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SCAD—A Fast Algorithm for Star Camera Attitude Determination

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Abstract

An earlier algorithm for multiple sensors is extended to provide three-axis attitude from multiple line-of-sight observations with a single optical sensor, typically a star camera. The algorithm, called SCAD, is simpler conceptually than either the QUEST, FOAM, or ESOQ algorithms and, although suboptimal, suffers only imperceptible loss of accuracy for typical star cameras with limited fields of view. A complete covariance analysis using the QUEST measurement model is presented.

Introduction: The Wahba Problem

A central problem in Spacecraft Attitude Determination has been that of determining the three-axis attitude which minimizes the cost function

$$J(A) = \frac{1}{2} \sum_{k=1}^{N} a_k |\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k|^2 \tag{1}$$

where A is the direction-cosine matrix [1], $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$, are directions (lines of sight, observation vectors) observed in the spacecraft body frame, $\hat{\mathbf{V}}_k$, $k = 1, \ldots, N$, are the corresponding directions known in an inertial frame (the reference vectors) and a_k, $k = 1, \ldots, N$, are a set of positive weights. A caret in this work will be used to denote a unit vector. This cost function was first proposed by G. Wahba [2] in 1965 and has been the starting point of many algorithms, of which the most popular has been the QUEST algorithm [3], although other attractive algorithms exist [4].

Of particular importance is the fact that the Wahba cost function can be derived from maximum-likelihood estimation provided one assumes the following measurement model [5], which has been called the QUEST model, because it was first used in an early accuracy study of the QUEST algorithm [3]

$$\hat{\mathbf{W}}_k = A\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k \tag{2}$$

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with the measurement error $\Delta\hat{\mathbf{W}}_k$ having first and second moments[2]

$$E\{\Delta\hat{\mathbf{W}}_k\} = \mathbf{0} \tag{3}$$

$$E\{\Delta\hat{\mathbf{W}}_k \Delta\hat{\mathbf{W}}_k^{\mathrm{T}}\} = \sigma_k^2[I - (A\hat{\mathbf{V}}_k)(A\hat{\mathbf{V}}_k)^{\mathrm{T}}] \tag{4}$$

where $E\{\cdot\}$ denotes the expectation, T denotes the matrix transpose, I is the 3×3 identity matrix, and one chooses the weights to be

$$a_k = \frac{\sigma_{\mathrm{tot}}^2}{\sigma_k^2} \tag{5}$$

with

$$\frac{1}{\sigma_{\mathrm{tot}}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{6}$$

The common constant in the numerators of equation (5) is arbitrary, of course, but the choice of equation (6) makes

$$\sum_{k=1}^{N} a_k = 1 \tag{7}$$

One defines the attitude covariance matrix $P_{\theta\theta}$ [1, 3, 5] as the covariance of the attitude error vector, which is the rotation vector [1] of the small rotation carrying the true attitude into the estimated attitude. Assuming the QUEST model for the measurements, this leads to the following expression for the attitude covariance matrix

$$P_{\theta\theta}^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2}(I - \hat{\mathbf{W}}_k^{\mathrm{true}}\hat{\mathbf{W}}_k^{\mathrm{true\,T}}) \tag{8}$$

and

$$\hat{\mathbf{W}}_k^{\mathrm{true}} \equiv A^{\mathrm{true}}\hat{\mathbf{V}}_k \tag{9}$$

In actual computations we must replace $\hat{\mathbf{W}}_k^{\mathrm{true}}$ by $\hat{\mathbf{W}}_k$, because the former is not known in general. Since we will be interested in calculating quantities only to lowest nonvanishing order in $\Delta\hat{\mathbf{W}}_k$ this replacement will not lead to important errors in general. We assume throughout this work that the uncertainties in $\hat{\mathbf{V}}_k$ are negligible compared to those in $\hat{\mathbf{W}}_k$.

In a previous work [6] a method was presented which simplified the attitude estimation process for an Earth albedo sensor. In that work, an approximate measurement for the direction of the Earth albedo centroid was determined by taking an average of the centroid of the directions of individual elements of the Earth albedo sensor weighted by the measured intensity, which was compared with a simulated model centroid. The effective vector measurement was combined with a measurement of the Sun direction and used as input to the TRIAD algorithm [3]. It could equally well have been used as input to the QUEST algorithm, but the miniscule improvement in accuracy was not justified by the additional computational burden. Brozenec and Bender [7] used a similar averaging of multiple star directions in a

[2]In fact, because of the unity constraint on the norm of $\hat{\mathbf{W}}_k$, the mean of $\Delta\hat{\mathbf{W}}_k$ will have a small nonvanishing part [5] equal to $-\sigma_k^2\hat{\mathbf{W}}_k$. This may be safely neglected in our discussion.

star camera to generate a reduced set of measurements for the QUEST algorithm. A careful covariance analysis of the algorithm of Bender and Brozenec was presented in reference [8]. In the present work we present a method for retaining full three-axis attitude information from multiple data from a single optical sensor, typically a star camera. In addition, rather than relying on heuristic arguments, we will develop the algorithm in a rigorous manner.

