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Stellar Aberration and Parallax: A Tutorial

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Abstract

Formulas for the observed distortions of star directions due to the motion of the spacecraft are developed within a framework suited to attitude determination activities. In particular, the expressions for these distortions are given in terms of a direction-cosine matrix and a rotation vector. Different mechanizations of the stellar aberration and parallax effects and their correction are discussed, as are details and trade-offs in the implementation of the algorithms in attitude determination.

Introduction

The use of star cameras[2] in spacecraft attitude determination requires that account be taken of the effect of the motion of the spacecraft on measured star directions. To ignore such effects for Earth-orbiting spacecraft is to introduce errors as large as 26 arc seconds into the attitude estimate. Unfortunately, while most three-axis attitude-determination methods nowadays require vector inputs, the effect on these vectors from the aberration of starlight and of parallax are usually presented in books on Observational Astronomy [1] in terms of scalar quantities, for which the correct interpretation of signs is not always transparent. Also, the correct derivation of stellar aberration is possible only within the framework of Special Relativity, which requires a familiarity with Physics usually foreign to aerospace engineers.

A purely Newtonian computation of the aberration of starlight will, in fact, yield the same result (to order v/c) as Special Relativity. The classical Newtonian result has been known since 1727 [2]. A Newtonian derivation appears also in the *Explanatory Supplement to the Astronautical Almanac*[3] [3]. Such an approach,

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[2]The term "star camera" denotes here any sensor which measures the direction of a star.

[3]The *Explanatory Supplement to the Astronautical Almanac* gives both Newtonian and relativistic derivations, both of which are too short to be really useful *explanatorily*.

unfortunately, rests on intermediate steps which contain physical errors as large as the stellar aberration effect itself. For example, while Newtonian Mechanics gives the correct direction of the velocity for the aberrant photon, it also gives a value for the magnitude of that velocity which can be wrong (fractionally) by the same amount as the real effect on the direction. This writer does not wish to present incorrect intermediate results simply because one final result happens to agree with the correctly derived expression. Such an approach in 1727 can be excused but not the application of that approach a century after the development of Special Relativity. Some physicists may even argue that the correct calculation requires General Relativity. However, the additional improvements due to General Relativity are of order υ^2/c^2, which can be ignored in attitude if not in orbit work.

The present work derives the effect of the aberration of starlight from very simple considerations and presents this effect in terms which are immediately relevant to attitude determination, that is, as a direction-cosine matrix and a rotation vector. For completeness, the much simpler effect of parallax is presented in a similar fashion. The phenomena of aberration and parallax are given a more rigorous and detailed treatment than is usual for these topics in an Engineering publication. This article is tutorial, but it is not necessarily elementary. Beyond greater rigor and detail, this note makes little claim to originality except that it makes maximal use of attitude representations rather than the older conventions of Observational Astronomy and addresses implementation issues specific to attitude determination systems.

The presentation of stellar aberration takes the Lorentz transformation as given and does not derive this transformation. The formula is simple enough, and elementary books abound which derive this equation from first principles, one of the best of which, by Einstein himself [4], is listed among the references.

There are two approaches to dealing with parallax and aberration effects in attitude determination. Either one removes the effects from the star-camera data or one adds them to the star-catalog directions, which are free of these effects due to the painstaking efforts of observational astronomers. Since one begins, naturally, with a derivation of the effects themselves, it makes more sense to begin with the second alternative.

Reference Frames

Generally, three-axis attitude is determined by comparing the 3×1 matrix representations $\hat{\mathbf{V}}_i$, $i = 1, \ldots, N$, of a set of directions with respect to some inertial reference frame with the 3×1 matrix representations $\hat{\mathbf{W}}_i$, $i = 1, \ldots, N$, of this same set of directions with respect to a reference frame fixed in the spacecraft (the body frame) [5, 6]. A caret will always denote a unit vector. When the attitude determination system includes a star camera, a convenient "fixed" inertial frame is the frame of the star catalog, whose origin and axes are determined from astronomical observations [7]. The (non-inertial) spacecraft body frame frequently has its origin at the center of mass (the barycenter) of the spacecraft. Thus, both the position and the velocity of the origin of the spacecraft body frame as well as its attitude change with time.

Consider the following four *inertial* reference frames:

1. $\mathfrak{I} = \{\boldsymbol{O}^{\mathfrak{I}}, \boldsymbol{v}^{\mathfrak{I}}, \hat{\boldsymbol{x}}^{o}, \hat{\boldsymbol{y}}^{o}, \hat{\boldsymbol{z}}^{o}\}$ is the inertial reference frame of the star catalog.[4] Here $\boldsymbol{O}^{\mathfrak{I}}$ is the origin of $\mathfrak{I}$ at t_k, and $\boldsymbol{v}^{\mathfrak{I}}$ is the velocity of $\boldsymbol{O}^{\mathfrak{I}}$ (which, of course, makes sense only with respect to some unspecified position). $\{\hat{\boldsymbol{x}}^{o}, \hat{\boldsymbol{y}}^{o}, \hat{\boldsymbol{z}}^{o}\}$ are the directions of the coordinate axes, assumed to be right-hand orthonormal. It would have been more consistent to write $\{\hat{\boldsymbol{x}}^{\mathfrak{I}}, \hat{\boldsymbol{y}}^{\mathfrak{I}}, \hat{\boldsymbol{z}}^{\mathfrak{I}}\}$, but the axis directions appear frequently in the text, and a lighter notation has been chosen for them.[5]
2. $\mathfrak{S} = \{\boldsymbol{O}^{\mathfrak{S}}, \boldsymbol{v}^{\mathfrak{S}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\} \equiv \{\boldsymbol{O}^{\mathfrak{S}}, \boldsymbol{v}^{\mathfrak{S}}, \hat{\boldsymbol{x}}^{d}, \hat{\boldsymbol{y}}^{d}, \hat{\boldsymbol{z}}^{d}\}$ (the superscript "d" (for "displaced") will be used at times to avoid possible confusion with a representation with respect to an unspecified basis) is the inertial frame whose axes are parallel to the axes of $\mathfrak{I}$ at the given epoch, but whose origin is at the origin of the spacecraft frame at time t_k, and whose velocity is the same as that of the star-catalog frame. $\mathfrak{S}$ differs from $\mathfrak{I}$ by a simple translation.
3. $\mathfrak{S}' = \{\boldsymbol{O}^{\mathfrak{S}'}, \boldsymbol{v}^{\mathfrak{S}'}, \hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}'\}$ is the inertial frame whose axes are parallel to the axes of $\mathfrak{I}$ at the given epoch, whose origin at time t_k coincides with the position of the spacecraft at time t_k, and whose velocity coincides with the velocity of the spacecraft at time t_k (relative to the star-catalog frame). $\mathfrak{S}'$ differs from $\mathfrak{S}$ by a simple boost, i.e., an instantaneous change of velocity.
4. $\mathfrak{B} = \{\boldsymbol{O}^{\mathfrak{B}}, \boldsymbol{v}^{\mathfrak{B}}, \hat{\boldsymbol{x}}^{b}, \hat{\boldsymbol{y}}^{b}, \hat{\boldsymbol{z}}^{b}\}$ is the inertial frame whose origin coincides with that of the spacecraft body frame at time t_k, whose velocity is that of the spacecraft at time t_k, and whose axes are parallel to those of the spacecraft at time t_k. $\mathfrak{B}$ differs from $\mathfrak{S}'$ by a rotation, namely the attitude rotation. $\mathfrak{B}$ is not the spacecraft body frame, which may have angular motion and linear acceleration.[6]

