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Uniform Attitude Probability Distributions[1]

Malcolm D. Shuster[2]

Abstract

We address the problem of constructing the probability density function (pdf) of a random attitude when there is no *a priori* knowledge of the attitude of any kind. We first define generally, on the basis of purely physical arguments, what is meant by such a uniform pdf of the attitude and develop a general expression from which the pdf for most three-dimensional attitude representations can be determined. We then calculate explicit expressions for this completely *a priori* pdf for the attitude quaternion, both for the vectorial components alone and as a function of all four components treated independently in R^4. On the basis of this last pdf we develop a universal expression for the uniform pdf of any three-dimensional representation of the attitude. Explicit expressions for the uniform pdf of the more useful attitude representations are presented and also methods for computing samples of these uniformly distributed attitude representations. The possible consequences of this work for attitude estimation are discussed briefly. The connection of the present work to the classical mathematical literature is also discussed briefly.

Introduction

Generally, in spacecraft attitude work, one needs to model a totally random attitude for which there is no *a priori* information in two situations:

1. In simulation, when one wishes to test an algorithm for all attitudes in a "uniform" and unbiased manner; and
2. In Bayesian attitude estimation, when there is no *a priori* information on the attitude.

In the present work, we address ourselves more to the first topic. The second topic will be treated briefly at the conclusion of this work. The bulk of this paper, however, is devoted to neither of these applications but to the general problem of how to define and construct a uniform probability density function (pdf) [1] for an attitude representation [2].

Within the framework of formal mathematics, probability is a measure. Hence, some of the results here can be found scattered throughout the literature of Measure Theory and the Theory of Continuous Groups of a century ago, especially that

[1]This paper is dedicated to Dr. James R. Wertz, long-time friend and father of Julie Wertz.

[2]Director of Research, Acme Spacecraft Company, 13017 Wisteria Drive, Box 328, Germantown, Maryland, 20874. email: mdshuster@comcast.net.

dealing with the Haar measure [3, 4], a measure defined on a continuous group and satisfying an invariance property which will be shown to be equivalent to our definition of uniformity below. Many of the results in this paper, however, cannot be located in the mathematical literature, and those that are are generally presented from a very abstract point of view for which most astronauticists, the present writer included, will have little sympathy. The results of the present paper, on the other hand, were derived by the author using only the most commonly known results of Riemann-Stieltjes integration, and without any foreknowledge of the measure-theoretic results. Since then, the author has become aware of the earlier work by mathematicians, largely through the persistent efforts of Robert Bauer. Those results of this work which have correspondences in early twentieth-century mathematics are indicated in footnotes, but they are not necessary to follow this article. Every attempt has been made to make this article self-contained at the risk of being somewhat tutorial.

The present work (except for the brief material on attitude estimation) was presented at the AAS/AIAA Space Flight Mechanics Meeting, held in Santa Barbara, California on February 11–14, 2001. The two papers [5, 6] presented there contain significant (and really unnecessary) pedagogical material, alternate derivations, and derivational detail that would be inappropriate to a journal article. The reader who might find such material helpful or interesting is referred to those two papers. The most important part of this additional material has been collected in the appendix of this work. Some minor errors in those works have also been corrected in the present work.

It is important to note that the results of this work for the uniform distribution of three-axis attitude run counter to the intuition of most astronauticists, who tend to model a uniform attitude distribution incorrectly. The impatient reader may read about this point in the first paragraph of the Discussion section.

The Rotation Group

The construction of a uniform pdf of an attitude representation can depend only on the nature of the group properties and the specific form of the composition rule, since, in general, that is the only information available about the attitude representation when its distribution is uniform. Thus, it will be useful to review these.

Every attitude representation, together with its composition rule, forms a group [7]. A *group* $\boldsymbol{G} = \{G, \circ\}$ consists of a set of elements $G = \{\alpha, \beta, \ldots\}$ and an operation $\circ$ which satisfy the following conditions:

(1) For any two α and β in G, $\alpha \circ \beta$ is also in G.

(2) The operation is associative

$$\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma. \tag{1}$$

(3) There exists an identity element ι in G which satisfies

$$\iota \circ \alpha = \alpha = \alpha \circ \iota \tag{2}$$

for every α in G.

(4) For every α in G there exits an inverse element α^{-1} such that

$$\alpha^{-1} \circ \alpha = \iota = \alpha \circ \alpha^{-1} \tag{3}$$

An immediate consequence of these rules is that the equations

$$\alpha \circ \xi = \beta \quad \text{and} \quad \zeta \circ \alpha = \beta \tag{4}$$

always have solutions, namely $\xi = \alpha^{-1} \circ \beta$ and $\zeta = \beta \circ \alpha^{-1}$, respectively.[3]

The canonical representation of the rotation group is the space of special (i.e., unimodular) orthogonal matrices in three dimensions together with the operation of matrix multiplication, the group $SO(3)$. The identity element of this group is just the 3×3 identity matrix, and the inverse element is simply the matrix inverse, or, equivalently for an orthogonal matrix, the matrix transpose. We will refer to these equally as rotation matrices, attitude matrices or direction-cosine matrices (DCMs). The properties of these matrices are given in detail in reference [2].

Closely related to $SO(3)$ is the group of the Euler-Rodrigues symmetric parameters (or unit quaternions), consisting of the space of 4×1 arrays $\bar{\eta}$ with unit norm[4]

$$\bar{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}, \quad \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = 1 \tag{5}$$

together with the operation of quaternion multiplication [2]. The identity element is just the unit quaternion $\bar{1} = [0, 0, 0, 1]^{\mathrm{T}}$, and the inverse of $\bar{\eta}$ is $\bar{\eta}^*$ with

$$\bar{\eta}^* \equiv \begin{bmatrix} -\boldsymbol{\eta} \\ \eta_4 \end{bmatrix} \tag{6}$$

This group of unit quaternions is not strictly equivalent (isomorphic) to $SO(3)$, because both $\bar{\eta}$ and $-\bar{\eta}$ represent the same rotation. One generally sidesteps this unpleasant fact in formal mathematics by identifying $\bar{\eta}$ and $-\bar{\eta}$ as the same point on the unit sphere in four dimensions.

While the redundancy of the unit-quaternions can be neatly sidestepped by the artifice above, there is no happy way to remove the infinite redundancies at the singularities (typically at an extremum of one of the parameters) of the other attitude representations, such as any of the twelve sets of Euler angles, the Rodrigues parameters, the rotation vector, etc. Fortunately, while it will be necessary at the various stages of the present work to pay attention to whether we have chosen the unit quaternions to occupy the entire unit sphere in four dimensions or only half of it, we will not need to be concerned with the singular nature of the other representations.

Uniform Attitude Probability Densities

What does it mean for the pdf of an attitude representation to be uniform? Let $\boldsymbol{\xi}$ denote an arbitrary representation of the attitude random variable, $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}''$ its realizations,[5] i.e., its possible sampled values, and $p_{\boldsymbol{\xi}}(\boldsymbol{\xi}')$ the uniform pdf of $\boldsymbol{\xi}$. Then it is clear that uniformity cannot mean in general

$$p_{\boldsymbol{\xi}}(\boldsymbol{\xi}') = p_{\boldsymbol{\xi}}(\boldsymbol{\xi}'') \tag{7}$$

which would be impossible to expect to hold for every attitude representation.

[3]Mathematically informed readers will know that it is sufficient for a left identity and a left inverse to exist, from which it follows that these will also be the right identity and right inverse, respectively.

[4]Generally, we will use the symbol $\bar{\eta}$ for the unit quaternion and $\bar{q}$ for the quaternion without unit-norm constraint, though perhaps not with perfect consistency. We will sometimes alternate between these two notations within a single sequence of equations to emphasize the presence or absence of the constraint. In the text we will not always write "unit-quaternion" or "attitude quaternion" explicitly when the presence of the unit-norm constraint is obvious from the context.

[5]We will maintain this notational distinction between random variables and their realizations throughout this paper. When it is obvious that a variable is not random, e.g., when it is so defined, or when it is a variable of integration or differentiation, we will often not write the primes. Likewise, we will usually not write the primes when a relationship is true both for the random variable and for its realization.

An attitude representation is the representation of a rotation. Thus, if $\boldsymbol{\xi}$ is the representation of the uniformly distributed random attitude from the space coordinate frame to the body coordinate frame, then $\boldsymbol{\xi} \circ \boldsymbol{\chi}$ is the representation of the rotation from a new space frame to the body frame, with the representation of the (constant) rotation from the old space frame to the new space frame being $\boldsymbol{\chi}^{-1}$. Likewise, if $\boldsymbol{\zeta}^{-1}$ is the the representation of the (constant) rotation carrying the old body axes to new body axes, then the random representation of the attitude from the old space axes to the new body axes becomes $\boldsymbol{\zeta} \circ \boldsymbol{\xi}$. Thus, for it to be impossible to infer a attitude from the uniform pdf, we must have

$$p_{\xi}(\boldsymbol{\xi}') = p_{\xi\circ\chi}(\boldsymbol{\xi}') = p_{\zeta\circ\xi}(\boldsymbol{\xi}') \tag{8}$$

for all values of $\boldsymbol{\xi}'$ no matter what the choice of $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$, so long as they have fixed (i.e., non-random) values. Equation (8) then is our definition of uniformity for attitude representations.[6,7] Q.E.F.

According to equation (8) the uniform pdf is totally *uninformative* of the specific value of $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ within their allowed domain. Hence, there can be no way to infer $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ from the sampled attitudes if the distribution of samples is uniform. Within the framework of maximum likelihood estimation [8], for example, $\boldsymbol{\chi}$ and $\boldsymbol{\zeta}$ have something of the character of parameters of the pdf. But while the uniform pdf may have a maximum for a particular value of the attitude realization, the likelihood function will always be independent of $\boldsymbol{\chi}$ or $\boldsymbol{\zeta}$ as parameters. Thus, the maximum likelihood estimate of either of these parameters would be completely indeterminate. Clearly, under these conditions there can be no preferred attitude, which is what we mean by a uniform attitude distribution.