Construction of a Suboptimal Cost Function

Let us reexamine the Wahba cost function, which we write in the form of the data-dependent part of the negative-log-likelihood function [5], assuming that the measurement model of equations (2) through (4) is valid, namely

$$J(A) = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} |\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k|^2 \tag{10}$$

Let us now introduce vectors $\overline{\mathbf{W}}$ and $\overline{\mathbf{V}}$ into the cost function as

$$J(A) = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left| \left[\overline{\mathbf{W}} - A\overline{\mathbf{V}} \right] + \left[(\hat{\mathbf{W}}_k - \overline{\mathbf{W}}) - A(\hat{\mathbf{V}}_k - \overline{\mathbf{V}}) \right] \right|^2 \tag{11}$$

and expand the cost function as

$$\begin{aligned} J(A) = {} & \frac{1}{2} \frac{1}{\sigma_{\text{tot}}^2} |\overline{\mathbf{W}} - A\overline{\mathbf{V}}|^2 \\ & + (\overline{\mathbf{W}} - A\overline{\mathbf{V}})^{\mathrm{T}} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left((\hat{\mathbf{W}}_k - \overline{\mathbf{W}}) - A(\hat{\mathbf{V}}_k - \overline{\mathbf{V}}) \right) \\ & + \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left| (\hat{\mathbf{W}}_k - \overline{\mathbf{W}}) - A(\hat{\mathbf{V}}_k - \overline{\mathbf{V}}) \right|^2 \end{aligned} \tag{12}$$

If now $\overline{\mathbf{W}}$ and $\overline{\mathbf{V}}$ are chosen to have the values

$$\overline{\mathbf{W}} = \sum_{k=1}^{N} a_k \hat{\mathbf{W}}_k \quad \text{and} \quad \overline{\mathbf{V}} = \sum_{k=1}^{N} a_k \hat{\mathbf{V}}_k \tag{13}$$

with the a_k, $k = 1, \ldots, N$, given by equation (5), then the second line of equation (12) will vanish identically, and the third line will be a minimum (for given A) leaving

$$\begin{aligned} J(A) &= \frac{1}{2} \frac{1}{\sigma_{\text{tot}}^2} |\overline{\mathbf{W}} - A\overline{\mathbf{V}}|^2 + \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left| (\hat{\mathbf{W}}_k - \overline{\mathbf{W}}) - A(\hat{\mathbf{V}}_k - \overline{\mathbf{V}}) \right|^2 \\ &\equiv J'(A) + J''(A) \end{aligned} \tag{14}$$

For a focal-plane sensor with a field of view of ± 0.1 rad per axis (roughly ± 6 deg per axis), we anticipate that the effective contribution of the summation in equation (14) will be roughly $(0.1)^2$ or one percent of the first term. Thus, the estimation of the spacecraft attitude will be "dominated" by the first term. The second term, which could be discarded if another vector sensor of suitable accuracy were present [6, 7], is not unimportant, however, if data from this sensor alone must be used to construct the three-axis attitude. Minimizing only the first term is not sufficient to determine the spacecraft attitude. If A_o minimizes the first term, then so does $R(\hat{\overline{\mathbf{W}}}, \psi)A_o$, where $R(\hat{\overline{\mathbf{W}}}, \psi)$ denotes the direction-cosine matrix for a rotation through an arbitrary angle ψ about the direction $\overline{\mathbf{W}}$

$$\hat{\overline{\mathbf{W}}} = \frac{\overline{\mathbf{W}}}{|\overline{\mathbf{W}}|} \tag{15}$$

It is the second term which provides the information on ψ.

Since the overall weight of the first term in equation (14) will be so much greater than that of the second term, we can determine an approximate value for the optimal attitude by writing

$$A = R(\hat{\overline{\mathbf{W}}}, \psi)A_o \tag{16}$$

and seeking first A_o^*, a value of the (nonunique) proper orthogonal matrix A_o which minimizes

$$J'(A_o) \equiv \frac{1}{2}\frac{1}{\sigma_{\text{tot}}^2}|\overline{\mathbf{W}} - A_o\overline{\mathbf{V}}|^2 \tag{17}$$

and then the value ψ^* which minimizes

$$J''(\psi) \equiv \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left|(\hat{\mathbf{W}}_k - \overline{\mathbf{W}}) - R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*(\hat{\mathbf{V}}_k - \overline{\mathbf{V}})\right|^2 \tag{18}$$

Given these A_o^* and ψ^*, we anticipate that

$$A^* \equiv R(\hat{\overline{\mathbf{W}}}, \psi^*)A_o^* \tag{19}$$

will be a good approximation for the optimal direction-cosine matrix which minimizes the cost function of equation (10). This is the desired suboptimal algorithm.

Simplification of the Cost Functions

We can simplify the two cost functions, $J'(A_o)$ and $J''(\psi)$, without loss of accuracy. Examine first $J'(A_o)$. Defining

$$e \equiv \frac{|\overline{\mathbf{W}}| - |\overline{\mathbf{V}}|}{|\overline{\mathbf{W}}|} \tag{20}$$

we write

$$|\overline{\mathbf{V}}| = (1 - \epsilon)|\overline{\mathbf{W}}| \tag{21}$$

and we can recast $J'(A_o)$ accordingly in the form

$$J'(A_o) = \frac{1}{2}\frac{|\overline{\mathbf{W}}|^2}{\sigma_{\text{tot}}^2}\left|\hat{\overline{\mathbf{W}}} - (1 - \epsilon)A_o\hat{\overline{\mathbf{V}}}\right|^2 \tag{22}$$

An optimizing value of A_o will cause $A_o\hat{\overline{\mathbf{V}}}$ to be equal to $\hat{\overline{\mathbf{W}}}$ independently of the value of ϵ. Thus, we will achieve the identical value of A_o^* if we discard ϵ in equation (22).