[4]Without an origin, the position space, with typical elements $\boldsymbol{O}$ and $\boldsymbol{P}$, is an *affine space*. An affine space is not a vector space; there is no addition rule for positions. However, with every ordered pair of positions $(\boldsymbol{O}, \boldsymbol{P})$ can be associated an abstract vector $\boldsymbol{r}$ which satisfies $\boldsymbol{P} = \boldsymbol{O} + \boldsymbol{r}$, and we can write symbolically $\boldsymbol{r} = \boldsymbol{P} - \boldsymbol{O}$, the relative position of $\boldsymbol{P}$ with respect to $\boldsymbol{O}$. The relative positions constitute an abstract vector space. These concepts are reminiscent of the conventions of Euclidean Geometry, in which $\boldsymbol{O}$ and $\boldsymbol{P}$ denote points, and $\overrightarrow{\boldsymbol{OP}}$ denotes a directed line segment from $\boldsymbol{O}$ to $\boldsymbol{P}$. Thus, the typical element of an affine space has the form $\boldsymbol{O} + \boldsymbol{u}$, where $\boldsymbol{O}$ is any member of the affine space (serving as the origin) and $\boldsymbol{u}$ is an element of the vector space. Given a basis of axis directions and a scalar product, the abstract vector $\boldsymbol{u}$ can be associated with the (numerical) 3×1 matrix $\mathbf{u} \equiv [\hat{\boldsymbol{x}} \cdot \boldsymbol{u}, \hat{\boldsymbol{y}} \cdot \boldsymbol{u}, \hat{\boldsymbol{z}} \cdot \boldsymbol{u}]^{\mathrm{T}}$, where the superscript T denotes the matrix transpose. One cannot assign a numerical value directly to elements of the affine space, which is equivalent to the statement that there are no absolute positions or velocities in Special Relativity (or in Newtonian Mechanics). However, if $\boldsymbol{O}$ is fixed, there is a one-to-one correspondence between the affine quantities and abstract vectors (between $\boldsymbol{P}$ and $\boldsymbol{r}$) and vice-versa, and once the basis is fixed, a similar correspondence between the vector space and $\boldsymbol{R}^3$, the three-dimensional continuum of real numbers. The abstract vector space is useful, because it permits the study of vectors without reference to a particular basis, the affine space is useful, because it permits the study of positions without reference to a particular origin. Note that abstract velocity "vectors" do not form a vector space, because the composition rule for finite velocities ("addition of velocities," see below) in Special Relativity is not commutative. This is not surprising, because finite (Lorentz) boost transformations do not commute unless the boost velocities are parallel, just as finite rotations do not commute unless the axes of rotation are parallel. (See below.)

[5]These axes are abstract (physical) vectors and not numerical (representations), and are set, therefore, in a bold italic typeface. Representations of vectors, such as $\hat{\mathbf{V}}_i$ and $\hat{\mathbf{W}}_i$, are set in bold (non-italic) typeface. Both of these (apart from measurement noise) are representations of the direction of the same star. Matrices, for the most part, are denoted by non-bold upper-case italic letters.

[6]This *quasi* body frame has been introduced instead of the actual (non-inertial) body frame, because Special Relativity treats only transformations between inertial coordinate frames. This subterfuge is obviously a self-delusion, since at some point the body frame must be approximated by the quasi body frame if the results of Special Relativity are to have any application to an attitude problem. Special Relativity is "special," because it is specialized to inertial frames. The General Theory of Relativity does not have this restriction, a freedom which is purchased dearly in intellectual effort.

Thus, $\mathfrak{S}$, $\mathfrak{S}'$, and $\mathfrak{B}$ have a common origin ($\boldsymbol{O}^{\mathfrak{S}} = \boldsymbol{O}^{\mathfrak{S}'} = \boldsymbol{O}^{\mathfrak{B}}$) at time t_k, and $\boldsymbol{v}^{\mathfrak{I}} = \boldsymbol{v}^{\mathfrak{S}}$, and $\boldsymbol{v}^{\mathfrak{S}'} = \boldsymbol{v}^{\mathfrak{B}}$. The connections between the four frames are illustrated below.

$$\mathfrak{B} \xleftarrow{\text{rotation}} \mathfrak{S}' \xleftarrow{\text{boost}} \mathfrak{S} \xleftarrow{\text{translation}} \mathfrak{I} \tag{1}$$

With respect to the successive transformations of the star directions from the star catalog, the transformation from the star-catalog frame to $\mathfrak{S}$ is the parallax effect; the transformation from $\mathfrak{S}$ to $\mathfrak{S}'$ is the aberration effect; and the transformation from $\mathfrak{S}'$ to the body frame $\mathfrak{B}$ is the attitude transformation. Thus, equation (1) is equivalent ideally[7] to

$$\hat{\mathbf{W}}_{i,k} \xleftarrow{\text{attitude}} \hat{\mathbf{V}}_{i,k} \xleftarrow{\text{aberration}} \hat{\mathbf{U}}_{i,k} \xleftarrow{\text{parallax}} \hat{\mathbf{S}}_{i,k} \tag{2}$$

where $\hat{\mathbf{W}}_{i,k}$ denotes the representation *in body coordinates* of the direction of star i at time t_k. Note from equation (2) $\hat{\mathbf{W}}$, $\hat{\mathbf{V}}$, $\hat{\mathbf{U}}$, and $\hat{\mathbf{S}}$ are symbols for representations of vectors observed from a specific frame and coordinatized with respect to the axis directions of that frame.

A further level of transformation can be added, namely, sensor alignment, which (in the direction of the arrows of equations (1) and (2)) would come after the attitude transformation [8, 9]. Thus,

$$\mathfrak{E} \xrightarrow{\text{alignment}} \mathfrak{B} \quad \text{and} \quad \hat{\mathbf{E}}_{i,k} \xrightarrow{\text{alignment}} \hat{\mathbf{W}}_{i,k} \tag{3}$$

The star-camera frame $\mathfrak{E} = \{\hat{\boldsymbol{x}}^e, \hat{\boldsymbol{y}}^e, \hat{\boldsymbol{z}}^e\}$ is also a body frame, and the alignment transformation is also a rotation. The star-camera frame is assumed to be the fiducial[8] body frame in the derivations, so that the additional trivial complication of equations (3) need not burden the presentation. However, the final results (for implementation) will allow $\mathfrak{E}$ and $\mathfrak{B}$ to be distinct frames.[9]

Note the direction of the arrows in equation (3). For true star directions (i.e., uncorrupted by measurement noise) $\hat{\mathbf{W}}_{i,k} = A_k \hat{\mathbf{V}}_{i,k}$, and, if the fiducial body frame is *not* the star-camera frame, $\hat{\mathbf{W}}_{i,k} = S_k \hat{\mathbf{E}}_{i,k}$, where S_k is the alignment matrix of the star camera. Our convention [8, 9] is that the direction of all transformation matrices (without an explicit transpose sign) is "toward" the *fiducial* body frame. For that reason the arrows in equations (3) point to the right. The reader should have no difficulty distinguishing $\hat{\boldsymbol{S}}$ for an abstract star direction observed from $\mathfrak{I}$, $\hat{\mathbf{S}}$ for its representation with respect to $\mathfrak{I}$, and S for the star-camera alignment matrix.

Note with caution that the aberration and parallax effects are not frame transformations themselves, although they arise from frame transformations. This fact will be repeated more than once.

[7]"Ideally," because the noise in $\hat{\mathbf{W}}_{i,k}$ originates from a different source than the noise in $\hat{\mathbf{S}}_{i,k}$, so that the two cannot be related by a transformation. Although random noise corrupts all four vectors in equation (2), one neglects the noise, generally, in the first three directions compared to that of $\hat{\mathbf{W}}_{i,k}$. When this is not possible, the additional noise terms present only a minor inconvenience, as in Ref. 5.

[8]From Latin *fiducia,* confidence, reliance, from Latin *fidere,* to trust. In the present context the closest meaning is "by common consent," which implies a common trust.

[9]Following the practice of the earliest papers on Special Relativity, The reference frames have been labeled by German *Fraktur* letters. The use of the letter $\mathfrak{S}$ (German Fraktur "S") arises from the German *(Koordinaten) System*. Fraktur "B" has been chosen for the body frame rather than "K" (for German *Körper*). The letter "S" was no longer available for the star-camera frame ("Star" is *Stern* in German), so "E" from French *étoile* has been chosen instead. "E," especially in lower case, is used frequently to denote a unit vector (because unit is *Einheit* in German).

The reference frames $\mathfrak{I}$, $\mathfrak{S}$, $\mathfrak{S}'$, $\mathfrak{B}$, and $\mathfrak{E}$ are each members of a continuum of frames, each defined at the time t_k. Only the transformation from $\mathfrak{S}$ to $\mathfrak{S}'$ requires the use of Special Relativity. The relative velocities of $\mathfrak{S}$ to the star-catalog frame and of the body frame with respect to $\mathfrak{S}'$ are sufficiently small that their treatment within the framework of classical Newtonian Mechanics causes no perceptible error.