The group property of the attitude thus determined the general nature of the invariance which characterizes a uniformly distributed random attitude. We must now bring into play the only other piece of information that we have about the given attitude representation, namely, its composition rule.

A multivariate pdf, being a density, has the general transformation rule [1] under the change of variable $\mathbf{x} \rightarrow \mathbf{y}(\mathbf{x})$

$$p_{\mathbf{x}}(\mathbf{x}') = p_{\mathbf{y}}(\mathbf{y}'(\mathbf{x}')) \left| \frac{\partial \mathbf{y}'(\mathbf{x}')}{\partial \mathbf{x}'} \right| \tag{9}$$

where the second factor is the Jacobian determinant, defined as the absolute value of the determinant of the matrix

$$\left[\frac{\partial \mathbf{y}'}{\partial \mathbf{x}'} \right]_{ij} = \left[\frac{\partial(y_1', y_2', \dots, y_n')}{\partial(x_1', x_2', \dots, x_n')} \right]_{ij} \equiv \frac{\partial y_i'}{\partial x_j'} \tag{10}$$

Equation (9) assumes that $\mathbf{y}(\mathbf{x})$ is a bijective function of $\mathbf{x}$ almost everywhere. Generally, this can be accomplished by restricting the range (or domain) of one or the

[6]By *uniform in the strict sense,* however, we will mean the common definition of uniformity as having a constant pdf over an interval, equation (7), as opposed to the definition given in equation (8). Generally, "uniform over $[a, b]$" will mean uniform in the strict sense.

[7]It would be well to compare equation (8) with the corresponding statement for the Haar measure. If $\mathsf{P}(\Xi) = \int_{\Xi} p_{\xi}(\boldsymbol{\xi}')\, d\boldsymbol{\xi}'$ is the probability of an attitude "event" lying in a region Ξ of our attitude event space, then P is a Haar measure if it satisfies (in the notation of equation (8)) $\mathsf{P}(\Xi) = \mathsf{P}(\Xi \circ \boldsymbol{\chi}^{-1}) = \mathsf{P}(\boldsymbol{\zeta}^{-1} \circ \Xi)$, with $\Xi \circ \boldsymbol{\chi}^{-1} = \{\boldsymbol{\xi}' \circ \boldsymbol{\chi}^{-1} | \boldsymbol{\xi}' \in \Xi\}$, etc. If we now put labels on the probability functions to denote the random variable, then this is equivalent to $\mathsf{P}_{\xi}(\Xi) = \mathsf{P}_{\xi\circ\chi}(\Xi) = \mathsf{P}_{\zeta\circ\xi}(\Xi)$. Thus, our invariance condition of equation (8) is equivalent to that of the Haar measure.

other or both variables. Applying equation (9) to the equality of the first two members of equation (8) leads straightforwardly to

$$p_\xi(\boldsymbol{\xi}') = p_{\xi\circ\chi}(\boldsymbol{\xi}') = p_\xi(\boldsymbol{\xi}' \circ \boldsymbol{\chi}^{-1}) \left| \frac{\partial(\boldsymbol{\xi}' \circ \boldsymbol{\chi}^{-1})}{\partial \boldsymbol{\xi}'} \right| \tag{11}$$

We now note a restriction on the scope of the problem imposed by equation (11). Only for three-dimensional representations of the attitude can all of the attitude variables be independent. Hence, for attitude representations of dimension higher than three, the Jacobian determinant in equation (11) is ambiguous. Thus, equation (11) can be applied only to three-dimensional representations of the attitude.

We are now prepared to derive a general formula for the uniform pdf of a three-dimensional attitude representation. First, let us define a new variable $\boldsymbol{\alpha} \equiv \boldsymbol{\xi}' \circ \boldsymbol{\chi}^{-1}$. Then equation (11) becomes

$$p_\xi(\boldsymbol{\alpha} \circ \boldsymbol{\chi}) = p_\xi(\boldsymbol{\alpha}) \left| \frac{\partial \boldsymbol{\alpha}}{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\chi})} \right| \tag{12}$$

If we now let $\boldsymbol{\chi}$, which is arbitrary, have the value $\boldsymbol{\xi}'$, this becomes

$$p_\xi(\boldsymbol{\alpha} \circ \boldsymbol{\xi}') = p_\xi(\boldsymbol{\alpha}) \left| \frac{\partial \boldsymbol{\alpha}}{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\xi}')} \right| \tag{13}$$

And now, finally, taking the limit $\boldsymbol{\alpha} \to \boldsymbol{\iota}$ we obtain

$$p_\xi(\boldsymbol{\xi}') = p_\xi(\boldsymbol{\iota}) \left| \frac{\partial \boldsymbol{\alpha}}{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\xi}')} \right|_{\boldsymbol{\alpha}=\boldsymbol{\iota}} \tag{14}$$

Q.E.F.[8] Starting from the equality of the first and third members of equation (8) leads to an identical expression but with $\boldsymbol{\alpha} \circ \boldsymbol{\xi}'$ replaced by $\boldsymbol{\xi}' \circ \boldsymbol{\alpha}$.

We thus have a general expression for the uniform pdf in terms of the value of the uniform pdf at the identity rotation $\boldsymbol{\iota}$ and the Jacobian determinant. The expression can be useful only when $p_\xi(\boldsymbol{\iota})$ is different from zero. This is not always the case, as will be seen for the 3-1-3 Euler angles, for which a different approach must be found. Even for the 3-1-2 Euler angles, for which $p_\xi(\boldsymbol{\iota})$ is different from zero, equation (14) is not very useful, because the "multiplication" rule for that representation is very complicated [10]. This need not mean necessarily that the uniform pdf for the Euler angles has a complicated form.

When $p_\xi(\boldsymbol{\iota})$ is different from zero, its value is clearly given by

$$p_\xi(\boldsymbol{\iota}) = \left[\int \left| \frac{\partial \boldsymbol{\alpha}}{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\xi}')} \right|_{\boldsymbol{\alpha}=\boldsymbol{\iota}} d^3\boldsymbol{\xi}' \right]^{-1} \tag{15}$$

where the integration is over the entire domain of $\boldsymbol{\xi}'$.

Note from equation (14) that a necessary and sufficient condition for the uniform pdf to be a constant function of the realization is that the Jacobian determinant in equation (14) also be constant. A sufficient condition for the Jacobian determinant to be constant is that $\boldsymbol{\alpha} \circ \boldsymbol{\xi}' \to \boldsymbol{\xi}' + M(\boldsymbol{\alpha} - \boldsymbol{\iota}) + O(|\boldsymbol{\alpha} - \boldsymbol{\iota}|^2)$ as $\boldsymbol{\alpha} \to \boldsymbol{\iota}$ for some constant 3×3 matrix M (independent of $\boldsymbol{\xi}'$). This is not a necessary condition, as will be seen in the Discussion section.

[8]A similar result has been derived for a Haar measure [9].

The Uniform PDF for Vector Components of the Quaternion

Let us now do something practical. The uniform pdf for the vector components of the quaternion, with the convention that the scalar component be positive, is obtained directly from equations (14) and (15). The unit quaternion, we note, is related to the axis and angle of rotation by [2]

$$\bar{\eta} = \begin{bmatrix} \boldsymbol{\eta} \\ \eta_4 \end{bmatrix} = \begin{bmatrix} \sin(\theta/2)\hat{\mathbf{n}} \\ \cos(\theta/2) \end{bmatrix} \tag{16}$$

with θ the angle of rotation and $\hat{\mathbf{n}}$ the axis of rotation, a unit vector. The unit quaternion composition rule for the vector components alone under the constraint that the scalar component be non-negative is [2]

$$\boldsymbol{\alpha} \circ \boldsymbol{\eta}' = \mathrm{sgn}(\boldsymbol{\alpha}, \boldsymbol{\eta}')\,(\alpha_4 \boldsymbol{\eta}' + \eta'_4 \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \boldsymbol{\eta}') \tag{17}$$

where α_4 and η'_4 are here shorthand for $+\sqrt{1 - |\boldsymbol{\alpha}|^2}$ and $+\sqrt{1 - |\boldsymbol{\eta}'|^2}$, respectively, which by our stated convention are always nonnegative, and

$$\mathrm{sgn}(\boldsymbol{\alpha}, \boldsymbol{\eta}') = \mathrm{sign}(\alpha_4 \eta'_4 - \boldsymbol{\alpha} \cdot \boldsymbol{\eta}') \tag{18}$$

Equation (14) for $\eta'_4 \neq 0$ leads to[9]

$$\left| \frac{\partial \boldsymbol{\alpha}}{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\eta}')} \right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left| \frac{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\eta}')}{\partial \boldsymbol{\alpha}} \right|^{-1}_{\boldsymbol{\alpha}=\mathbf{0}} = \frac{1}{\eta'_4} = \frac{1}{\sqrt{1 - |\boldsymbol{\eta}'|^2}} \tag{19}$$

and from equation (22), $p_{\boldsymbol{\eta}}(\mathbf{0}) = 1/\pi^2$. Hence, the uniform pdf for the vector components of the quaternion is

$$p_{\boldsymbol{\eta}}(\boldsymbol{\eta}') = \frac{1}{\pi^2 \sqrt{1 - |\boldsymbol{\eta}'|^2}}, \quad 0 \leq |\boldsymbol{\eta}'| < 1 \tag{20}$$

Q.E.F. The author derived this result (by heuristic arguments) for the first time in 1993, when the problem was first posed to him casually by Markley. Note that near $\boldsymbol{\eta}' = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{0}$ the composition rule is approximately simple addition and consequently the uniform pdf is constant there to within terms of order $|\boldsymbol{\eta}'|^2$, as predicted in the discussion following equation (15). Note also that the same uniform pdf would have been obtained had we insisted instead that the scalar component of the quaternion be negative, a fact that will be used in the sequel.