Likewise, substituting equation (21) into equation (18) leads to

$$J''(\psi) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left|\left(\hat{\mathbf{W}}_k - R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*\hat{\mathbf{V}}_k\right) - \left(\overline{\mathbf{W}} - R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*\overline{\mathbf{V}}\right)\right|^2$$

$$= \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left|\left(\hat{\mathbf{W}}_k - R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*\hat{\mathbf{V}}_k\right) - e\overline{\mathbf{W}}\right|^2 \tag{23}$$

Expanding equation (23) gives directly

$$J''(\psi) = -\frac{\epsilon_2}{2}\frac{|\overline{\mathbf{W}}|^2}{\sigma_{\text{tot}}^2} + \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left|\hat{\mathbf{W}}_k - R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*\hat{\mathbf{V}}_k\right|^2 \tag{24}$$

Only the second term of equation (24) depends on ψ. The first term, therefore, may be discarded from the cost function, so that for the purpose of locating the minimizing arguments we may replace $J'(A_0)$ and $J''(\psi)$ with

$$L'(A_o) = \frac{1}{2}\left|\hat{\overline{\mathbf{W}}} - A_o\hat{\overline{\mathbf{V}}}\right|^2 \tag{25a}$$

$$L''(\psi) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left|\hat{\mathbf{W}}_k - R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*\hat{\mathbf{V}}_k\right|^2 \tag{25b}$$

Note that the simplification of equations (17) and (18) to obtain equations (25) did not rely on any approximation for the value of ϵ. Note, in particular, that we have discarded an uninteresting factor as well as uninteresting terms to obtain equations (25). Note also that $L''(\psi)$ is equal to $J(R(\hat{\overline{\mathbf{W}}}, \psi)A_o^*)$ but $L''(\psi)$ and $J(A)$ are certainly not identical, however, since the latter is a least-squares cost function defined over all of $SO(3)$. Neither $L'(A_0)$ nor $L''(\psi)$ (nor their sum) has any statistical significance.

We determine the suboptimal attitude equivalently by minimizing the two cost functions of equation (25), $L'(A_o)$ and $L''(\psi)$, in sequence.

Construction of the Suboptimal Attitude

The cost function of equation (25a) can be made to vanish exactly for a continuum of solutions A_o^*. Except for the special case $\hat{\overline{\mathbf{W}}} = -\hat{\overline{\mathbf{V}}}$, for which an A_o^* may be found trivially [9, 10], a suitable A_o^* is given by [9, 10]

$$\begin{aligned} A_o^* &= (\hat{\overline{\mathbf{W}}} \cdot \hat{\overline{\mathbf{V}}})I + \frac{(\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{V}}})(\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{V}}})^{\mathrm{T}}}{1 + \hat{\overline{\mathbf{W}}} \cdot \hat{\overline{\mathbf{V}}}} + [\![\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{V}}}]\!] \\ &= I + [\![\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{V}}}]\!] + \frac{1}{1 + \hat{\overline{\mathbf{W}}} \cdot \hat{\overline{\mathbf{V}}}}[\![\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{V}}}]\!]^2 \end{aligned} \tag{26}$$

where

$$[\![\mathbf{v}]\!] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \tag{27}$$

This corresponds to the quaternion [1]

$$\bar{q}_o^* = \sqrt{\frac{1 + \hat{\overline{\mathbf{W}}} \cdot \hat{\overline{\mathbf{V}}}}{2}}\begin{bmatrix} \left(\dfrac{\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{V}}}}{1 + \hat{\overline{\mathbf{W}}} \cdot \hat{\overline{\mathbf{V}}}}\right) \\ 1 \end{bmatrix} \tag{28}$$

and the Rodrigues vector [1]

$$\boldsymbol{\rho}_o^* = \frac{\hat{\bar{\mathbf{W}}} \times \hat{\bar{\mathbf{V}}}}{1 + \hat{\bar{\mathbf{W}}} \cdot \hat{\bar{\mathbf{V}}}} \tag{29}$$

The particular A_o^* that we chose is of no consequence, provided that it satisfies

$$A_o^* \hat{\bar{\mathbf{V}}} = \hat{\bar{\mathbf{W}}} \tag{30}$$

It remains only to find the angle ψ^* which minimizes the cost function of equation (25b).

The vanishing of the denominator in equation (26) when $\hat{\bar{\mathbf{W}}} = -\hat{\bar{\mathbf{V}}}$, which requires a rotation through π, is reminiscent of a similar phenomenon in the construction of the QUEST attitude quaternion [3] and is sidestepped in the same way, namely, by means of the Method of Sequential Rotations. This is treated in a later section.

To determine ψ^* we rewrite $L''(\psi)$, using techniques developed by Davenport, which have become part of the development of the QUEST algorithm [3], as

$$\begin{aligned} L''(\psi) &= \frac{1}{\sigma_{\text{tot}}^2} - \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k^{\mathrm{T}} R(\hat{\bar{\mathbf{W}}}, \psi) A_o^* \hat{\mathbf{V}}_k \\ &= \frac{1}{\sigma_{\text{tot}}^2} - \operatorname{tr}\left[B^{\mathrm{T}} R(\hat{\bar{\mathbf{W}}}, \psi)\right] \end{aligned} \tag{31}$$

where $\operatorname{tr}(\cdot)$ denotes the trace operation, and

$$B = \left(\sum_{k=1}^{N} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^{\mathrm{T}} \right) A_o^{*\mathrm{T}} \equiv C A_o^{*\mathrm{T}} \tag{32}$$

