left(\frac{1}{2}\left[\sum_{k=1}^{3} d_k - 1\right]\right) \tag{61}$$

The inverse cosine is two-valued, but only the principal value need be taken, since θ can be restricted without loss of generality to the interval $0 \leq \theta \leq \pi$ assuming $-1 < d_k < 1$. Solving for the components of $\hat{\mathbf{n}}$ then leads to

$$n_k = \pm\sqrt{\frac{d_k - \cos\theta}{1 - \cos\theta}} \tag{62}$$

revealing the sign ambiguity in each component of $\hat{\mathbf{n}}$. The attitude computed from three arc lengths, therefore, will generally display an eight-fold ambiguity.

To construct the attitude we must distinguish two cases: (1) that two of the $\hat{\mathbf{V}}_k$, $k = 1, 2, 3$, are identical, and (2) that the three are distinct. The case that all three $\hat{\mathbf{V}}_k$ are identical may be excluded as that case is equivalent to knowing only the direction cosines of a single unit vector, which leads to a continuous degeneracy in the attitude solution.

Case 1: Two Reference Vectors Identical[7]

If, say, $\hat{\mathbf{V}}_1 = \hat{\mathbf{V}}_2$, then we can construct $\hat{\mathbf{W}}_3^*$, defined by equation (42) from

$$\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{W}}_3^* = d_1 \quad \text{and} \quad \hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_3^* = d_2 \tag{63}$$

We therefore know two direction cosines of $\hat{\mathbf{W}}_3^*$ and we can write in a manner similar to the earlier calculation above

$$\hat{\mathbf{W}}_3^* = a\hat{\mathbf{S}}_1 + b\hat{\mathbf{S}}_2 + c\frac{\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|} \tag{64}$$

with

[7]This situation might arise from a spacecraft equipped like the Oscar-30 spacecraft if one axis of the vector magnetometer were to become defective.

$$a = \frac{1}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|^2}(d_1 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)\, d_2) \tag{65a}$$

$$b = \frac{1}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|^2}(d_2 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)\, d_1) \tag{65b}$$

$$c = \pm\sqrt{1 - (a^2 + 2ab(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) + b^2)} \tag{65c}$$

and

$$a^2 + 2ab(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) + b^2 = \frac{1}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|^2}[d_1^2 + 2(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)\, d_1 d_2 + d_2^2] \tag{66}$$

There are two possible solutions for $\hat{\mathbf{W}}_3^*$. Apart from this two-fold degeneracy, the problem now reduces to Scenario 2. Therefore, in this case, there are four possible solutions, from which the true attitude solution can be determined only on the basis of additional information.

The above case includes also the situation where $\hat{\mathbf{V}}_1 = -\hat{\mathbf{V}}_2$, since one may simultaneously change the signs of $\hat{\mathbf{V}}_2$ and $\hat{\mathbf{S}}_2$ without changing the attitude.

Case 2: Reference Vectors Distinct

To construct a solution in this case,[8] we note first that the attitude matrix may be written in terms of Davenport angles [12, 13] as

$$A = R(\hat{\mathbf{S}}_3, \psi)\, R(\hat{\mathbf{m}}_o, \vartheta)\, R(\hat{\mathbf{V}}_3, \varphi) \tag{67}$$

provided that

$$\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{m}}_o = \hat{\mathbf{V}}_3 \cdot \hat{\mathbf{m}}_o = 0 \tag{68}$$

Assuming that we can order our inputs so that $\hat{\mathbf{S}}_3 \neq \pm\hat{\mathbf{V}}_3$, we can choose

$$\hat{\mathbf{m}}_o = \frac{\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3}{|\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3|} \tag{69}$$

In analogy with Scenario 2, we have now that

$$\hat{\mathbf{S}}_3^{\mathrm{T}} R(\hat{\mathbf{m}}_o, \vartheta)\hat{\mathbf{V}}_3 = d_3 \tag{70}$$

whence

$$\vartheta = \gamma \pm \arccos d_3 \equiv \vartheta(\pm) \tag{71}$$

where

$$\gamma = \arctan_2(\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{V}}_3, |\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3|) \tag{72}$$