The Lorentz Transformation and Addition of Velocities

Consider the two inertial frames $\mathfrak{S}$ and $\mathfrak{S}'$ with axes $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ and $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$, respectively. Suppose that frame $\mathfrak{S}'$ has a velocity $\mathbf{v}$ relative to $\mathfrak{S}$ of magnitude v along the x-axis with respect to frame $\mathfrak{S}$. The two frames coincide at t_k and identical clocks at the origin of each frame are synchronized at t_k. This means that the origins of the two frames are located necessarily at a point of the spacecraft trajectory only at t_k, since the two frames, having constant velocity, must move along straight lines, while the orbit of a spacecraft is certainly not straight. For simplicity, and because the final expressions will not contain the time explicitly, t_k can be set equal to zero and the connection between frame coordinates examined as a function of t.

If the speed of light c were infinite, then the transformation of coordinates from one frame to the other could be treated within the framework of classical Newtonian Mechanics with the result

$$x' = x - vt, \quad y' = y, \quad t' = t \tag{4}$$

The last equation above states that the two clocks remain synchronized. Within the framework of Special Relativity [4] the result will be qualitatively different, namely

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \tag{5}$$

which becomes equation (4), when c becomes infinite. This result, known as Lorentz transformation, was known empirically (but imperfectly—he was unaware of the time-offset term) to the Dutch physicist Henrik Lorentz at the end of the nineteenth century. Its correct form, equation (5), was first derived from basic principles by Einstein in 1905 [11] (also [4]). It is the most important result of Special Relativity.[10]

The present work considers only the lowest-order corrections due to Special Relativity. At some point general-relativistic effects may have to be considered, as they are already in orbit determination, but attitude determination has not yet reached that level of precision. The reader interested in learning more about the importance of General Relativity in Astronautics and Astrometrics is referred to reference [12].

In the following, the common notation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{6}$$

[10]The denominators $\sqrt{1 - v^2/c^2}$ in equations (5) are the expression of the celebrated phenomena of Lorentz contraction (or Fitzgerald-Lorentz contraction) and time delation. The offset term $-vx/c^2$ in the equation for time transformation expresses the phenomenon that clocks that are synchronized as observed from one inertial frame are not necessarily synchronized as observed from another.

is used to simplify the expressions.

The perceived direction of a star is the direction opposite to that of the velocity of a photon from the star reaching the observer. Hence, attention will now be directed to the transformation of velocities due to a boost, or, as it is more commonly called, the *addition of velocities* in Special Relativity.

The representation of the velocity of the photon or any object as observed from frame $\mathfrak{S}$ is $\mathbf{u} \equiv d\mathbf{s}/dt$ while that observed from frame $\mathfrak{S}'$ is $\mathbf{u}' \equiv d\mathbf{s}'/dt'$, where $\mathbf{s}$ and $\mathbf{s}'$ are the respective 3×1 matrices of coordinates of the position of the object observed from the two frames. To relate these note from equation (5) that

$$\frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \frac{dx'}{dt}\bigg/\frac{dt'}{dt} = \frac{\gamma\left(\dfrac{dx}{dt} - v\right)}{\gamma\left(1 - v\dfrac{dx}{dt}\Big/c^2\right)} = \frac{\left(\dfrac{dx}{dt} - v\right)}{\left(1 - v\dfrac{dx}{dt}\Big/c^2\right)} \tag{7}$$

In a similar manner

$$\frac{dy'}{dt'} = \frac{\dfrac{dy}{dt}}{\gamma\left(1 - v\dfrac{dx}{dt}\Big/c^2\right)}, \quad \frac{dz'}{dt'} = \frac{\dfrac{dz}{dt}}{\gamma\left(1 - v\dfrac{dx}{dt}\Big/c^2\right)} \tag{8}$$

Writing

$$\mathbf{u} = \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} \quad \text{and} \quad \mathbf{u}' = \begin{bmatrix} dx'/dt' \\ dy'/dt' \\ dz'/dt' \end{bmatrix} \tag{9}$$

the representations of the velocity of the object as observed from the two inertial frames, equations (7) and (8) become

$$u_x' = \frac{u_x - v}{1 - u_x v/c^2}, \quad u_y' = \frac{1}{\gamma}\frac{u_y}{1 - u_x v/c^2}, \quad u_z' = \frac{1}{\gamma}\frac{u_z}{1 - u_x v/c^2} \tag{10}$$

This is the formula for the addition of velocities (in this instance "subtraction" of velocities) in Special Relativity.[11]

Equation (10) holds for the transformation of the velocity of any object. The reader may verify from equation (10) that if the magnitude of $\mathbf{u}$ is c, then so is the magnitude of $\mathbf{u}'$, which is the statement that the speed of light has the same value in all inertial reference frames. This is one of the basic hypotheses of the Theory of Special Relativity, based empirically on the Michelson-Morley experiment or, at somewhat greater remove from experiment, on Maxwell's equations, and crucial in the derivation of equation (5) [4].

Denote the direction of $\mathbf{v}$ by $\hat{\mathbf{v}}$, then the projection operator onto the one-dimensional space parallel to $\mathbf{v}$ is $\hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathrm{T}}$. Likewise, the projection operator onto the two-dimensional space perpendicular to $\mathbf{v}$ is $(I_{3\times3} - \hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathrm{T}})$. Noting this, we may write the formula for the transformation of velocities due to an arbitrary boost $\mathbf{v}$, $|\mathbf{v}| < c$, and arbitrary velocity $\mathbf{u}$, $\mathbf{u} \leq c$, of the object with respect to $\mathfrak{I}$ as

[11] In the example treated in textbooks on Special Relativity, one solves for $\mathbf{u}$ as a function of $\mathbf{u}'$ and $\mathbf{v}$; hence, the minus signs in equation (10) correspond to plus signs in the usual textbook example.

$$\mathbf{u}' = \frac{1}{1 - \mathbf{u}\cdot\mathbf{v}/c^2}\left[\hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathrm{T}}(\mathbf{u} - \mathbf{v}) + \frac{1}{\gamma}(I_{3\times 3} - \hat{\mathbf{v}}\hat{\mathbf{v}}^{\mathrm{T}})u\right] \tag{11}$$

Aberration of Starlight

Let $\hat{\mathbf{V}}$ denote the direction of a star as observed from frame $\mathfrak{S}'$, and let $\hat{\mathbf{U}}$ be the direction of the same star as observed from frame $\mathfrak{S}$. Then, since the direction of a star from an observer is opposite to the direction of the velocity of a photon from a star to the observer, it follows that (again suppressing subscripts)

$$\mathbf{u} = -c\hat{\mathbf{U}}, \quad \text{and} \quad \mathbf{u}' = -c\hat{\mathbf{V}} \tag{12}$$

the speed of light being the same in the frames $\mathfrak{S}$ and $\mathfrak{S}'$. Substituting these expressions into equation (11) leads to

$$\hat{\mathbf{V}} = \frac{1}{1 + \boldsymbol{\beta}\cdot\hat{\mathbf{U}}}\left[\hat{\mathbf{U}} + \boldsymbol{\beta} - \left(\frac{1}{\gamma} - 1\right)\hat{\boldsymbol{\beta}} \times (\hat{\boldsymbol{\beta}} \times \hat{\mathbf{U}})\right] \tag{13}$$

where $\boldsymbol{\beta}$, following common practice, has been defined as

$$\boldsymbol{\beta} \equiv \mathbf{v}/c \tag{14}$$

The magnitude of $\boldsymbol{\beta}$ is never greater than unity.

The speed of a spacecraft in circular low-Earth orbit (say, 450 km) with respect to a frame fixed in the Earth is approximately 7.6 km/sec. The speed of the Earth in its orbit about the Sun is approximately 30 km/sec. Hence, the speed of the spacecraft with respect to the frame $\mathfrak{S}$ is necessarily less than 38 km/sec. The speed of light is approximately 3×10^5 km/sec. Thus, for spacecraft in low Earth orbit

$$\beta \lesssim 10^{-4} \tag{15}$$

Expanding equation (13) to linear order in β leads first to

$$\hat{\mathbf{V}} = [1 - \boldsymbol{\beta}\cdot\hat{\mathbf{U}}][\boldsymbol{\beta} + \hat{\mathbf{U}}] + O(\beta^2) \tag{16}$$

and finally

$$\hat{\mathbf{V}} = \hat{\mathbf{U}} - \hat{\mathbf{U}} \times (\hat{\mathbf{U}} \times \boldsymbol{\beta}) + O(\beta^2) \tag{17}$$

or

$$\hat{\mathbf{V}} = (I_{3\times 3} + [[\boldsymbol{\beta} \times \hat{\mathbf{U}}]])\hat{\mathbf{U}} + O(\beta^2) \tag{18}$$

where $[[\mathbf{u}]]$ is the 3×3 antisymmetric matrix[12]

$$[[\mathbf{u}]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{19}$$

The first expression in parentheses in equation (18) is simply (to first order) the direction-cosine matrix of an infinitesimal rotation [10] with rotation vector [10]

$$\delta\boldsymbol{\theta}_{\text{aberration}} = \boldsymbol{\beta} \times \hat{\mathbf{U}} \tag{20}$$

[12]Note $[[\mathbf{u}]]\mathbf{v} = -\mathbf{u} \times \mathbf{v}$. Some authors prefer the matrix $[\mathbf{u}\times] \equiv -[[\mathbf{u}]]$.