We emphasize that it is only through the nature of the composition rule that any information about the particular attitude representation enters the calculation of the uniform pdf. Equation (16), although a helpful geometrical reminder, plays no role in the derivation of equation (20).

The Uniform PDF for the Four-Component Quaternion:

We present here a mathematically rigorous derivation of the uniform pdf of the quaternion on the whole of the unit sphere in four dimensions, i.e., S^3, the unit 3-sphere.[10] Previously, we determined $p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')$ restricted to the open hemihypersphere $\eta'_4 > 0$. We now consider the domain of the pdf to be the entire hypersurface S^3, so that probability will be spread over twice the domain and therefore be half as "dense."

[9]Note that $\eta'_4 > 0$, implies the existance of an $\epsilon > 0$ such that $\mathrm{sgn}(\boldsymbol{\alpha}, \boldsymbol{\eta}') = +1$ for $|\boldsymbol{\alpha}| < \epsilon$. Thus, the sgn-function does not enter the calculation of the Jacobian determinant in equation (19).

[10]S^2, the unit 2-sphere, is the familiar spherical surface in three-dimensional space (the surface of the solid unit 3-ball); S^1 is the unit circle.

Let us define now eight open hemihyperspheres (hyperhemispheres?) $H(i,\kappa)$ according to

$$H(i,\kappa) = \begin{cases} \{\bar{\eta}' | \eta_i' > 0\}, & \text{for } i = 1,2,3,4, \quad \kappa = 1 \\ \{\bar{\eta}' | \eta_i' < 0\}, & \text{for } i = 1,2,3,4, \quad \kappa = 2 \end{cases} \tag{21}$$

Almost every point in S^3 belongs to four open hemihyperspheres, and every point of S^3 belongs to at least one open hemihypersphere. (All but eight points in S^3 belong to at least two open hemihyperspheres.) Thus, these eight open hemihyperspheres constitute an open covering of S^3.

Define now

$$\boldsymbol{\eta}(1) \equiv \begin{bmatrix} \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix}, \quad \boldsymbol{\eta}(2) \equiv \begin{bmatrix} \eta_1 \\ \eta_3 \\ \eta_4 \end{bmatrix}, \quad \boldsymbol{\eta}(3) \equiv \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_4 \end{bmatrix}, \quad \boldsymbol{\eta}(4) \equiv \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \boldsymbol{\eta} \tag{22}$$

It follows trivially, in a manner similar to the derivation of equation (20), that for every $\bar{\eta} \in H(i,\kappa)$

$$p_{\boldsymbol{\eta}(i),\kappa}(\boldsymbol{\eta}'(i)) = \frac{C(i,\kappa)}{|\eta_i'|}, \quad i = 1,2,3,4, \quad \kappa = 1,2 \tag{23}$$

for some (positive) constant $C(i,\kappa)$. Since the probability density function at every point of S^3 is finite when written in terms of the appropriate coordinates, it follows that the uniform pdf of $\bar{\eta}$ on S^3 cannot have a point, curve or area of concentration, i.e., a point, curve or (two-dimensional) area in S^3 which by itself has finite probability (as opposed to finite probability density).

We note further that wherever it is finite

$$\left|\frac{\partial \boldsymbol{\eta}'(i)}{\partial \boldsymbol{\eta}'(j)}\right| = \frac{|\eta_i'|}{|\eta_j'|}, \quad i = 1,2,3,4 \tag{24}$$

where $\boldsymbol{\eta}'(i)$ and $\boldsymbol{\eta}'(j)$ refer to the same point on S^3. Hence, if $\bar{\alpha}$ belongs to both $H(i,\kappa)$ and $H(j,\lambda)$, then

$$\frac{C(i,\kappa)}{|\alpha_i|} = p_{\boldsymbol{\eta}(i),\kappa}(\boldsymbol{\alpha}(i)) = p_{\boldsymbol{\eta}(j),\lambda}(\boldsymbol{\alpha}(j)) \left|\frac{\partial \boldsymbol{\alpha}(j)}{\partial \boldsymbol{\alpha}(i)}\right| = \frac{C(j,\lambda)}{|\alpha_j|}\frac{|\alpha_j|}{|\alpha_i|} = \frac{C(j,\lambda)}{|\alpha_i|} \tag{25}$$

It follows that $C(i,\kappa) = C(j,\lambda)$ for any two hemihyperspheres. Therefore, for any point of S^3 for which $\eta_i' \neq 0$

$$p_{\boldsymbol{\eta}(i),\kappa}(\boldsymbol{\eta}'(i)) = \frac{C}{|\eta_i'|}, \quad i = 1,2,3,4, \quad \kappa = 1,2 \tag{26}$$

for a common constant C. Since there are no points, curves or areas of concentration on S^3 and $S^3 = H(4,1) \cup H(4,2) \cup \partial H(4,1)$, with $\partial H(4,1)$, the boundary of $H(4,1)$ (or of $H(4,2)$), a set of measure zero, we can determine C from

$$1 = \int_{H(4,1)\cup H(4,2)} p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')\, d^3\boldsymbol{\eta}' = 2C\int_{|\eta'|<1} \frac{d^3\boldsymbol{\eta}'}{\sqrt{1-|\boldsymbol{\eta}'|^2}} = 2\pi^2 C \tag{27}$$

whence

$$p_{\boldsymbol{\eta}(i),\kappa}(\boldsymbol{\eta}'(i)) = \frac{1}{2\pi^2|\eta_i'|}, \quad i = 1,2,3,4, \quad \kappa = 1,2 \tag{28}$$

on every hemihypersphere of S^3. Note the additional factor 2 in the denominator of equation (28), the result of the probability being spread now over twice as much hypersurface.

We can describe the probability density function for the quaternion as a single function of the quaternion defined on all of S^3 and avoid the changes of variables associated with equation (28). To accomplish this task we write the general quaternion as a function of the vector components of the Euler-Rodrigues symmetric parameters (the unit quaternion) and the quaternion length as

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} q\eta_1 \\ q\eta_2 \\ q\eta_3 \\ q\sqrt{1 - |\boldsymbol{\eta}|^2} \end{bmatrix} \tag{29}$$

The Jacobian determinant of the transformation from Cartesian quaternion coordinates to $(q, \eta_1, \eta_2, \eta_3)$ is given by

$$\left| \frac{\partial(q_1, q_2, q_3, q_4)}{\partial(q, \eta_1, \eta_2, \eta_3)} \right| = \frac{q^3}{|\eta_4|} \tag{30}$$

and, therefore

$$d^4\bar{q} \equiv dq_1\, dq_2\, dq_3\, dq_4 = \frac{q^3}{|\eta_4|}\, dq\, d^3\boldsymbol{\eta} \equiv q^3\, dq\, d^3\boldsymbol{\sigma} \tag{31}$$

which implies that

$$d^3\boldsymbol{\sigma} = \frac{1}{|\eta_4|}\, d^3\boldsymbol{\eta} \tag{32}$$

is the invariant "hyper-area" element on S^3 (which should be called more properly the invariant "volume" element). It must be equally true that

$$d^3\boldsymbol{\sigma} = \frac{1}{|\eta_1|}\, d^3\boldsymbol{\eta}(1) = \frac{1}{|\eta_2|}\, d^3\boldsymbol{\eta}(2) = \frac{1}{|\eta_3|}\, d^3\boldsymbol{\eta}(3) \tag{33}$$

Writing $d^3\mathsf{P}_{\bar{\eta}}(\bar{\eta}') \equiv p_{\boldsymbol{\eta}(i),\kappa}(\boldsymbol{\eta}'(i))\, d^3\boldsymbol{\eta}'(i)$, $i = 1, 2, 3, 4$, $\kappa = 1, 2$, it follows that

$$d^3\mathsf{P}_{\bar{\eta}}(\bar{\eta}') = \frac{1}{2\pi^2}\, d^3\boldsymbol{\sigma}' \tag{34}$$

at any point of S^3. Q.E.F.

Equation (34) shows that $1/2\pi^2$ is the invariant hypersurface probability density on S^3. Note again that the value of the invariant hypersurface probability density is $1/2\pi^2$ and not $1/\pi^2$, because we consider the entire hypersphere to be the domain of the pdf and not just a hemi-hypersphere on which each attitude corresponds to a unique quaternion. The uniform pdf of the quaternion in four dimensions may be said to have a three-sphere of concentration.

Equation (34) still describes the uniform pdf for the quaternion in terms of *three*-dimensional hyper-area elements on the hypersphere. We now develop an expression for the quaternion density in the full quaternion space R^4 in which all four components of the quaternion are manifest in a symmetrical way.

Equation (20) can be be extended trivially to the four-dimensional half-space $q_4 > 0$ according to

$$p_{\bar{q}}(\bar{q}') = \frac{1}{\pi^2\sqrt{1 - |\mathbf{q}'|^2}}\,\delta\left(q_4' - \sqrt{1 - |\mathbf{q}'|^2}\right) \tag{35}$$

where $\delta(x)$ is the Dirac δ-function [11].