Writing Euler's formula [1] in the form

$$R(\hat{\bar{\mathbf{W}}}, \psi) = \hat{\bar{\mathbf{W}}} \hat{\bar{\mathbf{W}}}^{\mathrm{T}} + \sin\psi \, [\![\hat{\bar{\mathbf{W}}}]\!] + \cos\psi \, (I - \hat{\bar{\mathbf{W}}} \hat{\bar{\mathbf{W}}}^{\mathrm{T}}) \tag{33}$$

we have

$$\begin{aligned} L''(\psi) &= \frac{1}{\sigma_{\text{tot}}^2} - \hat{\bar{\mathbf{W}}}^{\mathrm{T}} B \hat{\bar{\mathbf{W}}} - \sin\psi \, \operatorname{tr}\left[B^{\mathrm{T}} [\![\hat{\bar{\mathbf{W}}}]\!]\right] \\ &\quad - \cos\psi \, \operatorname{tr}\left[B^{\mathrm{T}} (I - \hat{\bar{\mathbf{W}}} \hat{\bar{\mathbf{W}}}^{\mathrm{T}})\right] \\ &\equiv \frac{1}{\sigma_{\text{tot}}^2} - \hat{\bar{\mathbf{W}}}^{\mathrm{T}} B \hat{\bar{\mathbf{W}}} - s \sin\psi - c \cos\psi \end{aligned} \tag{34}$$

with

$$\mathbf{Z} \equiv \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix}, \quad s \equiv \left(\mathbf{Z}^{\mathrm{T}} \hat{\bar{\mathbf{W}}} \right), \quad \text{and} \quad c \equiv \left(\operatorname{tr}[B] - \hat{\bar{\mathbf{W}}}^{\mathrm{T}} B \hat{\bar{\mathbf{W}}} \right) \tag{35abc}$$

Minimization of $L''(\psi)$ leads straightforwardly to

$$-s \cos\psi^* + c \sin\psi^* = 0 \tag{36}$$

or

$$\psi^* = \arctan_2(s, c) \tag{37a}$$

Here $\arctan_2(s, c)$ is the function which returns the arc tangent of s/c in the correct quadrant. In the FORTRAN language this function is called ATAN2. The angle ψ^*

will be indeterminate if both s and c vanish. This is possible, however, only if all of the $\hat{\mathbf{W}}_k$ are identical.

Note that the evaluation of $R(\hat{\bar{\mathbf{W}}}, \psi^*)$ does not require the computation of ψ^*. From equation (37) it follows directly that

$$\sin \psi^* = \frac{s}{\sqrt{c^2 + s^2}}, \quad \cos \psi^* = \frac{c}{\sqrt{c^2 + s^2}} \tag{37bc}$$

which may then be substituted into equation (33). For the evaluation of the Rodrigues vector and the quaternion for the second rotation see the appendix to this work.

The parallelism of the calculation of ψ^* in the present algorithm with that of $\bar{q}^*$ in the QUEST algorithm is apparent. However, these methods are applied here to a single angle variable and not to the four components of the quaternion of rotation.

We call this algorithm SCAD for Star-Camera Attitude Determination.

Covariance Analysis of the Algorithm

We define the attitude error vector $\Delta\boldsymbol{\theta}$ by

$$A^* = e^{[\![\Delta\boldsymbol{\theta}]\!]}A^{\text{true}} = (I + [\![\Delta\boldsymbol{\theta}]\!])A^{\text{true}} + O(|\Delta\boldsymbol{\theta}|^2) \tag{38}$$

and the attitude covariance matrix by

$$P_{\theta\theta} \equiv E\{\Delta\boldsymbol{\theta}\Delta\boldsymbol{\theta}^{\text{T}}\} \tag{39}$$

where $E\{\cdot\}$ denotes the expectation. The choice of the initial reference frame to which the attitude is referenced, and hence the value of A^{true}, is immaterial to the attitude error by this definition, and the attitude error vector depends only on the measurement errors.[3] Thus, if the errors in the reference vectors are very small compared to those in the observation vectors, we may make the substitutions

$$\hat{\mathbf{V}}_k \rightarrow \hat{\mathbf{W}}_k^{\text{true}} \qquad k = 1, \ldots, N \tag{40}$$

in calculating the attitude error covariance matrix. Here $\hat{\mathbf{W}}_k^{\text{true}}$ is the true value of $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$. We will then have

$$A^{\text{true}} = I, \quad \text{and} \quad A^* = I + [\![\Delta\boldsymbol{\theta}]\!] + O(|\Delta\boldsymbol{\theta}|^2) \tag{41}$$

while the value of $P_{\theta\theta}$ will be unchanged. In this case, clearly

$$\hat{\bar{\mathbf{V}}} = \hat{\bar{\mathbf{W}}}^{\text{true}} \tag{42}$$

$$A_o^{\text{true}} = I, \qquad A_o^* = I + [\![\Delta\boldsymbol{\theta}_o]\!] + O(|\Delta\boldsymbol{\theta}_o|^2) \tag{43ab}$$

$$\psi^{\text{true}} = 0, \qquad \psi^* = \Delta\psi \tag{44ab}$$

$$R(\hat{\bar{\mathbf{W}}}, \psi^*) = I + [\![\Delta\psi\hat{\bar{\mathbf{W}}}^{\text{true}}]\!] + O(|\Delta\psi|^2, |\Delta\psi|\,\Delta\boldsymbol{\theta}_o|) \tag{45}$$

The calculation of the suboptimal attitude now leads directly to the attitude error vector. It follows that to within quadratic terms in the errors

$$\Delta\boldsymbol{\theta} = \Delta\boldsymbol{\theta}_o + \Delta\psi\hat{\bar{\mathbf{W}}}^{\text{true}} \tag{46}$$

[3]This is because a change in A^{true} will cause a corresponding change in A^* which will leave the value of $\Delta\boldsymbol{\theta}$ unchanged. This assumes, however, that the reference vectors are known perfectly. When this is not the case, the errors in the reference vectors can be transferred to the observation vectors, as in reference [3]. The measurement error covariance matrix will then depend on A^{true}, but this dependence will not affect the discussion which follows.