Thus, the effect of stellar aberration is

$$\begin{aligned}\hat{\mathbf{V}} &= \hat{\mathbf{U}} + [[\delta\boldsymbol{\theta}_{\text{aberration}}]]\hat{\mathbf{U}} \\ &= \hat{\mathbf{U}} - \delta\boldsymbol{\theta}_{\text{aberration}} \times \hat{\mathbf{U}}\end{aligned} \tag{21}$$

The expression in parentheses in equation (18) can be made exactly proper orthogonal by recomputing the direction-cosine matrix from the appropriate formula [10] as a function of $\delta\boldsymbol{\theta}_{\text{aberration}}$, for example, and the rotation performed accordingly. Thus

$$C(\delta\boldsymbol{\theta}) = I + \frac{\sin|\delta\boldsymbol{\theta}|}{|\delta\boldsymbol{\theta}|}[[\delta\boldsymbol{\theta}]] + \frac{(1 - \cos|\delta\boldsymbol{\theta}|)}{|\delta\boldsymbol{\theta}|^2}[[\delta\boldsymbol{\theta}]]^2 \tag{22a}$$

and

$$\hat{\mathbf{V}} = C(\delta\boldsymbol{\theta}_{\text{aberration}})\hat{\mathbf{U}} \tag{22b}$$

or $\hat{\mathbf{V}}$ as given by equation (18) or (21) after discarding the terms of order β^2 and higher may be simply normalized without loss of accuracy.

No accuracy is lost by linearizing equation (13). This follows not only from the smallness of β (because at the present limits of technology attitude sensors cannot detect terms of order β^2) but even more fundamentally from the fact that Special Relativity itself is correct only to linear order in β when applied to orbiting bodies. This is because for an orbiting body (in this case primarily the spacecraft about the Earth) quadratic effects in Special Relativity are on the same order as the gravitational effects, which can be computed only within the framework of General Relativity.

The axis vectors of $\mathfrak{S}$ and $\mathfrak{S}'$ are parallel, yet the representations $\hat{\mathbf{U}}$ and $\hat{\mathbf{V}}$ are different. This is because the physical vector itself has changed. Thus, the boost causes an *active* rotation of the star direction [10]. At the same time, this change is due entirely to a change in the perspective of the observer; hence, it is *passive*. This is perplexing if one's intuition is based on experience with true rotations. To obtain a truly unambiguous notation for representations of vectors, one which admits non-rotational effects, both the frame of the observer and the axis directions for the coordinization of the vectors must be specified separately. Generally, we will refer to a vector or the representation of a vector as observed from a given frame. Unless otherwise indicated or implied, the representations will be with respect to the axis directions of the frame of observation. Thus $\mathbf{u}$ and $\mathbf{u}'$ were vector representations of the photon velocity as observed from $\mathfrak{S}$ and $\mathfrak{S}'$, respectively (but each was also a representation with respect to either of the two frames, since the axis directions in the two frames are identical).

The effect of the aberration of starlight is on the order of β, which, as has been noted, for low-Earth orbit is as much as 10^{-4} (including the orbital motion of the Earth about the Sun). This leads to aberrations of as much as 130 microradians or about 26 arc seconds.[13]

[13]For Earth-pointing spacecraft equipped with a star camera the latter is often mounted so that the boresight points toward the zenith in order to minimize light interference from the Earth albedo. This, however, maximizes the aberration of observed star directions, which would be minimized if the star camera boresight were parallel or antiparallel to the spacecraft velocity. For geostationary spacecraft equipped with a star camera, the star-camera boresight is often parallel to the orbit normal, so that it can observe the pole star, which also maximizes the aberration.

Parallax

Spacecraft attitude is estimated by comparing $\hat{\mathbf{V}}_{i,k}$, the direction of star i at time t_k observed from and coordinatized with respect to frame $\mathfrak{S}'$, with $\hat{\mathbf{W}}_{i,k}$, the star direction observed from and coordinatized with respect to the spacecraft body frame. The star catalog supplies the direction of this star, $\hat{\mathbf{S}}_{i,k}$, observed from $\mathfrak{I}$. In order to compute $\hat{\mathbf{V}}_{i,k}$ for attitude determination, it will be necessary first to compute $\hat{\mathbf{U}}_{i,k}$, the representation of the star direction observed from $\mathfrak{S}$. The transformation from $\hat{\mathbf{S}}_{i,k}$ to $\hat{\mathbf{U}}_{i,k}$ is the parallax effect. This effect is very simple in origin, far simpler than aberration.

Let $\boldsymbol{P}$ be the position of a star. Then in the vector space of relative positions (but expressed in terms of affine positions) (see footnote 4),

$$\boldsymbol{P} - \boldsymbol{O}^{\mathfrak{S}} = (\boldsymbol{P} - \boldsymbol{O}^{\mathfrak{I}}) - (\boldsymbol{O}^{\mathfrak{S}} - \boldsymbol{O}^{\mathfrak{I}}) \tag{23a}$$

or, equivalently,

$$\boldsymbol{Q} = \boldsymbol{R} - \boldsymbol{r} \tag{23b}$$

where $\boldsymbol{Q}$ is the (abstract) position of the star relative to $\boldsymbol{O}^{\mathfrak{S}}$, $\boldsymbol{R}$ is the (abstract) position of the star relative to $\boldsymbol{O}^{\mathfrak{I}}$, and $\boldsymbol{r}$ is the (abstract) position of the spacecraft relative to $\boldsymbol{O}^{\mathfrak{I}}$. ("Abstract," because they are not representations of a vector with respect to a basis.)

Writing $\boldsymbol{Q} = |\boldsymbol{Q}|\hat{\boldsymbol{Q}} \equiv Q\hat{\boldsymbol{Q}}$ and $\boldsymbol{R} = |\boldsymbol{R}|\hat{\boldsymbol{R}} \equiv R\hat{\boldsymbol{R}}$, it follows that

$$\begin{aligned}
\hat{\boldsymbol{Q}} &= \frac{\hat{\boldsymbol{R}} - \boldsymbol{r}/R}{|\hat{\boldsymbol{R}} - \boldsymbol{r}/R|} \\
&= \frac{\hat{\boldsymbol{R}} - \boldsymbol{r}/R}{\sqrt{1 - 2(\hat{\boldsymbol{R}} \cdot \boldsymbol{r}/R) + r^2/R^2}} \\
&= \hat{\boldsymbol{R}} - \boldsymbol{r}/R + (\hat{\boldsymbol{R}} \cdot \boldsymbol{r}/R)\hat{\boldsymbol{R}} + \boldsymbol{O}(r^2/R^2) \\
&= \hat{\boldsymbol{R}} + \hat{\boldsymbol{R}} \times \left(\hat{\boldsymbol{R}} \times \frac{\boldsymbol{r}}{R}\right) + \boldsymbol{O}(r^2/R^2)
\end{aligned} \tag{24}$$