It is easy to show for the Davenport angles [12, 13] that

$$R(\hat{\mathbf{S}}_3, \psi)\, R(\hat{\mathbf{m}}_o, \vartheta(+))\, R(\hat{\mathbf{V}}_3, \varphi) = R(\hat{\mathbf{S}}_3, \psi - \pi)\, R(\hat{\mathbf{m}}_o, \vartheta(-))\, R(\hat{\mathbf{V}}_3, \varphi + \pi) \tag{73}$$

This equation has an obvious corresponding result for the 3-1-3 Euler angles. Thus, $\vartheta(+)$ and $\vartheta(-)$ each furnish equivalent parameterizations of the attitude.[9]

[8] If ever an engineer designs an attitude determination system like this, shoot him!
[9] This fact was not understood in reference [11].

Let us define

$$A_o \equiv R(\hat{\mathbf{m}}_o, \vartheta) \tag{74}$$

where ϑ may be either $\vartheta(+)$ or $\vartheta(-)$.

The remaining two equations of equation (11) may now be written

$$\hat{\mathbf{S}}_k^{\mathrm{T}} R(\hat{\mathbf{S}}_3, \psi)\, A_o R(\hat{\mathbf{V}}_3, \varphi)\hat{\mathbf{V}}_k = d_k \qquad k = 1, 2 \tag{75}$$

If we define now

$$\hat{\mathbf{U}}_k \equiv A_o \hat{\mathbf{V}}_k \qquad k = 1, 2, 3 \tag{76}$$

Then equations (75) become

$$\hat{\mathbf{S}}_k^{\mathrm{T}} R(\hat{\mathbf{S}}_3, \psi)\, R(\hat{\mathbf{U}}_3, \varphi)\hat{\mathbf{U}}_k = d_k \qquad k = 1, 2 \tag{77}$$

eliminating one rotation. We are now left with two nonlinear equations to solve in two unknowns.

Let us define the 3×1 matrices

$$\Phi \equiv \begin{bmatrix} 1 \\ \sin \varphi \\ \cos \varphi \end{bmatrix} \quad \text{and} \quad \Psi \equiv \begin{bmatrix} 1 \\ \sin \psi \\ \cos \psi \end{bmatrix} \tag{78ab}$$

Rewriting equation (29) as

$$\begin{aligned} R(\hat{\mathbf{n}}, \theta) &= \hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} + \sin\theta [\![\hat{\mathbf{n}}]\!] + \cos\theta (I_{3\times 3} - \hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}}) \\ &\equiv F_1(\hat{\mathbf{n}}) + \sin\theta\, F_2(\hat{\mathbf{n}}) + \cos\theta\, F_3(\hat{\mathbf{n}}) \end{aligned} \tag{79}$$

we define 3×3 matrices $M(k)$ with elements

$$M_{ij}(k) = \hat{\mathbf{S}}_k^{\mathrm{T}} F_i(\hat{\mathbf{S}}_3) F_j(\hat{\mathbf{U}}_3)\hat{\mathbf{U}}_k - d_k \delta_{i1}\delta_{j1} \qquad k = 1, 2 \tag{80}$$

and δ_{ij} is the usual Kronecker symbol, which is unity when the two indices are equal and zero otherwise. With this new notation equations (77) takes the form

$$\Psi^{\mathrm{T}} M(k)\, \Phi = 0 \qquad k = 1, 2 \tag{81}$$

which can be solved iteratively for the two variables φ and ψ. The numerical search of these equations for all eight possible solutions is tedious and uninstructive and will not be presented here.

Figure 1 presents the numerical solution of the eight values of (Φ, Ψ) for the case of equation (59). The graph shows the value of

$$\frac{a}{b + J(\Phi, \Psi)}$$

for $0 \leq \Phi < 2\pi$ and $0 \leq \Psi < 2\pi$. Here a and b are chosen simply to make the most attractive looking graph and the form to be plotted has been chosen, because peaks are easier to see than valleys. The eight solutions are very much in evidence.