From equations (26) and (40) it follows immediately that to first order

$$A_o^* = I + [\![\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{W}}}^{\text{true}}]\!] = I + [\![\Delta\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{W}}}^{\text{true}}]\!] \tag{47}$$

where

$$\begin{aligned}\Delta\hat{\overline{\mathbf{W}}} &= \hat{\overline{\mathbf{W}}} - \hat{\overline{\mathbf{W}}}^{\text{true}} \\ &= |\hat{\overline{\mathbf{W}}}^{\text{true}}|^{-1}(I - \hat{\overline{\mathbf{W}}}^{\text{true}}\hat{\overline{\mathbf{W}}}^{\text{true T}})\Delta\overline{\mathbf{W}}\end{aligned} \tag{48}$$

and

$$\Delta\overline{\mathbf{W}} = \sum_{k=1}^{N} a_k \Delta\hat{\mathbf{W}}_k \tag{49}$$

Hence

$$\Delta\boldsymbol{\theta}_o = \Delta\hat{\overline{\mathbf{W}}} \times \hat{\overline{\mathbf{W}}}^{\text{true}} = [\![\hat{\overline{\mathbf{W}}}^{\text{true}}]\!]\Delta\hat{\overline{\mathbf{W}}} \tag{50}$$

To determine $\Delta\psi$ we write

$$\begin{aligned}L''(\psi) &= \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}|\hat{\mathbf{W}}_k - R(\hat{\overline{\mathbf{W}}}^{\text{true}}, \psi)A_o^*\hat{\mathbf{W}}_k^{\text{true}}|^2 \\ &= \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\left|\Delta\hat{\mathbf{W}}_k - [\![\Delta\boldsymbol{\theta}_o + \psi\hat{\overline{\mathbf{W}}}^{\text{true}}]\!]\hat{\mathbf{W}}_k^{\text{true}}\right|^2\end{aligned} \tag{51}$$

and solve for $\Delta\psi$, the value of ψ which minimizes this cost function, which is given by

$$\Delta\psi = -c_o/c_1 \tag{52}$$

with

$$c_1 = \sum_{k=1}^{N}\frac{1}{\sigma_k^2}\hat{\mathbf{W}}_k^{\text{true T}}[\![\hat{\overline{\mathbf{W}}}^{\text{true}}]\!][\![\hat{\overline{\mathbf{W}}}^{\text{true}}]\!]^{\text{T}}\hat{\mathbf{W}}_k^{\text{true}} \tag{53a}$$

$$c_o = \sum_{k=1}^{N}\frac{1}{\sigma_k^2}\hat{\mathbf{W}}_k^{\text{true T}}[\![\hat{\overline{\mathbf{W}}}^{\text{true}}]\!](\Delta\hat{\mathbf{W}}_k - [\![\Delta\boldsymbol{\theta}_o]\!]\hat{\mathbf{W}}_k^{\text{true}}) \tag{53b}$$

These may be rewritten

$$c_1 = \hat{\overline{\mathbf{W}}}^{\text{trueT}}F\hat{\overline{\mathbf{W}}}^{\text{true}} \tag{54a}$$

$$c_o = \hat{\overline{\mathbf{W}}}^{\text{trueT}}\left(F\Delta\boldsymbol{\theta}_o - \sum_{k=1}^{N}a_k[\![\hat{\mathbf{W}}_k^{\text{true}}]\!]\Delta\hat{\mathbf{W}}_K\right) \tag{54b}$$

with

$$F \equiv \sum_{k=1}^{N}\frac{1}{\sigma_k^2}(I - \hat{\mathbf{W}}_k^{\text{true}}\hat{\mathbf{W}}_k^{\text{true T}}) = (P^{\text{QUEST}})^{-1} \tag{55}$$

The covariance matrix is given now by

$$\begin{aligned}P_{\theta\theta} &= E\{(\Delta\boldsymbol{\theta}_o + \Delta\psi\hat{\overline{\mathbf{W}}})(\Delta\boldsymbol{\theta}_o + \Delta\psi\hat{\overline{\mathbf{W}}})^{\text{T}}\} \\ &= E\{\Delta\boldsymbol{\theta}_o\Delta_o^{\text{T}}\} + (E\{\Delta\boldsymbol{\theta}_o\Delta\psi\}\hat{\overline{\mathbf{W}}}^{\text{T}} + \text{transpose}) \\ &\quad + \hat{\overline{\mathbf{W}}}E\{(\Delta\psi)^2\}\hat{\overline{\mathbf{W}}}^{\text{T}}\end{aligned} \tag{56}$$

In equation (56) and in the remainder of this section we have not written the superscript "true" on whole vectors, to avoid making our formulae too cumbersome. In calculating the covariance matrix, we have no recourse but to use the observed values in any event. Evaluating equation (56) one obtains in succession

$$E\{\Delta\boldsymbol{\theta}_o\Delta\boldsymbol{\theta}_o^{\mathrm{T}}\} = \frac{1}{|\overline{\mathbf{W}}|^2}[\![\hat{\overline{\mathbf{W}}}]\!]F[\![\hat{\overline{\mathbf{W}}}]\!]^{\mathrm{T}} \tag{57a}$$

$$E\{\Delta\boldsymbol{\theta}_o\Delta\psi\} = -\frac{1}{|\overline{\mathbf{W}}|^2}\frac{1}{\left(\hat{\overline{\mathbf{W}}}^{\mathrm{T}}F\hat{\overline{\mathbf{W}}}\right)}[\![\hat{\overline{\mathbf{W}}}]\!]F[\![\hat{\overline{\mathbf{W}}}]\!]^{\mathrm{T}}F\hat{\overline{\mathbf{W}}} \tag{57b}$$

$$E\{(\Delta\psi)^2\} = \frac{1}{|\overline{\mathbf{W}}|^2}\frac{1}{\left(\hat{\overline{\mathbf{W}}}^{\mathrm{T}}F\hat{\overline{\mathbf{W}}}\right)^2}\hat{\overline{\mathbf{W}}}^{\mathrm{T}}F[\![\hat{\overline{\mathbf{W}}}]\!]F[\![\hat{\overline{\mathbf{W}}}]\!]^{\mathrm{T}}F\hat{\overline{\mathbf{W}}} + (\hat{\overline{\mathbf{W}}}^{\mathrm{T}}F\hat{\overline{\mathbf{W}}})^{-1} \tag{57c}$$