The representation of $\hat{\boldsymbol{R}}$ with respect to $\mathfrak{I}$ is $\hat{\mathbf{S}}$, and the representation of $\hat{\boldsymbol{Q}}$ with respect to $\mathfrak{S}$ is $\hat{\mathbf{U}}$. However, since the axis directions of $\mathfrak{S}$ are parallel to those of $\mathfrak{I}$, it follows also that the representation of $\hat{\boldsymbol{Q}}$ with respect to $\mathfrak{I}$ is also $\hat{\mathbf{U}}$. The representation of $\boldsymbol{r}$ with respect to $\mathfrak{I}$ is just $\mathbf{r}$. Thus, discarding terms of order r^2/R^2 and higher ($r/R < 4 \times 10^{-6}$, see below), the representation of equation (24) with respect to $\mathfrak{I}$ (or $\mathfrak{S}$) becomes

$$\begin{aligned}
\hat{\mathbf{U}} &= \hat{\mathbf{S}} + \hat{\mathbf{S}} \times \left(\hat{\mathbf{S}} \times \frac{\mathbf{r}}{R}\right) \\
&= \left(I_{3\times3} + \left[\left[\hat{\mathbf{S}} \times \frac{\mathbf{r}}{R}\right]\right]\right)\hat{\mathbf{S}}
\end{aligned} \tag{25}$$

so that the rotation vector for the parallax effect is given by

$$\delta\boldsymbol{\theta}_{\text{parallax}} = \hat{\mathbf{S}} \times \frac{\mathbf{r}}{R} \tag{26}$$

and, similarly to equation (21)

$$\hat{\mathbf{U}} = \hat{\mathbf{S}} + [[\delta\boldsymbol{\theta}_{\text{parallax}}]]\hat{\mathbf{S}}$$
$$= \hat{\mathbf{S}} - \delta\boldsymbol{\theta}_{\text{parallax}} \times \hat{\mathbf{S}} \tag{27}$$

The displacement $\mathbf{r}$ is the position of the spacecraft measured from the origin of the star-catalog reference frame, which is located at the barycenter of the solar system at some epoch.[14] Thus, the magnitude of $\mathbf{r}$ is roughly the radius of the Earth's orbit, or 1.5×10^8 km. The distance to α-Centauri, the nearest star, is roughly 4.3 light-years or about 4.0×10^{13} km. Hence, $r/R \lesssim 3.7 \times 10^{-6}$, and the magnitude of the parallax effect can be as large as 3.7 microradians, or 0.7 arc seconds. Star catalogs for attitude work sometimes exclude stars closer to the Earth than 100 light-years, so that the parallax effect for mission-catalog stars is smaller than 0.03 arc seconds. All the same, even for α-Centauri the effect of parallax is much smaller, in general, than that of aberration.[15]

Combined Effects

Since both distortions are very small, they may be combined with little error into a single rotation with rotation vector given by

$$\delta\boldsymbol{\theta}_{\text{aberration-parallax}} = \left(\boldsymbol{\beta} - \frac{\mathbf{r}}{R}\right) \times \hat{\mathbf{S}} = \left(\frac{\mathbf{v}}{c} - \frac{\mathbf{r}}{R}\right) \times \hat{\mathbf{S}} \tag{28}$$

The commutation error from this approximation is on the order of 10^{-4} arc seconds. The reference direction $\hat{\mathbf{V}}$ which must be compared with the direction $\hat{\mathbf{W}}$ observed by the star camera for attitude determination is then

$$\hat{\mathbf{V}} = \text{unit}(\hat{\mathbf{S}} - \delta\boldsymbol{\theta}_{\text{aberration-parallax}} \times \hat{\mathbf{S}}) \tag{29}$$

where unit$(\cdot)$ denotes the function which unitizes its "vector" argument. To emphasize the fact that these corrections are frame-dependent, that $\boldsymbol{\beta}_k$, $\mathbf{v}_k$, and $\mathbf{r}_k$ depend on the time, that $R_{i,k}$ depends on the time and the star, and that the vectors are representations with respect to a particular frame, equations (20), (26), (28), and (29) can be written with greater precision in their fully subscripted and superscripted splendor as

$$\delta\boldsymbol{\theta}^d_{\text{aberration},\,i,\,k} = \boldsymbol{\beta}^d_k \times \hat{\mathbf{U}}_{i,k}, \quad \delta\boldsymbol{\theta}^o_{\text{parallax},\,i,\,k} = -\frac{\mathbf{r}^o_k}{R_{i,k}} \times \hat{\mathbf{S}}_{i,k}, \tag{20$'$, 26$'$}$$

$$\delta\boldsymbol{\theta}^o_{\text{aberration-parallax},\,i,\,k} = \left(\boldsymbol{\beta}^o_k - \frac{\mathbf{r}^o_k}{R_{i,k}}\right) \times \hat{\mathbf{S}}_{i,k} = \left(\frac{\mathbf{v}^o_k}{c} - \frac{\mathbf{r}^o_k}{R_{i,k}}\right) \times \hat{\mathbf{S}}_{i,k}, \tag{28$'$}$$

and

$$\hat{\mathbf{V}}_{i,k} = \text{unit}(\hat{\mathbf{S}}_{i,k} - \delta\boldsymbol{\theta}^o_{\text{aberration-parallax},\,i,\,k} \times \hat{\mathbf{S}}_{i,k}) \tag{29$'$}$$

The superscripts o and d here denote that the representations are with respect to $\mathfrak{I}$ and $\mathfrak{S}$, respectively. There is no need for a superscript on the star directions, since they are already defined to be representations of vectors observed from and with

[14]Before 1984 the origin of the star-catalog reference frame was usually the geocenter at some epoch, as noted in Wertz [7]. This doubled the possible size of the parallax effect.

[15]"In general," because there is always a geometry, not necessarily achievable for the given spacecraft orbit, for which the aberration of a particular star direction is zero.

respect to specific frames. Note that equations (20′), (26′) and (28′), and their unprimed counterparts, are consistent only to linear order in β_k and $\mathbf{r}_k/R_{i,k}$.

Note once again that equation (28) or (28′) gives the distortion due to aberration and parallax of the star-catalog direction $\hat{\mathbf{S}}_{i,k}$, so that the attitude matrix from frame $\mathfrak{S}'$ to the spacecraft body frame can be computed. However, the rotation vector of equation (20′), (26′) or (28′) is not the rotation vector for the rotation of *coordinate axes* from one frame to another. Since the transformations of the *frames* are not rotations, no such quantities exist. The direction-cosine matrix for each effect depends, for example, on the direction of the individual star. Also, because $\delta\boldsymbol{\theta}^o_{\text{parallax},i,k}$, for example, is applicable only to $\hat{\mathbf{S}}_{i,k}$, it follows that $\delta\boldsymbol{\theta}^o_{\text{parallax},i,k}$ is not unique.[16] Clearly, a term $a\hat{\mathbf{S}}_{i,k}$ with a arbitrary (but infinitesimal so that the linear approximation for the direction-cosine matrix holds) may be added to the right member of equation (26′) without altering the action of $\delta\boldsymbol{\theta}^o_{\text{parallax},i,k}$ on $\hat{\mathbf{S}}_{i,k}$, and similarly for the rotation vectors of equations (20′) and (28′). An analogous result holds for the corresponding direction-cosine matrices.

Likewise, because the distortion due to parallax and aberration is different for every star, the spacecraft attitude cannot be estimated accurately by first neglecting the lack of distortion in the star-catalog directions and then "adjusting" the computed attitude afterward, say, by a rotation corresponding to the distortion of a (hypothetical) star observed along the star-camera boresight. Such an approximate treatment of aberration applied to a star camera with an 8-deg-by-8-deg field of view can introduce errors as large as 2.0 arc seconds into the attitude estimate if the measurements are largely clustered in one corner of the field of view. Similar arguments apply also if, instead of not distorting catalog directions, the star-camera measurements are not individually corrected for aberration and parallax. (See below).