The marginal pdf for $\boldsymbol{\eta}$, obtained by integrating over q_4' from 0 to ∞, is just $p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')$ as given by equation (20). Likewise, had we chosen to make our attitude quaternion event space $H(4, 2)$, our four-dimensional generalization restricted to the negative half-space $q_4' < 0$ would have been

$$p_{\bar{q}}(\bar{q}') = \frac{1}{\pi^2\sqrt{1 - |\mathbf{q}'|^2}}\,\delta\left(q_4' + \sqrt{1 - |\mathbf{q}'|^2}\right) \tag{36}$$

To construct a uniform pdf for the quaternion over all of R^4 we note that uniformity in the sign of q_4 (the parameter κ introduced in equation (21)) means that the probability for each of the two values of κ must be $1/2$. Hence interpreting $p_{\bar{q}}(\bar{q}')$ in equations (35) and (36) more correctly as $p_{\bar{q}|\kappa}(\bar{q}'|\kappa')$, a conditional pdf, leads to an expression for the uniform pdf in the full space of R^4 (except for the hyperplane $q_4' = 0$) which is the average of these two pdf's, namely

$$p_{\bar{q}}(\bar{q}') = \frac{1}{2\pi^2\sqrt{1 - |\mathbf{q}'|^2}}\left[\delta\left(q_4' - \sqrt{1 - |\mathbf{q}'|^2}\right) + \delta\left(q_4' + \sqrt{1 - |\mathbf{q}'|^2}\right)\right] \tag{37}$$

We now note a general result for the δ-function [11]

$$\delta(g(x)) = \sum_i \left|\frac{dg}{dx}(x_i)\right|^{-1} \delta(x - x_i) \tag{38}$$

where the x_i are the roots of $g(x)$. When applied to the two δ-functions of equation (37) this yields

$$\delta((q_4')^2 - 1 + |\mathbf{q}'|^2) = \frac{1}{2\sqrt{1 - |\mathbf{q}'|^2}} \left[\delta\left(q_4' - \sqrt{1 - |\mathbf{q}'|^2}\right) + \delta\left(q_4' + \sqrt{1 - |\mathbf{q}'|^2}\right)\right] \tag{39}$$

whence equation (37) becomes

$$p_{\bar{q}}(\bar{q}') = \frac{1}{\pi^2}\,\delta(\bar{q}'^{\mathrm{T}}\bar{q}' - 1) \tag{40}$$

where the superscript "T" denotes the matrix transpose. This is certainly a very elegant expression, if not obviously useful. Note that equation (40) holds over all of R^4. If q denotes the magnitude of $\bar{q}$, then equation (40) is equivalent to[11]

$$p_{\bar{q}}(\bar{q}') = \frac{1}{2\pi^2}\,\delta(q' - 1) \tag{40$'$}$$

in more obvious analogy to equation (34). Interestingly, the quaternion, which is often denigrated by engineers, because it is not easily visualized, supplies the simplest geometrical picture of the uniform attitude pdf.

[11]Note that the term in $\delta(q' + 1)$ is absent in equation (40′) because the root of its argument lies outside the specified domain of the pdf.

A Universal Formula for the Uniform PDF for the Three-Dimensional Attitude Representations

We saw earlier that the uniform pdf of many three-dimensional attitude representations could be derived from equation (14). We call this method the *group method,* because it relies only on the knowledge of the particular form of the group operation. This equation is not always of practical use. For one thing, it may happen that $p_\xi(\boldsymbol{\iota})$ vanishes, in which case the expression is undefined. Also, if the composition rule for the representation, as for the Euler angles [10], is very complicated, then the Jacobian determinant will be extremely difficult to evaluate. Hence, we must find another method for constructing the uniform pdf for some representations.

Such a method, for example, is to use our knowledge of the uniform pdf of the vector components of the quaternion as given by equation (20) or of any other three-dimensional attitude representation. It follows then from the general rule for transforming densities that for any three-dimensional attitude representation $\boldsymbol{\xi}$ we can write the uniform pdf as[12]

$$p_\xi(\boldsymbol{\xi}') = p_\eta(\boldsymbol{\eta}'(\boldsymbol{\xi}')) \left| \frac{\partial \boldsymbol{\eta}'(\boldsymbol{\xi}')}{\partial \boldsymbol{\xi}'} \right| \tag{41}$$

Frequently, it is much easier to evaluate the Jacobian determinant in equation (41) than the one in equation (14). We call this method the *transformation method.*

We develop now a third method, which uses the result for the uniform pdf for the quaternion with the four components treated as independent, as given by equation (40′). The advantage here is that the pdf of the four-component quaternion is so simple. We derive this new method now.[13]

Let us begin by restricting the attitude quaternion $\bar{\eta}'$ to the hemihypersphere $H(4,1)$ and the four-dimensional quaternion with arbitrary norm to the half-space $q_4' > 0$, so that the representation $\bar{\eta}'$ will be non-redundant, and the ratio q_4'/η_4' will always be positive. We may write the quaternion in the positive half-space as

$$\bar{q}(q, \boldsymbol{\eta}(\boldsymbol{\xi})) = q\bar{\eta}(\boldsymbol{\eta}(\boldsymbol{\xi})) \tag{42}$$

Here q is a scale factor, not necessarily unity in the half-space $q_4 > 0$, and $\boldsymbol{\xi}$ is the three-dimensional attitude representation for which we wish to construct the uniform pdf. Explicitly

$$\bar{q}(q, \boldsymbol{\eta}(\boldsymbol{\xi})) = \begin{bmatrix} q\eta_1(\boldsymbol{\xi}) \\ q\eta_2(\boldsymbol{\xi}) \\ q\eta_3(\boldsymbol{\xi}) \\ q\sqrt{1 - |\boldsymbol{\eta}(\boldsymbol{\xi})|^2} \end{bmatrix} \tag{43}$$

where, since $\boldsymbol{\eta}' \in H(4,1)$ and $q_4' > 0$, the positive square root is always chosen and $q' > 0$. Equation (43) defines the change of variables

$$(q_1, q_2, q_3, q_4) \rightarrow (\xi_1, \xi_2, \xi_3, q) \tag{44}$$

In similar fashion to equation (41) we can now write in four dimensions

[12]Again we assume that $\boldsymbol{\eta}(\boldsymbol{\xi})$ is a bijective function. When this is not the case, then we must replace equation (41) by an equation similar to equation (38). (Recall that the δ-function is also a density.)
[13]The earlier derivation of this result in reference 6 was rather murky.

$$p_{q,\xi}(q', \boldsymbol{\xi}') = p_{\bar{q}}(\bar{q}'(q', \boldsymbol{\xi}')) \left| \frac{\partial(\bar{q}'(q', \boldsymbol{\xi}'))}{\partial(q', \boldsymbol{\xi}')} \right| \tag{45}$$

Clearly

$$p_{q,\xi}(q', \boldsymbol{\xi}') = p_{\xi}(\boldsymbol{\xi}')\, \delta(q' - 1) \tag{46}$$

and

$$p_{\bar{q}}(\bar{q}') = \frac{2}{\pi^2}\, \delta(\bar{q}'^{\mathrm{T}}\bar{q}' - 1), \quad q'_4 > 0 \tag{47}$$

where the factor "2" in equation (47) occurs because the domain of the quaternion is restricted to the positive half-space. Thus (recall the footnote preceding equation (40′))

$$p_{\bar{q}}(\bar{q}') = \frac{2}{\pi^2}\, \delta((q')^2 - 1) = \frac{1}{\pi^2}\, \delta(q' - 1) \tag{48}$$

Substituting equations (46) and (48) into equation (45) and integrating over q' from 0 to ∞ leads directly to an expression for the uniform pdf for an arbitrary three-dimensional representation $\boldsymbol{\xi}$ of the attitude

$$p_{\xi}(\boldsymbol{\xi}') = \frac{1}{\pi^2} \left| \frac{\partial(\bar{q}'(q', \boldsymbol{\xi}'))}{\partial(q', \boldsymbol{\xi}')} \right|_{q'=1} \tag{49}$$

Q.E.F.

Equation (49) is certainly the "royal road" to calculating uniform pdf's for three-dimensional attitude representations. We call this the *quaternion method* for deriving the uniform pdf for a three-dimensional attitude representation.

The Uniform PDF for the Other Attitude Representations

While the uniform pdf can be computed straightforwardly for any three-dimensional representation from equation (49), in practice we have used the other two methods or special tricks. This was the case, because equation (49) was practically the very last result of this work to be derived. Equation (49) was used, however, to check several of the results (and to check equation (49)).[14] That said, the uniform pdf's for the various representations were derived in Appendix A and reference [6] in the following manner:

- Rodrigues Parameters: derived both by the group method and by the transformation method (from $p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')$).
- Modified Rodrigues Parameters: by the transformation method starting with the uniform pdf for the Rodrigues vector.
- Rotation Vector: by the transformation method (from $p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')$).
- Axis and Angle of Rotation: by explicit factorization of the uniform pdf for the rotation vector.
- Symmetric Sequence of Euler Angles: by the transformation method (from $p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')$).
- Asymmetric Sequence of Euler Angles: by a novel transformation of the result for a symmetric sequence of Euler Angles.

[14]The reader may quickly check equation (49) for the case of the vector components of the unit quaternion. The necessary Jacobian determinant has been given in equation (30).

Detailed derivations of the uniform pdf's are given in reference [6] and in the appendix. The final results follow. The notation and definitions follow exactly reference [2].

The Rodrigues Parameters (Gibbs Vector)

$$p_{\boldsymbol{\rho}}(\boldsymbol{\rho}') = \frac{1}{\pi^2(1 + |\boldsymbol{\rho}'|^2)^2}, \quad 0 \leq |\boldsymbol{\rho}'| < \infty \tag{50}$$

The Modified Rodrigues Parameters

For the positive form of the modified Rodrigues parameters

$$p_{\mathbf{p}}(\mathbf{p}') = \frac{8}{\pi^2(1 + |\mathbf{p}'|^2)^3}, \quad 0 \leq |\mathbf{p}'| \leq 1 \tag{51}$$

Likewise, for the negative form of the modified Rodrigues parameters we find equally

$$p_{\mathbf{m}}(\mathbf{m}') = \frac{8}{\pi^2(1 + |\mathbf{m}'|^2)^3}, \quad 1 \leq |\mathbf{m}'| < \infty \tag{52}$$

The Rotation Vector

$$p_{\boldsymbol{\theta}}(\boldsymbol{\theta}') = \frac{1 - \cos(|\boldsymbol{\theta}'|)}{4\pi^2|\boldsymbol{\theta}'|^2}, \quad 0 \leq |\boldsymbol{\theta}'| \leq \pi \tag{53}$$

The Axis and Angle of Rotation

$$p_{\theta}(\theta') = \frac{1 - \cos(\theta')}{\pi}, \quad 0 \leq \theta' \leq \pi \tag{54}$$

and

$$p_{\hat{\mathbf{n}}}(\hat{\mathbf{n}}') \equiv p_{\alpha,\beta}(\alpha',\beta') = p_{\alpha}(\alpha')p_{\beta}(\beta') \tag{55}$$

with

$$p_{\alpha}(\alpha') = \frac{\sin\alpha}{2}, \quad p_{\beta}(\beta') = \frac{1}{2\pi} \tag{56ab}$$

where α and β are the spherical angles of $\hat{\mathbf{n}}$ defined by

$$\hat{\mathbf{n}} = \begin{bmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{bmatrix}, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \beta < 2\pi \tag{57}$$

(Equation 54 appears also in Hammermesh [12].)