Covariance Analysis for the Case of Three Arc Lengths

The covariance matrix may be calculated straightforwardly using the results of Scenario 2.

$$P_{\theta\theta}^{-1} = \sum_{k=1}^{3} \frac{1}{\sigma_{d_k}^2} (\hat{\mathbf{S}}_k \times A\hat{\mathbf{V}}_k)(\hat{\mathbf{S}}_k \times A\hat{\mathbf{V}}_k)^{\mathrm{T}} \tag{82}$$

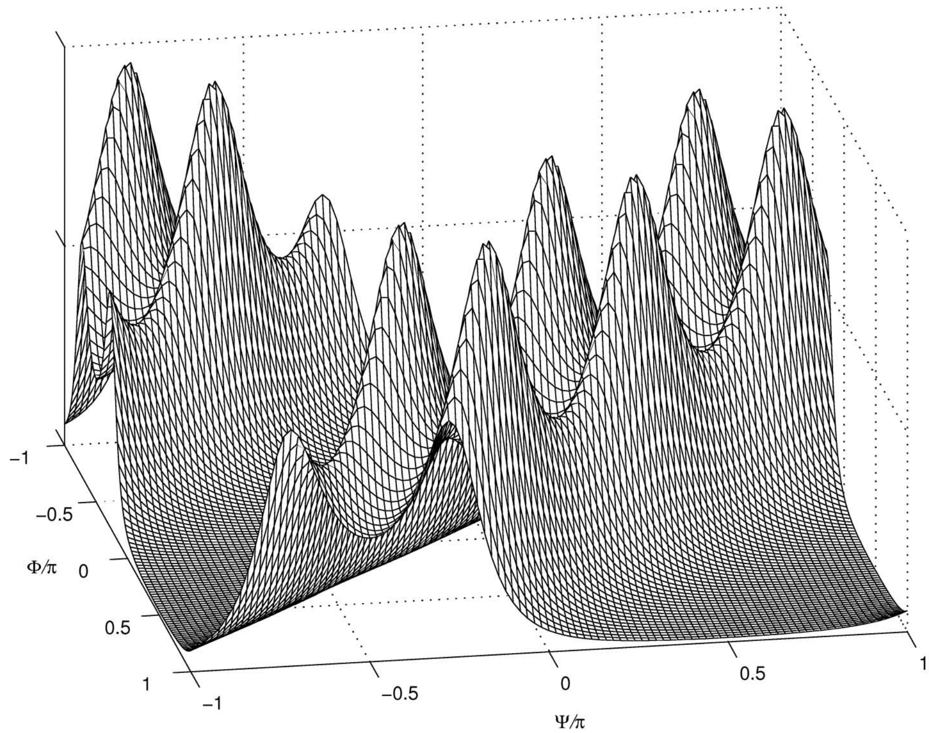

FIG. 1. Attitude Solutions from the Measurement of Three Arc Lengths.

whose calculation requires, of course, that we know the correct value of A. Note that the covariance matrix above takes account only of random noise and not of the errors due to choosing the wrong attitude solution.

Dihedral Angles

We have largely sidestepped the issue of dihedral angles, passing them off as only an intermediate step to the computation of arc lengths. This is generally a practical necessity, but it belies the fact that important information is discarded in the process.

A unit vector may be represented with respect to right-hand orthonormal axes either by three arc lengths (the direction cosines) or by two spherical angles. These last consist of the declination ϑ, also called the coelevation (an arc length), and the right-ascension φ, also called the azimuth (a dihedral angle). This nomenclature has its origin in the pointing of telescopes and artillery. Thus, an arc length and a dihedral angle accomplish the task of three arc lengths in describing a direction, so we may say roughly that one dihedral angle is worth two arc-lengths. This makes further sense when we consider that the useful range of an arc length is $[0, \pi]$, while that for a dihedral angle is $[0, 2\pi]$. Of course, for the pointing of a telescope or an artillery piece two arc lengths suffice, because the false direction would point unphysically to the back of the device. However, mechanical necessities dictate that the direction be instrumented as a dihedral angle and an arc length. One could not, however, determine the direction of the magnetic field from the knowledge of two arc lengths.

The attitude can be described unambiguously by three Euler angles (although the angles themselves may be ambiguous). Hence, if the Euler angles are measured directly, by, say, a three-axis gimballed gyro, then the attitude is uniquely specified (within statistical error) by three angle measurements. From the usable range of these angles, it is clear that the outer Euler angles must be dihedral angles, and the middle Euler angle an arc length.