Body-Referenced and Sensor-Referenced Effects

A different point of view can be exercises for the calculation of both the stellar aberration and parallax effects. The inertial coordinate system of the star catalog is arbitrary to a large degree. Consider, therefore, a different inertial coordinate system for the star catalog which has the same origin and velocity as $\mathfrak{I}$ but whose axes are rotated by the attitude matrix. Thus, the axis vectors of the new inertial reference frame $\mathfrak{S}''$ are

$$\hat{\mathbf{x}}'' = (A_k)_{11}\hat{\mathbf{x}}^o + (A_k)_{12}\hat{\mathbf{y}}^o + (A_k)_{13}\hat{\mathbf{z}}^o \tag{30a}$$

$$\hat{\mathbf{y}}'' = (A_k)_{21}\hat{\mathbf{x}}^o + (A_k)_{22}\hat{\mathbf{y}}^o + (A_k)_{23}\hat{\mathbf{z}}^o \tag{30b}$$

$$\hat{\mathbf{z}}'' = (A_k)_{31}\hat{\mathbf{x}}^o + (A_k)_{32}\hat{\mathbf{y}}^o + (A_k)_{33}\hat{\mathbf{z}}^o \tag{30c}$$

where A_k is the true value of the attitude matrix, not the estimate of the attitude matrix (which would be distinguished by an additional marker), which is corrupted by measurement noise. The operation above is not matrix multiplication, because two different sets of abstract vectors are being related and not the representations of a given abstract vector with respect to the axes of two reference frames, as in equation (33) below.[17] See reference [10] for further details of this distinction. The axes of $\mathfrak{S}''$ differ from the true body axes by the absence of parallax

[16]This is nothing more than the statement that in the absence of error the action on at least two vectors is necessary for the determination of a proper orthogonal matrix.
[17]The relation of $\{\hat{\mathbf{x}}'', \hat{\mathbf{y}}'', \hat{\mathbf{z}}''\}$ to $\{\hat{\mathbf{x}}^o, \hat{\mathbf{y}}^o, \hat{\mathbf{z}}^o\}$ is identical to the relation of $\{\hat{\mathbf{x}}^b, \hat{\mathbf{y}}^b, \hat{\mathbf{z}}^b\}$ to $\{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$.

and stellar aberration effects. For lack of a better name $\mathfrak{S}''$ may be called the *untranslated unboosted body frame*.[18]

Obviously, the stellar-aberration and parallax effects can be calculated just as well starting with this new untranslated unboosted body or star-camera frame as the previous inertial one, and the result must be identical in form. Thus, it must be true that

$$\delta\boldsymbol{\theta}''_{\text{aberration-parallax}, i, k} = \left(\boldsymbol{\beta}''_k - \frac{\mathbf{r}''_k}{R_{i,k}}\right) \times \hat{\mathbf{S}}''_{i,k} = \left(\frac{\mathbf{v}''_k}{c} - \frac{\mathbf{r}''_k}{R_{i,k}}\right) \times \hat{\mathbf{S}}''_{i,k} \tag{31}$$

and

$$\hat{\mathbf{W}}^{\text{true}}_{i,k} = \text{unit}(\hat{\mathbf{S}}''_{i,k} - \delta\boldsymbol{\theta}''_{\text{aberration-parallax}, i, k} \times \hat{\mathbf{S}}''_{i,k}) \tag{32}$$

Here

$$\hat{\mathbf{S}}''_{i,k} = A_k \hat{\mathbf{S}}_{i,k} \tag{33}$$

is the representation of the star-catalog direction with respect to $\mathfrak{S}''$, and $\hat{\mathbf{W}}^{\text{true}}$ is the value of $\hat{\mathbf{W}}_{i,k}$ without the measurement noise.[19]

The two approaches must be equivalent. If not, the most basic principle of Special Relativity, that all inertial reference frames are equivalent for the formulation of physical laws, would be violated. Thus, it must be true that

$$C(\delta\boldsymbol{\theta}''_{\text{aberration-parallax}, i, k}) A_k = A_k C(\delta\boldsymbol{\theta}^{o}_{\text{aberration-parallax}, i, k}) \tag{34}$$

or

$$C(\delta\boldsymbol{\theta}''_{\text{aberration-parallax}, i, k}) = A_k C(\delta\boldsymbol{\theta}^{o}_{\text{aberration-parallax}, i, k}) A_k^{\text{T}} \tag{35}$$

which is indeed true provided that (see equations (106) and (107) of reference [10]

$$\delta\boldsymbol{\theta}''_{\text{aberration-parallax}, i, k} = A_k \delta\boldsymbol{\theta}^{o}_{\text{aberration-parallax}, i, k} \tag{36}$$

This is certainly the case, since the action of A_k transforms every representation of a vector with respect to $\mathfrak{I}$ into a representation with respect to $\mathfrak{S}''$. Likewise, if the star-camera reference frame is different from the fiducial body reference frame, the stellar aberration and parallax effects may be computed in either of those two frames.[20]

Distortion and Correction

If the distortion of a star-catalog direction $\hat{\mathbf{S}}_{i,k}$ for the effects of stellar aberration and parallax is

$$\hat{\mathbf{V}}_{i,k} = \text{unit}(\hat{\mathbf{S}}_{i,k} + (\Delta\hat{\mathbf{S}}_{i,k})_{\text{distortion}}) \tag{37a}$$

$$(\Delta\hat{\mathbf{S}}_{i,k})_{\text{distortion}} = -\delta\boldsymbol{\theta}^{o}_{\text{aberration-parallax}, i, k} \times \hat{\mathbf{S}}_{i,k} \tag{37b}$$

[18] When the star-camera frame is not the fiducial body frame, then the axes of the *untranslated unboosted* star-camera frame are derived from equations (30) with A_k replaced by $S_k^{\text{T}} A_k$.

[19] All star directions labeled $\hat{\mathbf{S}}_{i,k}$ in this work, with or without superscripts, are distorted neither by aberration nor by parallax nor corrupted by measurement noise. Thus, $\hat{\mathbf{S}}_{i,k}$, $\hat{\mathbf{S}}''_{i,k}$ and $\hat{\mathbf{S}}'''_{i,k}$ (below). The transformation of equation (32) is not of practical use, although it illustrates the concept best. Of greater practical interest is the inverse transformation, which computes $\hat{\mathbf{W}}''_{i,k}$ from $\hat{\mathbf{W}}_{i,k}$, which will appear shortly.

[20] Although equations (28′) and (31) rely on $\delta\boldsymbol{\theta}_{\text{aberration-parallax}}$ being infinitesimal, equations (34) through (36) do not. They are correct even if the distortions are macroscopic, a fact that will be more useful to the crew of the future Starship Enterprise than to present terrestrials. Again, if the star-camera frame is different from the body frame the transformation matrix from $\mathfrak{I}$ to $\mathfrak{S}'''$, the untranslated unboosted star-camera frame, is $S_k^{\text{T}} A_k$.

then the *correction* to the *distorted* star direction to remove these effects is[21]

$$\hat{\mathbf{S}}_{i,k} = \text{unit}(\hat{\mathbf{V}}_{i,k} + (\Delta\hat{\mathbf{V}}_{i,k})_{\text{correction}}) \tag{38a}$$

$$(\Delta\hat{\mathbf{V}}_{i,k})_{\text{correction}} = +\delta\boldsymbol{\theta}'_{\text{aberration-parallax},\,i,k} \times \hat{\mathbf{V}}_{i,k} \tag{38b}$$

Equation (38b) presents the essence of the operation performed by observational astronomers when preparing entries for the star catalog.[22] Identical equations, but with a different superscript on $\delta\boldsymbol{\theta}$ (and different symbols for the star directions) hold for the correction of star-camera- or fiducial-body-referenced star directions. (See equations (40) and (42) below.)

The earlier discussion examined the effects of stellar aberration and parallax on the *star-catalog star directions*. The star camera on board the spacecraft supplies the observed star directions with the aberration and parallax effects already built into the data by nature. Therefore, to compute the attitude either: (1) undistorted (i.e., corrected) star-camera star directions must be compared with undistorted star-catalog directions or (2) uncorrected star-camera star directions must be compared with distorted star-catalog directions. As demonstrated by equation (34), either approach will lead to the same estimate of the attitude. Thus, either the star-catalog star directions must be *distorted* or the star-camera star directions must be *corrected*. Philosophical arguments exist for the intrinsic superiority of either approach. Since numerous corrections must be made to the directions measured by the star camera (misalignment, physical distortion of the star-camera focal plane, temperature corrections, star-intensity corrections, etc.) it makes sense to perform the corrections to the star-camera data rather than to distort the catalog, so that at least the star-catalog values of the directions will always be the same. On the other hand, the calculation of the corrections for stellar aberration in any body-fixed frame (fiducial body or star-camera) requires an *a priori* value of the spacecraft attitude, which is not always immediately available. Thus, both approaches have their respective advantages and disadvantages. Both are equally correct.

Summary: Attitude Estimation

Assume in this section that the star-camera and fiducial body frames are distinct (otherwise, $S_k = I$). There are then three approaches to attitude estimation using star data.