with the result that[4]

$$P^{\mathrm{SCAD}} = \frac{1}{|\overline{\mathbf{W}}|^2}(I - G\,F)[\![\hat{\overline{\mathbf{W}}}]\!]F[\![\hat{\overline{\mathbf{W}}}]\!]^{\mathrm{T}}(I - G\,F)^{\mathrm{T}} + G \tag{58}$$

with

$$G = \hat{\overline{\mathbf{W}}}\,(\hat{\overline{\mathbf{W}}}^{\mathrm{T}}F\hat{\overline{\mathbf{W}}})^{-1}\hat{\overline{\mathbf{W}}}^{\mathrm{T}} \tag{59}$$

In deriving the above formulae we have made liberal use of the result

$$R_{\hat{\overline{\mathbf{W}}}} = E\{\Delta\hat{\overline{\mathbf{W}}}\Delta\hat{\overline{\mathbf{W}}}^{\mathrm{T}}\} = \frac{1}{|\overline{\mathbf{W}}|^2}(I - \hat{\overline{\mathbf{W}}}\hat{\overline{\mathbf{W}}}^{\mathrm{T}})\,F\,(I - \hat{\overline{\mathbf{W}}}\hat{\overline{\mathbf{W}}}^{\mathrm{T}}) \tag{60}$$

Model Covariance Analysis

It follows from the Cramér-Rao Theorem [11] that

$$P^{\mathrm{QUEST}} \leq P^{\mathrm{SCAD}} \tag{61}$$

The important question is: how large is the difference between the two attitude covariance matrices? To answer this question, we examine the two covariances in a simple model, in which the star camera is assumed to have a circular field of view of angular radius ρ, and the stars are assumed to be distributed uniformly (in the probabilistic sense) over the field of view of the sensor. We will assume for convenience that the star camera has its boresight along the spacecraft z-axis. We assume in addition that the covariance matrix of every line-of-sight observation is characterized by the same variance σ^2.

In the limit that N is large we may replace the summation over the observations by an integral. Thus, if $f(\hat{\mathbf{W}})$ is any function of an observed direction, we may write

$$\sum_{k=1}^{N} f(\hat{\mathbf{W}}_k) \rightarrow \frac{N}{\Omega}\int_0^{2\pi}\int_0^{\rho} f(\hat{\mathbf{W}}(\vartheta, \varphi))\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \tag{62}$$

with Ω the solid angle subtended by the star camera field of view

$$\Omega = 2\pi(1 - \cos\rho) \tag{63}$$

and

$$\hat{\mathbf{W}}(\vartheta, \varphi) = \begin{bmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{bmatrix} \tag{64}$$

[4]When no subscript appears in the covariance matrix P, the latter may be assumed to mean $P_{\theta\theta}$.

With these substitutions we obtain

$$\overline{\mathbf{W}} = \left(\frac{1 + \cos\rho}{2}\right)\hat{\mathbf{z}} \tag{65a}$$

$$F = \frac{1}{\sigma_{\text{tot}}^2}\,\text{diag}(a, a, b) \tag{65b}$$

$$R_{\hat{\overline{\mathbf{W}}}} = \sigma_{\text{tot}}^2\left(\frac{2}{1 + \cos\rho}\right)^2 \text{diag}(a, a, 0) \tag{65c}$$

$$\sigma_{\text{tot}}^2 = \sigma^2/N \tag{65d}$$

where

$$\text{diag}(a, b, c) \equiv \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{66}$$

and

$$a = (4 + \cos\rho + \cos^2\rho)/6 \tag{67a}$$

$$b = (2 - \cos\rho - \cos^2\rho)/3 \tag{67b}$$

Note that as $\rho \longrightarrow 0$ we have that $a \longrightarrow 1$ and $b \longrightarrow 0$.

From these results we may compute the covariance matrix for the QUEST and SCAD algorithms given by equations (8) and (58) to obtain

$$P^{\text{QUEST}} = \frac{\sigma^2}{N}\,\text{diag}(1/a, 1/a, 1/b) \tag{68a}$$

$$P^{\text{SCAD}} = \frac{\sigma^2}{N}\left\{\left(\frac{2}{1 + \cos\rho}\right)^2 \text{diag}(a, a, 0) + \text{diag}(0, 0, 1/b)\right\} \tag{68b}$$

The two covariance matrices are both diagonal in the model case examined.

We note first that the standard deviation about the boresight is identical for this example for both the QUEST and the SCAD algorithms

$$\frac{\sigma_b^{\text{SCAD}}}{\sigma_b^{\text{QUEST}}} = 1 \tag{69}$$

where the subscript b stands for "boresight." Thus, not only do we recover the information on the attitude about the boresight, we recover it completely.

The ratio of the standard deviation of the SCAD algorithm to that of the QUEST algorithm for attitude errors about axes normal to $\hat{\overline{\mathbf{W}}}$ is

$$\frac{\sigma_t^{\text{SCAD}}}{\sigma_t^{\text{QUEST}}} = \frac{a}{|\overline{\mathbf{W}}|} = \frac{2}{1 + \cos\rho}\,\frac{4 + \cos\rho + \cos^2\rho}{6} \tag{70}$$

where the subscript t stands for "transverse." Note that the boresight variance for this special case, receives its entire contribution from the second term of equation (58), while the transverse variance arises entirely from the first term.