In spin-axis attitude estimation, the Earth-width measurement, a dihedral angle, is used only to construct the nadir angle, with loss of information. This is done for no other reason than to avoid the computational complexity of using a dihedral angle measurement directly in the estimator. To see this, examine the Sun-Earth dihedral angle, the angle from the plane containing the spin-axis and the Sun vector to the plane containing the spin-axis and the Earth vector. In obvious notation

$$\sin \psi_{\mathrm{SE}} = \frac{(\hat{\mathbf{S}} \times \hat{\mathbf{E}}) \cdot \hat{\mathbf{n}}}{\sqrt{1 - (\hat{\mathbf{S}} \cdot \hat{\mathbf{n}})^2}\sqrt{1 - (\hat{\mathbf{E}} \cdot \hat{\mathbf{n}})^2}} \tag{83a}$$

$$\cos \psi_{\mathrm{SE}} = \frac{(\hat{\mathbf{S}} \times \hat{\mathbf{n}}) \cdot (\hat{\mathbf{E}} \times \hat{\mathbf{n}})}{\sqrt{1 - (\hat{\mathbf{S}} \cdot \hat{\mathbf{n}})^2}\sqrt{1 - (\hat{\mathbf{E}} \cdot \hat{\mathbf{n}})^2}} \tag{83b}$$

with $\hat{\mathbf{n}}$ the spacecraft spin axis. The denominators can be calculated directly from measurements of the Sun and nadir angles, leaving us with a linear measurement for $\sin \psi_{\mathrm{SE}}$ but a quadratic measurement for $\cos \psi_{\mathrm{SE}}$. The use of the quadratic measurement in the estimator would cause enormous computational hardship, so we discard it and put up with the loss of accuracy.

Discussion

We have examined three minimum data cases for constructing spacecraft attitude deterministically when the measurement consists of directions or arc lengths. The simplest case, when the data consists of two directions, is that of the well-known TRIAD algorithm. The case when the data consist of one direction and one arc length is only slightly more complicated and shows a two-fold degeneracy in the attitude solutions. The case when the data consist of three arc lengths is much more complicated. The attitude solution in this case is more elaborate and displays, in general, an eight-fold degeneracy. While the case of one direction and one arc length has been implemented in actual mission support, it is unlikely that this will ever be the case when only three arc lengths are measured.

In extreme cases, we have seen, the degeneracy may be of a lower order. For example, in the case where the measurement set consists of one direction and one arc length or of three arc lengths, there will be no ambiguity in the solution if the arc length is 0 or π. Likewise, the degeneracy is of lower order in the case of three arc lengths when the reference vectors are the same for two of the measurements. How many additional arc length measurements are needed to remove the degeneracies is also not easy to determine. One can easily construct cases where the measurement of two arc lengths leads to a unique attitude (for example, if the two arc lengths are each 0) or even an overdetermined attitude (for example, if the two arc lengths are each 0 or π) as well as cases where a unique attitude cannot be constructed from the measurement of five arc lengths, all of which are essential, but an unambiguous attitude solution is obtainable if a sixth arc length measured is introduced.

For example, consider the case where we are given four geometrically and statistically independent arc-length measurements, three of which have the same reference vector $\hat{\mathbf{V}}_1$ and the fourth with a different reference direction $\hat{\mathbf{V}}_2$. This is equivalent to being given a direction measurement $\hat{\mathbf{W}}_1$ and an arc-length measurement. The attitude estimate, therefore, will have a two-fold degeneracy. A fifth arc-length measurement, even with reference direction $\hat{\mathbf{V}}_2$, will not necessarily remove the degeneracy, because the reconstructed observation $\hat{\mathbf{W}}_2$ will still have a two-fold degeneracy. It may be possible, however, to decide which one is correct if the two reconstructed directions are separated by much more than measurement error level in equivalent angle or if the measured arc-length is 0 or π. The question of optimality and degeneracy for this case is treated in detail in a recent work in connection with an unconventional analysis of the TRIAD algorithm as a maximum maximum likelihood estimator [14].

Acknowledgment

The author is grateful to F. Landis Markley of NASA Goddard Space Flight Center and Yang Cheng of the University of Buffalo for helpful criticisms. The author is grateful also to Dr. Cheng for supplying Fig. 1, which he accomplished for the author while a graduate student at the Harbin Institute of Technology in the People's Republic of China.

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