Symmetric Sequences of Euler Angles

For the 3-1-3 Euler angles

$$p_{313}(\varphi', \vartheta', \psi') = p_{\varphi}(\varphi')p_{\vartheta}(\vartheta')p_{\psi}(\psi') \tag{58}$$

with

$$p_\varphi(\varphi') = \frac{1}{2\pi}, \qquad p_\vartheta(\vartheta') = \frac{\sin\vartheta'}{2}, \qquad p_\psi(\psi') = \frac{1}{2\pi} \tag{59abc}$$

and

$$0 \le \varphi' < 2\pi, \qquad 0 \le \vartheta' \le \pi, \qquad 0 \le \psi' < 2\pi \tag{59def}$$

The same pdf will be obtained for any symmetric sequence of Euler angles.

Asymmetric Sequences of Euler Angles

For the 3-1-2 Euler angles

$$p_{312}(\varphi', \vartheta', \psi') = p_\varphi(\varphi')p_\vartheta(\vartheta')p_\psi(\psi') \tag{60}$$

with

$$p_\varphi(\varphi') = \frac{1}{2\pi}, \qquad p_\vartheta(\vartheta') = \frac{\cos\vartheta'}{2}, \qquad p_\psi(\psi') = \frac{1}{2\pi} \tag{61abc}$$

and

$$0 \le \varphi' < 2\pi, \qquad -\pi/2 \le \vartheta' \le \pi/2, \qquad 0 \le \psi' < 2\pi \tag{61def}$$

The same result will be obtained for any asymmetric sequence of Euler angles.

Davenport Angles

The result for the Davenport angles (generalized Euler angles) [13] will be the same as that for the 3-1-3 Euler angles but with $\sin\vartheta'$ replaced by $|\sin(\vartheta' - \lambda)|$. No absolute value signs are needed for the sine function if $\vartheta' - \lambda$ is restricted to the interval $[0, \pi]$.

Marginal Uniform PDF's for the DCM and Other Representations

A formulation of the joint probability density function for the elements of the direction-cosine matrix (DCM) is of uncertain value, since the variables would be subject to six constraints, which would need to be implicit in the functional form of the joint pdf of all nine elements. While a complete description of this joint pdf lies outside our capacities and our interest, a partial description is still possible and enlightening. For the symmetric sets of Euler angles, we have seen that $\cos\vartheta$ is uniform on the interval $[-1, 1]$, while for the symmetric sets of Euler angles it is $\sin\vartheta$ which has a uniform distribution on that interval. If we examine the formulae for the DCM as a function of the twelve sets of Euler angles [2], we find that

$$\begin{array}{lll} R_{11} = \cos\vartheta_{121}, & R_{12} = \sin\vartheta_{231}, & R_{13} = -\sin\vartheta_{321} \\ R_{21} = -\sin\vartheta_{132}, & R_{22} = \cos\vartheta_{232}, & R_{23} = \sin\vartheta_{312} \\ R_{31} = \sin\vartheta_{123}, & R_{32} = -\sin\vartheta_{213}, & R_{33} = \cos\vartheta_{313} \end{array} \tag{62}$$

where the subscript on ϑ labels the Euler-angle sequence. Thus, it follows that each matrix element of a uniformly distributed random DCM, must be uniformly distributed in the strict sense (sic) on $[-1, 1]$. It is obvious also from any table of the DCM as a function of the twelve sets of Euler angles [2] that the correlation between any two different elements of the DCM must vanish. In summary,

$$R_{ij} \sim U(-1, 1), \quad i, j = 1, 2, 3 \tag{63}$$

$$E\{R_{ij}R_{kl}\} = \frac{1}{3}\,\delta_{ik}\,\delta_{jl}, \quad i,j,k,l = 1,2,3 \tag{64}$$

where $E\{\cdot\}$ denotes the expectation. Note that although the nine elements are uncorrelated, they are certainly not independent.

Consider now the four components of the unit quaternion. If we integrate equation (20) over any two of the components of $\boldsymbol{\eta}'$, we will find for the marginal (uniform) pdf of the remaining component

$$p_{\eta_i}(\eta'_i) = \frac{2}{\pi}\sqrt{1-(\eta'_i)^2}, \quad i = 1,2,3,4 \tag{65}$$

That this is true also for η_4 has not been proved but should be obvious by now. Thus, the components of the uniformly distributed random unit quaternion are not distributed uniformly in the strict sense. However, the four components of the uniformly distributed random quaternion are uncorrelated

$$E\{\eta_i\eta_j\} = \frac{1}{4}\,\delta_{ij}, \quad i,j = 1,2,3,4 \tag{66}$$

Although the four components of the uniformly distributed random quaternion are uncorrelated, they are not independent.

In general, for the attitude representations which are proportional to the axis of rotation, the Cartesian components are uncorrelated while the spherical components are statistically independent. This last statement follows trivially from the fact that the uniform pdf of the axis of rotation is constant on S^2. Hence, the uniform pdf of an attitude representation proportional to the axis of rotation factors trivially in spherical coordinates. Note that the variances of the Cartesian components of the Rodrigues vector are infinite, showing that the moments of these uniform pdf's are not of much practical value. The three uniformly distributed random Euler angles, for all twelve sets, are, as we have stated, statistically independent.

The Uniform PDF for a Rigid Body

For a rigid body we must consider the uniform pdf not only of the attitude but also of the angular velocity. In terms of the attitude matrix and the angular velocity, the composition rule, using the DCM as the attitude representation, is

$$(A_3, \boldsymbol{\omega}_3) = (A_2, \boldsymbol{\omega}_2) \circ (A_1, \boldsymbol{\omega}_1) = (A_2 A_1, \boldsymbol{\omega}_2 + A_2\boldsymbol{\omega}_1) \tag{67}$$

Thus, if we write the 6×1 state vector $\boldsymbol{\xi}$ for the rigid body as

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\omega} \end{bmatrix} \tag{68}$$

we obtain the Jacobian determinant

$$\left|\frac{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\xi}')}{\partial \boldsymbol{\alpha}}\right|_{\boldsymbol{\alpha}=\boldsymbol{\iota}} = \eta'_4 \tag{69}$$

with now $\boldsymbol{\iota} = [0,0,0,0,0,0]^{\mathrm{T}}$. Since the Jacobian determinant of equation (69) is independent of $\boldsymbol{\omega}$, the pdf is uniform in $\boldsymbol{\omega}$ in the strict sense. Thus,

$$p_{\boldsymbol{\eta},\boldsymbol{\omega}}(\boldsymbol{\eta}', \boldsymbol{\omega}') = \frac{p_{\boldsymbol{\eta},\boldsymbol{\omega}}(\mathbf{0},\mathbf{0})}{\sqrt{1-|\boldsymbol{\eta}'|^2}} = p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')p_{\boldsymbol{\omega}}(\mathbf{0}) \tag{70}$$

Clearly, $p_{\boldsymbol{\omega}}(\mathbf{0}) = 0$, since the range of $\boldsymbol{\omega}$ is infinite, so that we cannot write a uniform pdf for $\boldsymbol{\omega}$ as easily as we could for $\boldsymbol{\eta}$, the problem with all functions which are constant on infinite intervals. This difficulty can be sidestepped by restricting the range of $\boldsymbol{\omega}$ to the interior of a suitably large cube centered at the origin and identifying opposite sides in order to ensure a group structure. In that case the uniform $p_{\boldsymbol{\omega}}(\boldsymbol{\omega}')$ is equal to the (finite) inverse volume of the cube. This is artificial and obviously violates our naive understanding of uniformity (while preserving its formal aspect), since the size of this cube must depend on our preconceived idea of the probable range of $\boldsymbol{\omega}$, but it is what one must do in practice.

Generating a Uniformly Distributed Random Attitude Sequence

High-level computer languages generally have in their function libraries routines for computing samples of a random variable uniformly distributed on the interval $[0, 1]$. If x_i' is the ith sample of this random variable, then samples of an equivalent random variable y, distributed uniformly on the interval $[a, b]$ can be generated according to

$$y_i' = a + (b - a)x_i' \tag{71}$$

In the more general case, if we wish to transform samples of a random variable y uniformly distributed on $[a, b]$ to samples of an unknown random variable $z(y)$ with known pdf $p_z(z')$, then

$$\int_{z(a)}^{z(y')} p_z(z'')\, dz'' = \int_a^{y'} p_y(y'')\, dy'' = \frac{1}{b - a} \int_a^{y'} dy'' \tag{72}$$

If $p_z(z')$ is bounded everywhere and does not vanish over any finite interval of z, then $z(y)$ is a monotonically increasing function of y. Thus, the function $z(y)$ is obtained by solving the equation

$$\mathsf{P}_z(z') \equiv \int_{z(a)}^{z'} p_z(z'')\, dz'' = \frac{y' - a}{b - a} \tag{73}$$

for z'. Our ability to find a closed-form expression for $z(y)$, therefore, depends on our ability to find a closed-form expression for the probability distribution function $\mathsf{P}_z(z')$ which can be inverted.