Since we are interested in this algorithm primarily for a sensor of limited field of view, we define

$$\delta \equiv 1 - \cos\rho \tag{71}$$

Then

$$\frac{\sigma_t^{\text{SCAD}}}{\sigma_t^{\text{QUEST}}} = \frac{2}{2-\delta}\frac{6-3\delta+\delta^2}{6} = 1 + \frac{1}{6}\frac{\delta^2}{1-\delta/2} \tag{72}$$

As ρ increases from 0 to π, δ increases from 0 to 2. For narrow fields of view we have approximately $\delta \approx \rho^2/2$, and we obtain

$$\frac{\sigma_t^{\text{SCAD}}}{\sigma_t^{\text{QUEST}}} \approx 1 + \rho^4/24 + O(\rho^6/96) \tag{73}$$

For limited fields of view, the relative loss in accuracy compared to the QUEST algorithm is imperceptible. Table 1 gives the relative loss of accuracy for several fields of view. Note that star cameras with fields of view greater than ± 90 deg would require multiple heads, similarly to full-sphere vector Sun sensors, a situation very unlikely ever to occur. Nonetheless, the entries for the fields of view in the above table which are greater than 90 deg express a geometric truth if not a practical one.

Sequential Rotations for SCAD

The special case $\hat{\hat{\mathbf{W}}} = -\hat{\hat{\mathbf{V}}}$, noted just before equation (26) corresponds to a rotation through an angle of π about an axis perpendicular to $\hat{\hat{\mathbf{V}}}$. Thus, SCAD has the same singularity in the construction of the attitude solution as does the QUEST algorithm [3], and it is sidestepped in the same way, namely, by the method of sequential rotations [3]. The implementation of sequential rotations is simpler for SCAD than for QUEST. Define the four 3×3 matrices R_i, $i = 1, \ldots, 4$, according to

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{74ab}$$

$$R_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{74cd}$$

TABLE 1. Comparison of the SCAD and QUEST Algorithms

Field of View	$\sigma_t^{\text{SCAD}}/\sigma_t^{\text{QUEST}}$	$\sigma_b^{\text{SCAD}}/\sigma_b^{\text{QUEST}}$
± 6 deg	1.000004	1.000000
± 12 deg	1.00007	1.000000
± 30 deg	1.003	1.000000
± 60 deg	1.06	1.000000
± 90 deg	1.33	1.000000
±120 deg	2.50	1.000000
±150 deg	9.67	1.000000
±168 deg	19.	1.000000
±174 deg	75.	1.000000
±179 deg	2650.	1.000000
±180 deg	∞	1.000000

corresponding respectively to rotations through π about the x-, y-, or z-axes or the null rotation. We then examine successively

$$\hat{\hat{\mathbf{V}}}(i) \equiv R_i \hat{\hat{\mathbf{V}}}, \qquad i = 1, 2, 3, 4 \tag{75}$$

and compute

$$\alpha_i = \hat{\hat{\mathbf{W}}} \cdot \hat{\hat{\mathbf{V}}}(i), \qquad i - 1, 2, 3, 4 \tag{76}$$

For at least one of the four values of i, say j, we must have [12]

$$\alpha_j \geq -\cos(\pi/3) = -1/2 \tag{77}$$

We then compute $A_0^*(j)$ from equation (26) with $\hat{\hat{\mathbf{V}}}$ replaced with $\hat{\hat{\mathbf{V}}}(j)$. The desired A_0^* is then given by

$$A_0^* = A_0^*(j) R_j \tag{78}$$

The calculation of θ_0^* is as before.

Discussion

Despite its good properties, it is hard to imagine that SCAD will ever displace QUEST from the favored position it now holds or that it will displace any of the more recent optimal estimators for the Wahba Problem [4]. However, it is interesting as an estimator, and it gives us important insights to the workings of the optimal estimators. But SCAD is nonetheless a solid algorithm for star-camera attitude determination. For speed SCAD must be in the same ballpark as QUEST since the operations it performs are so similar (Markley finds that it runs somewhat slower than QUEST) and it is certainly much faster than some of the optimal estimators of reference [4].

The restriction of SCAD to star-cameras with fields of view less than 90 deg is hardly a disadvantage, since few star-cameras have a field of view which exceed a quarter of that. Fields of view even close to 90 deg are unlikely ever to appear since such a field of view would require optical systems that would lead to significant distortion at such large angles.

SCAD possesses a figure of merit like TASTE in QUEST for judging the quality of the fit achieved between the reference vectors and the observation vectors. Such a figure of merit has shown itself to be much more important in mission support than the speed of the attitude computation [13]. One constructs the figure of merit for SCAD from equations (24) and (34) through (37), namely

$$J(A_o^*, \psi^*) = \frac{1}{\sigma_{\text{tot}}^2} - \hat{\hat{\mathbf{W}}}^{\text{T}} B \hat{\hat{\mathbf{W}}} - \sqrt{c^2 + s^2} \tag{79}$$

which is no more unaesthetic than the computation of $1 - \lambda_{\text{max}}$ in QUEST. The random variable $J(A_o^*, \psi^*)$ should have a $\chi^2(2N - 3)$ distribution.

In summary, we have developed a suboptimal algorithm for estimating the attitude which is almost as accurate as QUEST for calculating three-axis attitude from star-camera observations and which avoids the iterative process for the overlap eigenvalue.

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References

[1] SHUSTER, M. D. "A Survey of Attitude Representations," *The Journal of the Astronautical Sciences,* Vol. 41, No. 4, October–December 1993, pp. 439–517.

[2] WAHBA, G. "Problem 65-1: A Least Squares Estimate of Spacecraft Attitude," *SIAM Review,* Vol. 7, No. 3, July 1965, p. 409.

[3] SHUSTER, M. D. and OH, S. D. "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control,* Vol. 4, No. 1, January–February 1981, pp. 70–77.