Star Catalog Directions Are Distorted

To estimate the attitude based on uncorrected star-camera measurements, the relevant equations are (assuming distinct star-camera and fiducial body frames):

$$\delta\boldsymbol{\theta}^{o}_{\text{aberration-parallax},\,i,k} = \left(\frac{\mathbf{v}^{o}_{k}}{c} - \frac{\mathbf{r}^{o}_{k}}{R_{i,k}}\right) \times \hat{\mathbf{S}}_{i,k} \tag{39a}$$

$$\hat{\mathbf{V}}_{i,k} = \text{unit}(\hat{\mathbf{S}}_{i,k} - \delta\boldsymbol{\theta}^{o}_{\text{aberration-parallax},\,i,k} \times \hat{\mathbf{S}}_{i,k}) \tag{39b}$$

$$\hat{\mathbf{W}}_{i,k} = S_k\hat{\mathbf{E}}_{i,k} \tag{39c}$$

$$\hat{\mathbf{W}}_{i,k} = A_k\hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}_{i,k} \tag{39d}$$

[21]Note the superscripts on $\delta\boldsymbol{\theta}$ in equations (37) and (38). From the numerical standpoint the distinction between $\delta\boldsymbol{\theta}^{o}$ and $\delta\boldsymbol{\theta}'$ is insignificant. However, in terms of the direction-cosine matrix as in equation (22) rather than the expansion to order $|\delta\boldsymbol{\theta}|$ the expressions will not be formally consistent unless these small differences are given proper attention.

[22]In fact, observational astronomers carry out many tasks akin to those of attitude-determination analysts, since the observatory is essentially a huge star camera. The corrections applied to observatory star data, however, are more complex than just equation (38b).

with S_k the star-camera alignment matrix at time t_k. This is the approach presented first in this work (and which occupied much of it). The first three lines of equations (39) present the transformations which must be applied to the star-catalog directions and the star-camera data. The fourth line is the principal equation for the attitude estimation step.

Star-Camera Measurements Are Corrected in the Star-Camera Frame

To calculate the attitude based on corrected star-camera measurements and undistorted star-catalog values for the star directions:

$$\delta\boldsymbol{\theta}^{e}_{\text{aberration-parallax}, i, k} = \left(\frac{\mathbf{v}^{e}_{k}}{c} - \frac{\mathbf{r}^{e}_{k}}{R_{i,k}} \right) \times \hat{\mathbf{E}}_{i,k} \tag{40a}$$

$$\hat{\mathbf{E}}'''_{i,k} = \text{unit}(\hat{\mathbf{E}}_{i,k} + \delta\boldsymbol{\theta}^{e}_{\text{aberration-parallax}, i, k} \times \hat{\mathbf{E}}_{i,k}) \tag{40b}$$

$$\hat{\mathbf{W}}''_{i,k} = S_k \hat{\mathbf{E}}'''_{i,k} \tag{40c}$$

$$\hat{\mathbf{W}}''_{i,k} = A_k \hat{\mathbf{S}}_{i,k} + \Delta\hat{\mathbf{W}}''_{i,k} \tag{40d}$$

Again, $\hat{\mathbf{W}}''_{i,k}$ above is not identical to $\hat{\mathbf{S}}''_{i,k}$ because of the presence of measurement noise in the former, but $\hat{\mathbf{W}}^{\text{true}}_{i,k}{}'' = \hat{\mathbf{S}}''_{i,k}$. The unboosted untranslated star-camera reference frame at t_k, $\mathfrak{S}'' = \{\boldsymbol{O}^{\mathfrak{S}'''}, \boldsymbol{v}^{\mathfrak{S}'''}, \hat{\boldsymbol{x}}''', \hat{\boldsymbol{y}}''', \hat{\boldsymbol{z}}'''\}$, with respect to which all of the $\hat{\mathbf{E}}'''_{i,k}$ are representations, is connected to $\mathfrak{S}''$ by the alignment rotation S_k^{T}. Thus, ideally

$$\hat{\boldsymbol{x}}''' = (S_k^{\text{T}})_{11}\hat{\boldsymbol{x}}'' + (S_k^{\text{T}})_{12}\hat{\boldsymbol{y}}'' + (S_k^{\text{T}})_{13}\hat{\boldsymbol{z}}'' \tag{41a}$$

$$\hat{\boldsymbol{y}}''' = (S_k^{\text{T}})_{21}\hat{\boldsymbol{x}}'' + (S_k^{\text{T}})_{22}\hat{\boldsymbol{y}}'' + (S_k^{\text{T}})_{23}\hat{\boldsymbol{z}}'' \tag{41b}$$

$$\hat{\boldsymbol{z}}''' = (S_k^{\text{T}})_{31}\hat{\boldsymbol{x}}'' + (S_k^{\text{T}})_{32}\hat{\boldsymbol{y}}'' + (S_k^{\text{T}})_{33}\hat{\boldsymbol{z}}'' \tag{41c}$$

and S_k is the true value of the alignment matrix, not the estimated value, which is corrupted by measurement noise. Thus, $\hat{\mathbf{E}}^{\text{true}}_{i,k}{}''' = \hat{\mathbf{S}}'''_{i,k}$.

Star-Camera Measurements Are Corrected in the Fiducial Body Frame

To calculate the attitude based on star-camera measurements corrected in the fiducial frame and undistorted star-catalog values for the star directions:

$$\hat{\mathbf{W}}_{i,k} = S_k \hat{\mathbf{E}}_{i,k} \tag{42a}$$

$$\delta\boldsymbol{\theta}^{b}_{\text{aberration-parallax}, i, k} = \left(\frac{\mathbf{v}^{b}_{k}}{c} - \frac{\mathbf{r}^{b}_{k}}{R_{i,k}} \right) \times \hat{\mathbf{W}}_{i,k} \tag{42b}$$

$$\hat{\mathbf{W}}''_{i,k} = \text{unit}(\hat{\mathbf{W}}_{i,k} + \delta\boldsymbol{\theta}^{b}_{\text{aberration-parallax}, i, k} \times \hat{\mathbf{W}}_{i,k}) \tag{42c}$$

$$\hat{\mathbf{W}}''_{i,k} = A_k \hat{\mathbf{S}}_{i,k} + \Delta\hat{\mathbf{W}}''_{i,k} \tag{42d}$$

This approach is less useful nowadays, since recent star cameras are capable of correcting or distorting for aberration effects internally and computing a batch attitude estimate in their resident software.

Frame Summary

Seven frames in two sequences have been defined: $\mathfrak{I}$, $\mathfrak{S}$, $\mathfrak{S}'$, $\mathfrak{B}$, $\mathfrak{E}$, and $\mathfrak{I}$, $\mathfrak{S}''$, $\mathfrak{S}'''$. The relationships of these seven frames to one another can be understood most easily from the commutative diagram of Fig. 1. (The elongated "equal" sign is just that.) The relationships of the representations of the star directions are presented in Fig. 2.

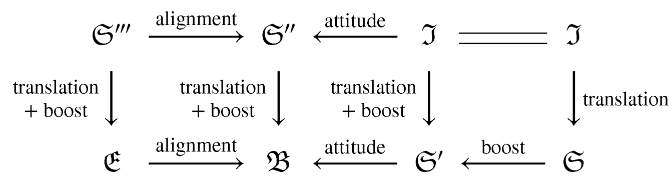

FIG. 1. Commutative Diagram for Reference Frames.

Horizontal arrows in the commutative diagrams have been written in the direction for which the transformation (direction-cosine) matrix bears no explicit transpose sign and vertical arrows have been written in the direction of distortion (from parallax or aberration effects) or of corruption (from measurement noise).[23] These direction-cosine matrices are S_k, A_k, $C_{\text{parallax}} \equiv C(\delta\boldsymbol{\theta}_{\text{parallax}})$, $C_{\text{aberration}} \equiv C(\delta\boldsymbol{\theta}_{\text{aberration}})$, and $C_{\text{aberration-parallax}} \equiv C_{\text{aberration}} C_{\text{parallax}}$. Note that $\hat{\mathbf{S}}'''_{i,k} = \hat{\mathbf{E}}'''^{\text{true}}_{i,k}$, and $\hat{\mathbf{S}}''_{i,k} = \hat{\mathbf{W}}''^{\text{true}}_{i,k}$.

Had measurement noise in the star-catalog directions been acknowledged, the commutative diagram would have been as in Fig. 3. The diagram assumes that noise in the system originates only in the star-camera and star-catalog data. (Note the prominent absence of an arrow at the lowest level.)