Frequently it is a uniformly random sequence of quaternions which are desired, or a uniformly random sequence of direction-cosine matrices, for which the computation of the quaternion is often an efficient intermediate step. Thus, we focus in this section on the generation of uniformly random sequences of quaternions.

For the vector components of the unit quaternion the inversion of equation (73) is not possible in closed-form. To see this we note that the pdf for the magnitude of the vector components of the unit quaternion is given by

$$p_\eta(\eta') \equiv \int p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')\, (\eta')^2\, d^2\hat{\boldsymbol{\eta}} = \frac{4(\eta')^2}{\pi\sqrt{1 - (\eta')^2}} \tag{74}$$

from which we obtain

$$\mathsf{P}_\eta(\eta') = \frac{2}{\pi}\left(\sin^{-1} \eta' - \eta'\sqrt{1 - (\eta')^2}\right) \tag{75}$$

The same is true for the probability distribution function of the angle of rotation

$$\mathrm{P}_\theta(\theta') = \frac{1}{\pi}(\theta' - \sin\theta') \tag{76}$$

The right member of neither equation (75) nor of equation (76) can be inverted in closed-form, but only by an infinite process.

Thus, the value of the vector components of the unit quaternion or the angle of rotation does not provide a convenient vehicle for generating uniformly distributed random samples of the attitude representations. A similarly non-invertible function will, in fact, appear equivalently in the probability distribution function of all the other attitude representations except for the Euler angles.

Let us consider next the 3-1-3 Euler angles. Clearly, both φ and ψ are uniformly distributed on the interval $[0, 2\pi)$. The probability distribution function for ϑ is trivial to determine, namely

$$\mathrm{P}_\vartheta(\vartheta') = \frac{1 - \cos\vartheta'}{2} \equiv \mu(\vartheta') \tag{77}$$

The random variable μ is uniformly distributed on the interval $[0, 1]$, and

$$\vartheta' = \mathrm{Arccos}(1 - 2\mu') \tag{78}$$

Thus, we need only compute random samples of φ, ψ, and μ, after which the quaternion may be calculated from the formula [2]

$$\bar{\eta} = \begin{bmatrix} s(\vartheta)c(\varphi - \psi) \\ s(\vartheta)s(\varphi - \psi) \\ c(\vartheta)s(\varphi + \psi) \\ c(\vartheta)c(\varphi + \psi) \end{bmatrix} \tag{79}$$

with

$$s(x) \equiv \sin(x/2), \qquad c(x) \equiv \cos(x/2) \tag{80}$$

In fact, it is unnecessary to compute ϑ' as an intermediate variable since

$$s(\vartheta') = \sqrt{\mu'}, \qquad c(\vartheta') = \sqrt{1 - \mu'} \tag{81}$$

with the positive sign always chosen for the square roots.

Note now that if φ and ψ are strictly uniform on $[0, 2\pi)$, then so are $(\varphi \pm \psi)/2$ equivalently when one treats the addition of angles as being modulo 2π. Thus, we may rewrite equation (79) in the form

$$\bar{\eta} = \begin{bmatrix} \sqrt{\mu}\cos(\sigma) \\ \sqrt{\mu}\sin(\sigma) \\ \sqrt{1 - \mu}\sin(\tau) \\ \sqrt{1 - \mu}\cos(\tau) \end{bmatrix} \tag{82}$$

and a uniform distribution of $\bar{\eta}$ is obtained by assuming μ that is distributed uniformly on $[0, 1]$ while σ and τ are distributed uniformly on $[0, 2\pi)$.

Thus, we may propose the following algorithm for the generation of uniformly random sequences of quaternions:

- Compute samples σ' and τ' of σ and τ, which are independent and are distributed uniformly on $[0, 2\pi)$.

- Compute a sample μ' of μ, which is distributed uniformly on $[0, 1]$.
- Compute the sample $\bar{\eta}'$ of $\bar{\eta}$ according to equation (82).

This algorithm is due, apparently, to Shoemake [14].

The simplest way to generate uniformly distributed samples of the direction cosine matrix is apparently also directly via the Euler angles.

It is amusing to note that the Euler angles, generally shunned in attitude studies because of their singularity and the cumbersome trigonometric functions, seem to be the superior representation for generating uniformly distributed random sequences of attitude in simulation. Other methods (reported to the author by Bauer) are presented in reference [6].

Consequences for Attitude Estimation

Clearly, from the above discussion, one ought to take account in any attitude estimation problem that there is an *a priori* probability distribution of the attitude even in the absence of earlier measurements, namely that given by the uniform pdf associated with the particular attitude representation.

Maximum likelihood estimation (MLE) will not be altered by this realization, because MLE ignores any *a priori* information. However, the related maximum *a posteriori* estimation (MAP) approach will be altered. If $p_{\mathbf{Z}|\boldsymbol{\xi}}(\mathbf{Z}'|\boldsymbol{\xi}')$ is the conditional pdf of the measurements $\mathbf{Z}$ given the attitude, then according to the MAP estimation criterion, the optimal attitude minimizes the MAP negative-log-likelihood function given by [8]

$$J(\boldsymbol{\xi}) = -\ln(p_{\mathbf{Z}|\boldsymbol{\xi}}(\mathbf{Z}'|\boldsymbol{\xi}')) - \ln(p_{\boldsymbol{\xi}}(\boldsymbol{\xi}')) \tag{83}$$

where the second pdf is the uniform pdf for the representation. However, the uniform attitude pdf cannot depend on the actual attitude. Thus, it cannot contribute to the MAP attitude estimate either.

What the uniform attitude pdf does provide in attitude estimation is a means for treating attitude estimation in a Bayesian framework. In Bayesian estimation one must construct $p_{\boldsymbol{\xi}|\mathbf{Z}}(\boldsymbol{\xi}'|\mathbf{Z}')$ from $p_{\mathbf{Z}|\boldsymbol{\xi}}(\mathbf{Z}'|\boldsymbol{\xi}')$. To do this one must have an *a priori* pdf for the attitude random variables $\boldsymbol{\xi}$, which is supplied by the uniform pdf when there is no prior attitude information. Given this *a priori* pdf one can then construct the joint pdf $p_{\mathbf{Z},\boldsymbol{\xi}}(\mathbf{Z}', \boldsymbol{\xi}')$ and thence the conditional pdf of the attitude given the measurements. The author has, in fact, succeeded in developing the Wahba problem [15–17] in just such a Bayesian framework. This will be the subject of a future publication [18].

It might also be asked what implication this work has for the initialization of the Kalman filter when there is no *a priori* information about the spacecraft attitude. The uniform pdf might then provide a means for calculating the initial covariance matrix. We first point out that the uniform pdf always gives a specific result for the initial mean of the attitude distribution which is not necessarily meaningful. For the quaternion, for example, the initial mean will be $[0, 0, 0, 0]$, which is not an allowed value for the quaternion at all and totally unsuitable as a point of linearization. For the three-dimensional "vectorial" representations, the initial mean will generally be $\mathbf{0}$. For the initial covariance, those representations whose range is bounded will obviously have finite and not very large covariance matrices. For the Rodrigues parameters, which are defined for all of R^3, the variances will all be infinite. Hence, the interpretation of the initial covariance (generally a useful concept only for linear systems, which macroscopically the attitude is not) is beset with difficulties, especially when one considers that these numerical results will be obtained no matter how the spacecraft and inertial coordinate frames are chosen. In particular, we note that even though the initial covariance matrix must be finite, its effect on the final estimate must be zero if there is no *a priori* attitude information, which will not be the case in the Kalman filter. The obvious conclusion, then, is that the uniform attitude pdf will be useful only in estimation regimes that work directly with the attitude pdf, as in the Bayesian estimation scheme outlined in the preceding paragraph, rather than with the attitude moments. This does not mean, however, that an approximate Kalman-filter-like estimation procedure may not be obtainable within this Bayesian estimation scheme in certain cases. In any event, if the choice of attitude representation parameters in the Kalman filter update is the rotation vector of the small rotation from the predicted to the updated attitude, then there will seldom be a *practical* reason to choose the initial filter variances to be much larger than π^2.

Discussion

We have attempted to give a fairly complete and mathematically simple description of uniform attitude distributions. The results presented here are very much in conflict with the conventional wisdom among engineers on how to generate a uniform attitude distribution. That wisdom was that one first picked a direction at random, distributed uniformly over the unit sphere (equations (55) through (57)), and took that as the axis of rotation. The angle of rotation was then assumed to have constant pdf in angle over $[0, \pi)$. That would be equivalent to positing a pdf for the rotation vector of

$$p_{\boldsymbol{\theta}}(\boldsymbol{\theta}') = \frac{1}{8\pi^2}, \qquad 0 \leq |\boldsymbol{\theta}'| \leq \pi \tag{53$'$}$$

which disagrees with the uniform pdf given by equation (53) above. Equivalently, the common wisdom assumes incorrectly that the angle of rotation has a constant pdf of $1/\pi$ on $[0, \pi]$ rather than the correct expression given by equation (54). Thus, attitude simulation testing has often overemphasized small rotations.

We can understand in another way that the pdf of the angle of rotation cannot be uniform. The angle of rotation is just the length variable in a spherical-geometric representation of the rotation vector. Since the composition rule for the rotation vector is approximately simple addition in a small region of the origin, it follows that the pdf of the rotation vector must be approximately constant in that region (as demonstrated by equation (53)). If one transforms to spherical coordinates, then the pdf of the angle of rotation θ near the origin must be proportional to θ^2 (as demonstrated by equation (54)). One must be cautious, clearly, in relying upon an insufficiently informed intuition.