[4] MARKLEY, F. L. and MORTARI, D. "Quaternion Estimation Using Vector Observations," *The Journal of the Astronautical Sciences,* Vol. 48, Nos. 2 and 3, April–September 2000, pp. 359–380.

[5] SHUSTER, M. D. "Maximum Likelihood Estimation of Spacecraft Attitude," *The Journal of the Astronautical Sciences,* Vol. 37, No. 1, January–March 1989, pp. 79–88.

[6] FISHER, H. L., MUSSER, K. L., and SHUSTER, M. D. "Coarse Attitude Determination from Earth Albedo Measurements," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. 29, No. 1, Jan.–Feb. 1993, pp. 22–26.

[7] BROZENEC, T. F. and BENDER, D. J. "A Simple Suboptimal Least-Squares Algorithm for Attitude Determination with Multiple Sensors," *Proceedings, Flight Mechanics/Estimation Theory Symposium,* NASA Goddard Space Flight Center, Greenbelt, Maryland, May 17–19, 1994, pp. 529–543.

[8] SHUSTER, M. D. "A Suboptimal Algorithm for Attitude Determination from Multiple Star Cameras," *Advances in the Astronautical Sciences,* Vol. 105, 2000, pp. 383–393.

[9] SHUSTER, M. D. "Attitude Estimation from the Measurement of a Direction and an Angle," *Revista Brasileira de Ciéncias Mecânicas,* Vol. 16, Special Issue, 1994, pp. 19–23.

[10] SHUSTER, M. D. "Deterministic Three-Axis Attitude Estimation," submitted to *The Journal of the Astronautical Sciences.*

[11] SORENSON, H. W. *Parameter Estimation,* Marcel Dekker, New York, 1980.

[12] SHUSTER, M. D. and NATANSON, G. A. "Quaternion Computation from a Geometric Point of View," *The Journal of the Astronautical Sciences,* Vol. 41, No. 4, October–December 1993, pp. 545–556.

[13] SHUSTER, M. D. "In Quest of Better Attitudes," *Advances in the Astronautical Sciences,* Vol. 108, 2001, pp. 2089–2117.

Appendix: Implementation of SCAD

The following are the steps for computing the optimal attitude using the SCAD algorithm:

- From the input data, $\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$, the corresponding reference vectors, $\hat{\mathbf{V}}_k$, $k = 1, \ldots, N$, and the sensor variances, σ_k^2, $k = 1, \ldots, N$, compute: (1) σ_{tot}^2 according to equation (6); (2) the weights a_k, $k = 1, \ldots, N$, according to equation (5); (3) $\overline{\mathbf{W}}$ and $\overline{\mathbf{V}}$ according to equation (13); and (4) the matrix C according to equation (32).
- From these quantities compute the unit vectors $\hat{\overline{\mathbf{W}}}$ and $\hat{\overline{\mathbf{V}}}$ according to equation (15) for $\hat{\overline{\mathbf{W}}}$ and similarly for $\hat{\overline{\mathbf{V}}}$.
- Compute F according to equation (55) using, of course, $\hat{\overline{\mathbf{W}}}$ for $\hat{\overline{\mathbf{W}}}^{\text{true}}$.

- Compute $\hat{\bar{\mathbf{W}}}^{\mathrm{T}} F \hat{\bar{\mathbf{W}}}$ and verify that it is does not vanish. If $\hat{\bar{\mathbf{W}}}^{\mathrm{T}} F \hat{\bar{\mathbf{W}}}$ is close to vanishing, then the attitude is not observable from the data and the computation is terminated. Otherwise, continue.
- Compute A_o^* according to the following method:
  — If $\hat{\bar{\mathbf{W}}} \cdot \hat{\bar{\mathbf{V}}} < -1 + \epsilon$ for some predetermined value of ϵ, set $A_o^* = R(\hat{\mathbf{n}}, \pi)$, where $\hat{\mathbf{n}} = \hat{\mathbf{m}} \times \hat{\bar{\mathbf{W}}} / |\hat{\mathbf{m}} \times \hat{\bar{\mathbf{W}}}|$, and $\hat{\mathbf{m}}$ is the representation of the sensor coordinate axis for which $|\hat{\mathbf{m}} \cdot \hat{\bar{\mathbf{W}}}|$ is smallest.
  — Otherwise, use any of equations (26a) through (29) to generate A_o^* either directly or via the quaternion or Rodrigues vector.
- Compute B according to equation (32), and $\mathbf{Z}$, s, and c according to equation (35).
- Compute ψ^* according to equation (37) and A^* according to equation (19).
- Compute $P_{\theta\theta}^{\mathrm{SCAD}}$ according to equation (58) or using the formula for the QUEST covariance matrix, to which it is exceedingly close.

This completes the SCAD algorithm.

The above implementation was given with a mind to generating the direction-cosine matrix as final output. If it is desired to generate instead either the quaternion or the Rodrigues vector as final output, one requires the formulae:

$$\cos(\psi^*/2) = \sqrt{\frac{1 + \cos\psi^*}{2}} \quad \text{and} \quad \sin(\psi^*/2) = \frac{\sin\psi^*}{2\cos(\psi^*/2)} \tag{A1}$$

whence

$$\bar{q}(\hat{\bar{\mathbf{W}}}, \psi^*) = \begin{bmatrix} \sin(\psi^*/2)\,\hat{\bar{\mathbf{W}}} \\ \cos(\psi^*/2) \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma}(\hat{\bar{\mathbf{W}}}, \psi^*) = \frac{\sin(\psi^*/2)}{\cos(\psi^*/2)}\,\hat{\bar{\mathbf{W}}} \tag{A2}$$

and combining these directly with $\bar{q}_0^*$ and $\boldsymbol{\rho}_o^*$ according to the prescriptions in [1].