The seven reference frames of Fig. 1 are, in fact, insufficient in practice because Earth-orbiting spacecraft orbit-determination activities require a geocentric inertial coordinate system for their practical implementation. Thus, in theory, parallax and aberration corrections should take place in two phases, one for Earth motion with respect to $\mathfrak{I}$ and one for spacecraft motion with respect to the Earth. The sequence of distortions will then be

$$\mathfrak{E} \longrightarrow \mathfrak{B} \longleftarrow \mathfrak{S}' \longleftarrow \mathfrak{S}''' \longleftarrow \mathfrak{G}' \longleftarrow \mathfrak{G} \longleftarrow \mathfrak{I} \tag{43}$$

where $\mathfrak{G}$ is Fraktur "G" (for Greek $\Gamma\tilde{\eta}$ = "Earth"). $\mathfrak{G}$ and $\mathfrak{G}'$ are geocentric reference frames analogous to $\mathfrak{S}$ and $\mathfrak{S}'$ but reflect distortion effects only from the motion of the Earth. $\mathfrak{S}'''$ is now the result of a boost due to the velocity of the Earth with respect to $\mathfrak{I}$ and both the displacement of the Earth from the origin of $\mathfrak{I}$ and the displacement of the spacecraft from the geocenter. It is, therefore, not identical

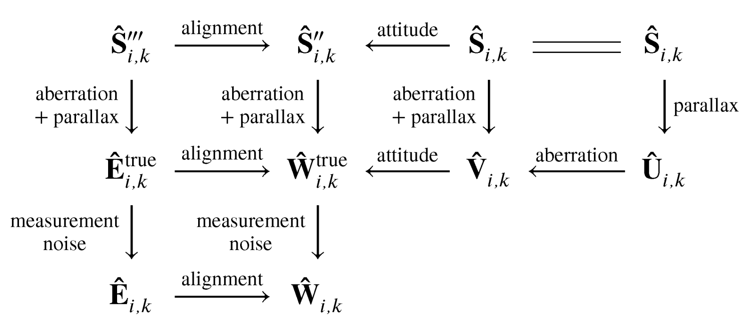

FIG. 2. Commutative Diagram for Star Directions.

[23]The commutative diagrams of Figs. 2 and 3 have been constructed so that at the highest level are star directions undistorted by parallax or aberration and uncorrupted by measurement noise, at the center level are star directions distorted by parallax or aberration and uncorrupted by noise, and at the lowest level are directions which are both distorted and corrupted. Thus, the diagram begins in "purity" at the highest level and descends into "corruption."

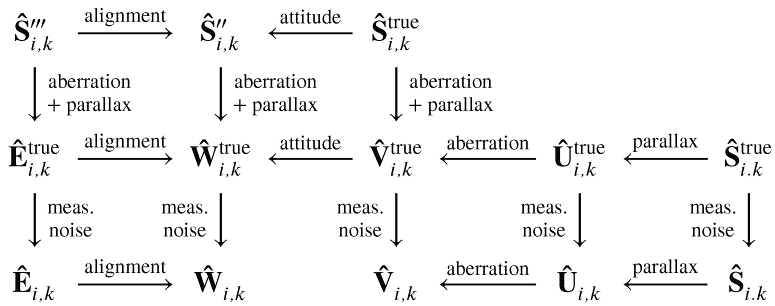

FIG. 3. Expanded Commutative Diagram for Star Directions.

to $\mathfrak{S}$ defined previously. $\mathfrak{S}'$ must be formally identical to its previous definition. Equation (28′) is still valid to first-order and the displacement and the velocity vectors are each now, as obviously they were previously, the trivial sum of two terms, one for Earth motion with respect to $\mathfrak{I}$ and one for spacecraft motion about the Earth.

To Distort or to Correct, That Is the Question!

In a Kalman-filter implementation, the *correction* for aberration and parallax can be computed using the predicted attitude $A_{k|k-1}$ in order to obtain $\mathbf{v}_k$ and $\mathbf{r}_k$ referenced to fiducial-body or star-camera axes, which figure in the computation of $\delta\boldsymbol{\theta}^b_{\text{aberration-parallax}}$ or $\delta\boldsymbol{\theta}^e_{\text{aberration-parallax}}$, respectively. Thus, in the Kalman filter, the option, exists with equal ease, of either *distorting* the star-catalog directions or *correcting* the measured directions. In a batch-estimation procedure, on the other hand, unless an iterative Newton-Raphson procedure is being carried out for a nonlinear estimation problem, there is usually no readily available *a priori* estimate of the attitude. In this case, the correction of the measured directions, rather than the distortion of the star-catalog directions, requires the initial computation of an *a priori* value for the attitude by neglecting the aberration and parallax effects in the data. This *a priori* estimate of the attitude is then used to compute $\delta\boldsymbol{\theta}^b_{\text{aberration-parallax}}$ or $\delta\boldsymbol{\theta}^e_{\text{aberration-parallax}}$. Thus, the batch estimation of the attitude with measurements corrected for aberration and parallax may require an additional attitude computation, while no additional attitude computation is required if the star-catalog directions are distorted instead. In batch attitude estimation, therefore, it is advantageous to distort the star-catalog directions for aberration and parallax rather than to correct the observed star directions.

For a deep-space mission currently in planning the star-camera microcomputer will distort its resident star catalog star directions for aberration based on a stored ephemeris and will output a batch estimate of the star-camera quaternion. This will then be combined with gyro data in a Kalman filter in the spacecraft computer.

An *a priori* estimate of the attitude is required not only for the correction of star-direction measurements but as an important component of star identification. With an *a priori* estimate of the attitude, the star search in the star-identification process can be limited to an area of the sky not much larger than 100 deg^2 (for a star-camera field of view of 8 deg by 8 deg), which is small compared with the area of the entire celestial sphere (approximately 40,000 deg^2). A full-sky search is possible, of course, but much more time-consuming. For both star identification and the computation of aberration and parallax corrections an attitude estimate from coarse attitude sensors (with accuracies of from 0.5 to 1.0 deg) is adequate. Fortunately, these distortion effects may be neglected still (in 2004) in the star-identification process.

Discussion: Star Catalog and Proper Motion

The analysis above has not considered the proper motion of the Solar System, to which the "inertial" frame $\mathfrak{I}$ is fixed. In fact, the Sun and 200 billion other stars are revolving around the center of the Milky Way Galaxy with an orbital speed of about 250 km/sec. It follows then that the magnitude of the velocity of a star on the other side of the Galaxy relative to the spacecraft can exceed 500 km/sec, which is more than an order of magnitude greater than the velocity of the spacecraft relative to $\mathfrak{I}$, which in Earth orbit, as has been seen, will not exceed 38 km/sec. Thus, in physical terms, the treatment of aberration above might seem at first glance to be inadequate. That, however, is not the case.

The principal requirement of a star catalog for attitude work is not that it provide the most detailed model of star motion but that it provide the means for computing reliable directions of its stars with respect to $\mathfrak{I}$ at t_k. The proper star motion behind the star direction recorded by the astronomical observatories is of no interest for attitude determination except in so far as it permits the extrapolation of the catalog directions forward or backward in time from the catalog epoch. Thus, astronomical star catalogs record not only the corrected observed directions ("positions") of stars at some epoch (currently the onset of the year 2000) but also the proper motion at epoch, visual magnitude, spectral information, distance from the Earth and more, not all of which is useful for attitude determination. Galactic coordinates offer no practical advantages for attitude determination and, considering how less accurately we can model Galactic motion than motion within the solar system, many obvious disadvantages.

The Sun is approximately 28,000 light-years from the Galactic center. Therefore, the angular rate of revolution of the Solar System about the Galactic center is only 0.6 are seconds/century, so that directions change very slowly due to mean Galactic rotation. Also, this angular rate corresponds to a centripetal acceleration of roughly 10^{-9} m/sec^2 or 10^{-10} gal, which hardly makes a reference frame fixed at the barycenter of the Solar System significantly less inertial than a Galactic coordinate system.

Of greater concern is the proper motion of proximate stars, whose observable proper motions are less regular than mean Galactic rotation. The star-catalog "positions" are simply corrected for known proper motion at whatever time interval is needed during the mission.

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[24]Einstein boasted that this book could be understood by Gymnasium students. Shortly after the book appeared, German reporters asked a young relative of Einstein if she had understood it. "Oh, yes," was her reply, "everything except the part about coordinate systems." The book is, nonetheless, very readable to any one who has completed freshman Physics.

[25]Readers should be warned that original copies of volume 17 of the *Annalen* are much sought-after collector's items and have largely disappeared from libraries.