If $(\varphi, \vartheta, \psi)$ is the 3-1-3 or any symmetric sequence of Euler angles, then $(\varphi, \mu(\vartheta), \psi)$, with $\mu(\vartheta)$ defined by equation (77) is also a three-dimensional attitude representation. This representation is remarkable in that over its domain $0 \leq \varphi' < 2\pi, 0 \leq \mu' \leq 1, 0 \leq \psi' < 2\pi$ its uniform pdf is constant (and equal to $1/4\pi^2$). The same holds true for the asymmetric sequences of Euler angles with new attitude variables $(\varphi, \nu(\vartheta), \psi)$ and

$$\nu(\vartheta') \equiv \frac{1 + \sin(\vartheta')}{2}, \qquad 0 \leq \nu(\vartheta') \leq 1 \tag{84}$$

The composition rule of the Euler angles is far from being at most linear, even when one rotation is in infinitesimal (see the comments following equation (15)). In fact, it is quite complicated [10]. Nonetheless, the Jacobian determinant is very simple. One might suspect from this that the composition rule for the asymmetric sequences of Euler angles is, in fact, not more complicated than that for the symmetric sequences and that the authors of reference [10] abandoned their search for a direct composition rule for the asymmetric sequences of Euler angles too early.

While the attitude variables (φ, μ, ψ) and (φ, ν, ψ) are remarkable for having a pdf which is both uniform as well as uniform in the strict sense, they are by no means unique in having this property. For any three-dimensional attitude representation proportional to the axis of rotation (the vector components of the unit quaternion, the Rodrigues vector, the two cases of the modified Rodrigues vector, the rotation vector), the spherical components under the assumption of uniformity are statistically independent. Hence, like the Euler angles, the uniform pdf factors as

$$p_{\boldsymbol{\xi}}(\boldsymbol{\xi}') = p_{\xi}(\xi')\, p_{\alpha}(\alpha')\, p_{\beta}(\beta') \tag{85}$$

with α and β the spherical angles defined by equation (57) and ξ the magnitude of $\boldsymbol{\xi}$. Since the univariate pdf $p_{\xi}(\xi')$ does not vanish over a finite interval, it follows that the associated random variable γ defined by

$$\gamma' \equiv \mathsf{P}_{\xi}(\xi') = \int_0^{\xi'} p_{\xi}(x')\, dx', \qquad 0 \leq \gamma' \leq 1 \tag{86}$$

with $\mathsf{P}_{\xi}(\xi')$ the *probability distribution function* of ξ, can replace ξ as a random variable. Then $(\gamma_1, \gamma_2, \gamma_3) \equiv (\mu(\alpha), \beta/2\pi, \gamma(\xi))$, also functions as a set of attitude random variables.[15] For this set of random variables the uniform attitude pdf has the constant value unity, and the new random variables are distributed uniformly over the unit cube. Thus, there is nothing extraordinary about a uniform attitude pdf also being uniform in the strict sense. However, only for the Euler angles does the reconstruction of the realization of the original familiar random attitude representation from that of these associated random variables (note equations (75) and (76)) impose a very small computational burden.

Robert Bauer has also addressed the problem of simulating totally random attitudes, and within the framework of Riemannian metrics (as opposed to the use of Jacobian methods in the present work). His work [19] was presented at the NASA Space Flight Mechanics Symposium, NASA Goddard Space Flight Center, Greenbelt, Maryland, on June 19–21, 2001. Despite a commonality of purpose, there is little overlap in our papers because: (1) the authors did not learn of each other's work until both were nearly complete (in the spring of 2000); (2) the methodologies are different; and (3) the thrusts of the two works are not identical. In particular, Bauer is primarily interested in non-probabilistic approaches to sampling attitude.

Bauer also offers a general formula for the uniform pdf based on his Riemannian metric approach [19]. He obtains a metric tensor g_{ij} for an arbitrary three-dimensional attitude representation according to (in the notation of this article)

$$g_{ij}(\boldsymbol{\xi}') = \left(\frac{\partial\, \bar{\eta}(\boldsymbol{\xi}')}{\partial \xi_i}\right)^{\mathrm{T}} \left(\frac{\partial\, \bar{\eta}(\boldsymbol{\xi}')}{\partial \xi_j}\right) \tag{87}$$

[15]Every component of $(\gamma_1, \gamma_2, \gamma_3)$, in fact, is a probability distribution function and the integration constant of $\nu(\vartheta')$ has been chosen so that this variable will be a probability distribution function as well.

The uniform measure density is then

$$\rho_{\xi}(\boldsymbol{\xi}') = \sqrt{(\det g(\boldsymbol{\xi}'))} \tag{88}$$

This is a fourth method to compute the uniform pdf of an arbitrary three-dimensional attitude representation. However, Bauer's formula does not yield the density of a probability measure but of a volume measure (the Riemannian volume metric density). Thus, $\rho_{\xi}(\xi')$ must be divided by the total volume of the attitude-representation space in order to obtain the uniform pdf for the attitude representation.

$$p_{\xi}(\boldsymbol{\xi}') = \frac{\rho_{\xi}(\boldsymbol{\xi}')}{\int \rho_{\xi}(\boldsymbol{\xi}'')\, d^3\boldsymbol{\xi}''} \tag{89}$$

Given the additional burden of computing the integral in equation (89), equation (49) would seem to be the simplest general expression for computing the *probability* density function of a three-parameter attitude representation.

Note Added in Proof:

Markley (private communication) has disclosed to the author a result for third-order moments of the elements of the direction-cosine matrix, namely,

$$E\{R_{ij}R_{kl}R_{mn}\} = \frac{1}{6}\,\boldsymbol{\epsilon}_{ikm}\boldsymbol{\epsilon}_{jln} \tag{90}$$

where $\boldsymbol{\epsilon}_{ijk}$ is the Levi-Civita symbol [2]. He has suggested it be published here.

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Appendix A: Details of the Derivations

We give here some details of the derivations of the uniform pdf for some of the attitude representations. Since the derivations are quite intricate, we hope the presentation of a few of the intermediate steps will be of service to the reader.

The Rodrigues Parameters

The 3×1 matrix of Rodrigues parameters (the Rodrigues vector) is related to the quaternion and the axis and angle of rotation by [2]

$$\boldsymbol{\rho} = \frac{\boldsymbol{\eta}}{\eta_4} = \tan(\theta/2)\hat{\mathbf{n}} \tag{A1}$$

The composition rule is [2]

$$\boldsymbol{\alpha} \circ \boldsymbol{\rho}' = \frac{\boldsymbol{\alpha} + \boldsymbol{\rho}' - \boldsymbol{\alpha} \times \boldsymbol{\rho}'}{1 - \boldsymbol{\alpha} \cdot \boldsymbol{\rho}'} \tag{A2}$$

whence

$$\left| \frac{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\rho}')}{\partial \boldsymbol{\alpha}} \right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left|\det(I_{3\times3} - [[\boldsymbol{\rho}']] + \boldsymbol{\rho}'\boldsymbol{\rho}'^{\mathrm{T}})\right| = (1 + |\boldsymbol{\rho}'|^2)^2 \tag{A3}$$

where [2]

$$[[\mathbf{v}]] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \tag{A4}$$

It follows again from equation (12), after computing $p_{\boldsymbol{\rho}}(\mathbf{0})$ from equation (15), that

$$p_{\boldsymbol{\rho}}(\boldsymbol{\rho}') = \frac{1}{\pi^2(1 + |\boldsymbol{\rho}'|^2)^2} \tag{A5}$$

The calculation of the determinant in equation (A3) is tedious. A simpler method, therefore, is to apply the implicit function theorem directly to equation (20), calculating the Jacobian determinant $|\partial(\eta_1, \eta_2, \eta_3)/\partial(\rho_1, \rho_2, \rho_3)|$ instead of the result of equation (A3), and then calculating the pdf for the Rodrigues parameters from the pdf already calculated for the vector components of the quaternion using the relations

$$p_\rho(\boldsymbol{\rho}') = p_\eta(\boldsymbol{\eta}(\boldsymbol{\rho}'))\left|\frac{\partial(\eta_1', \eta_2', \eta_3')}{\partial(\rho_1', \rho_2', \rho_3')}\right|, \quad \text{and} \quad \boldsymbol{\eta}(\boldsymbol{\rho}) = \frac{\boldsymbol{\rho}}{\sqrt{1 + |\boldsymbol{\rho}|^2}} \qquad \text{(A6ab)}$$

This calculation would be equally tedious except that the vector components of the unit quaternion and the Rodrigues parameters are both proportional to the axis of rotation $\hat{\mathbf{n}}$. Hence, the only interdependence of these two representations that should be of interest is that of $|\boldsymbol{\eta}|$, the magnitude of $\boldsymbol{\eta}$, on $|\boldsymbol{\rho}|$, the magnitude of $\boldsymbol{\rho}$. If we write in the usual short-hand

$$p_\eta(\boldsymbol{\eta}')\, d^3\boldsymbol{\eta}' = p_\rho(\boldsymbol{\rho}')\, d^3\boldsymbol{\rho}' \qquad \text{(A7)}$$

or

$$p_\eta(\boldsymbol{\eta}')\,|\boldsymbol{\eta}'|^2\, d|\boldsymbol{\eta}'|\, d^2\Omega_{\hat{\mathbf{n}}'} = p_\rho(\boldsymbol{\rho}')\,|\boldsymbol{\eta}'|^2\, d|\boldsymbol{\rho}'|\, d^2\Omega_{\hat{\mathbf{n}}'} \qquad \text{(A8)}$$

then it is evident that we must have

$$\left|\frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\rho}'}\right| = \frac{|\boldsymbol{\eta}'|^2}{|\boldsymbol{\rho}'|^2}\frac{\partial|\boldsymbol{\eta}'|}{\partial|\boldsymbol{\rho}'|} = \frac{1}{(1 + |\boldsymbol{\rho}'|^2)^{5/2}} \qquad \text{(A9)}$$

From equation (A9) and

$$p_\eta(\boldsymbol{\eta}(\boldsymbol{\rho}')) = \frac{\sqrt{1 + |\boldsymbol{\rho}'|^2}}{\pi^2} \qquad \text{(A10)}$$

we arrive again at equation (A5).

The Modified Rodrigues Parameters

For the modified Rodrigues parameters [2] we have (for the positive form)

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{1 + \eta_4} = \tan(\theta/4)\hat{\mathbf{n}}, \quad \text{and} \quad \boldsymbol{\rho}(\mathbf{p}) = 2\,\mathbf{p}/(1 - |\mathbf{p}|^2) \qquad \text{(A11ab)}$$

The pdf can be most easily calculated from the pdf of the Rodrigues vector, and again the Jacobian determinant can be obtained from the calculation of a single radial derivative with the results

$$p_\rho(\boldsymbol{\rho}(\mathbf{p}')) = \frac{(1 - |\mathbf{p}'|^2)^4}{\pi^4(1 + |\mathbf{p}'|^2)^4}, \qquad \left|\frac{\partial \boldsymbol{\rho}'}{\partial \mathbf{p}'}\right| = \frac{8(1 + |\mathbf{p}'|^2)}{(1 - |\mathbf{p}'|^2)^4} \qquad \text{(A12ab)}$$

Hence,

$$p_{\mathbf{p}}(\mathbf{p}') = \frac{8}{\pi^2(1 + |\mathbf{p}'|^2)^3} \qquad \text{(A13)}$$

where we restrict $|\mathbf{p}'|$ to the region $|\mathrm{p}'| \leq 1$. One could have extended the domain of $\mathbf{p}$ to all space, but we chose to avoid infinite values of the representation.

For the negative form of the vector

$$\mathbf{m} = \frac{\boldsymbol{\eta}}{1 - \eta_4} = \cot(\theta/2)\hat{\mathbf{n}} \qquad \text{(A14)}$$

we find equally easily

$$p_{\mathbf{m}}(\mathbf{m}') = \frac{8}{\pi^2(1 + |\mathbf{m}'|^2)^3} \tag{A15}$$

but we restrict $\mathbf{m}$ to the region $|\mathbf{m}'| \geq 1$.

The Rotation Vector

It is easiest to compute the pdf for the rotation vector [2] from that of the vector components of the quaternion. We note that

$$\boldsymbol{\theta} = \theta\hat{\mathbf{n}}, \quad \text{and} \quad \boldsymbol{\eta}(\boldsymbol{\theta}) = \sin(|\boldsymbol{\theta}|/2)\hat{\mathbf{n}} \tag{A16ab}$$

from which it follows that

$$p_{\boldsymbol{\eta}}(\boldsymbol{\eta}(\boldsymbol{\theta}')) = \frac{1}{\pi^2\cos(|\boldsymbol{\theta}'|/2)}, \qquad \left|\frac{\partial\boldsymbol{\eta}'}{\partial\boldsymbol{\theta}'}\right| = \frac{\sin^2(|\boldsymbol{\theta}'|/2)\cos(|\boldsymbol{\theta}'|/2)}{2|\boldsymbol{\theta}'|^2} \tag{A16cd}$$

and finally

$$p_{\boldsymbol{\theta}}(\theta') = \frac{\sin^2(|\boldsymbol{\theta}'|/2)}{2\pi^2|\boldsymbol{\theta}'|^2} = \frac{1 - \cos(|\boldsymbol{\theta}'|)}{4\pi^2|\boldsymbol{\theta}'|^2} \tag{A17}$$

defined in the region $0 \leq |\boldsymbol{\theta}'| \leq \pi$.

The Axis and Angle of Rotation

We see immediately that the result for the rotation vector can be factored as

$$p_{\boldsymbol{\theta}}(\boldsymbol{\theta}') = \left(\frac{1 - \cos(|\boldsymbol{\theta}'|)}{\pi|\boldsymbol{\theta}'|^2}\right)\left(\frac{1}{4\pi}\right) = p_{\theta}(\theta')p_{\hat{\mathbf{n}}}(\hat{\mathbf{n}}') \tag{A18}$$

We note that the pdf does not depend on $\hat{\mathbf{n}}'$ at all, hence, we must have

$$p_{\theta}(\theta') = \frac{1 - \cos(\theta')}{\pi(\theta')^2}, \quad p_{\hat{\mathbf{n}}}(\hat{\mathbf{n}}') = \frac{1}{4\pi} \tag{A19ab}$$

To understand equation (19b) we note that the unit axis vector $\hat{\mathbf{n}}$ is parameterized in terms of spherical angles as

$$\hat{\mathbf{n}} = \begin{bmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{bmatrix}, \quad 0 \leq \alpha \leq \pi, \quad 0 \leq \beta < 2\pi \tag{A20}$$

Here, α is the angle between the z-axis and $\hat{\mathbf{n}}$, and β is the dihedral angle about the positive z-axis from the xz-plane to the plane containing the z-axis and $\hat{\mathbf{n}}$. Thus,

$$d^3\boldsymbol{\theta}' = (\theta')^2\,d\theta'\,d^2\hat{\mathbf{n}} \quad \text{with} \quad d^2\hat{\mathbf{n}}' \equiv \sin\alpha'\,d\alpha'\,d\beta' \equiv d^2\Omega_{\hat{\mathbf{n}}'} \tag{A21}$$

and the integral of $d^2\Omega_{\hat{\mathbf{n}}'}$ over all directions is 4π.

The Symmetric Sequence of Euler Angles

For the 3-1-3 Euler angles [5] we begin again with the pdf for the vector components of the quaternion. Defining first, in the notation of reference [2]

$$R_{313}(\varphi, \vartheta, \psi) \equiv R(\hat{\mathbf{3}}, \psi)R(\hat{\mathbf{1}}, \vartheta)R(\hat{\mathbf{3}}, \varphi) \tag{A22}$$

where

$$\hat{\mathbf{1}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{2}} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{3}} \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{A23abc}$$

The vector components of the quaternion are then given by

$$\boldsymbol{\eta} = \begin{bmatrix} s(\vartheta)c(\varphi - \psi) \\ s(\vartheta)s(\varphi - \psi) \\ c(\vartheta)s(\varphi + \psi) \end{bmatrix} \tag{A24}$$

where

$$s(x) \equiv \sin(x/2), \qquad c(x) \equiv \cos(x/2) \tag{A25}$$

Hence, from $\eta_4' = c(\vartheta')c(\varphi + \psi)$ it follows that

$$p_{\boldsymbol{\eta}}(\boldsymbol{\eta}(\varphi', \vartheta', \psi')) = \frac{1}{\pi^2 |c(\vartheta') \cos(\varphi' + \psi')|} \tag{A26}$$

and

$$\left| \frac{\partial(\boldsymbol{\eta}')}{\partial(\varphi', \vartheta', \psi')} \right| = s(\vartheta)c^2(\vartheta')|c(\varphi' + \psi')|/4 \tag{A27}$$

whence

$$p_{313}(\varphi', \vartheta', \psi') = \frac{\sin \vartheta'}{8\pi^2} \tag{A28}$$

which is defined on the intervals $0 \leq \varphi' < 2\pi$, $0 \leq \vartheta' \leq \pi$, $0 \leq \psi' < 2\pi$. The identical result holds for the other five symmetric sets of Euler angles.

We may write this result equivalently as

$$p_{313}(\varphi', \vartheta', \psi') = p_{\varphi}(\varphi')p_{\vartheta}(\vartheta')p_{\psi}(\psi') \tag{A29}$$

with

$$p_{\varphi}(\varphi') = \frac{1}{2\pi}, \qquad p_{\vartheta}(\vartheta') = \frac{\sin \vartheta'}{2}, \qquad p_{\psi}(\psi') = \frac{1}{2\pi} \tag{A30abc}$$

Again, the same pdf will be obtained for any symmetric sequence of Euler angles.

Note that equation (A28) would be difficult to obtain from equation (12) because the pdf vanishes at $\boldsymbol{\iota} = (0, 0, 0)$. Either L'Hôpital's rule would need to be invoked when taking the limit of equation (13), or equation (13) would need to be evaluated at a different value of α. In addition, the application of the composition rule for the 3-1-3 Euler angles [10] would not be simple.

The Asymmetric Sequence of Euler Angles

For the 3-1-2 Euler angles the calculation of the Jacobian determinant for the transformation from the vector components of the unit quaternion to the 3-1-2 Euler angles is an ordeal. Therefore, we seek a method to avoid this calculation and rely instead on the invariance principle set forth in equation (8).

Consider the direction-cosine matrix generated by a 3-1-2 sequence of Euler angles:

$$R_{312}(\varphi, \vartheta, \psi) = R(\hat{\mathbf{2}}, \psi)R(\hat{\mathbf{1}}, \vartheta)R(\hat{\mathbf{3}}, \varphi) \tag{A31}$$

We note that

$$\hat{\mathbf{2}} = R(\hat{\mathbf{1}}, \pi/2)\hat{\mathbf{3}} \tag{A32}$$

from which it follows that [2]

$$\begin{aligned} R(\hat{\mathbf{2}}, \psi) &= R(R(\hat{\mathbf{1}}, \pi/2)\hat{\mathbf{3}}, \psi)) \\ &= R(\hat{\mathbf{1}}, \pi/2)R(\hat{\mathbf{3}}, \psi)R^{\mathrm{T}}(\hat{\mathbf{1}}, \pi/2) \end{aligned} \tag{A33}$$

Thus,

$$\begin{aligned} R_{312}(\varphi, \vartheta, \psi) &= R(\hat{\mathbf{1}}, \pi/2)R(\hat{\mathbf{3}}, \psi)R^{\mathrm{T}}(\hat{\mathbf{1}}, \pi/2)R(\hat{\mathbf{1}}, \vartheta)R(\hat{\mathbf{3}}, \varphi) \\ &= R(\hat{\mathbf{1}}, \pi/2)R(\hat{\mathbf{3}}, \varphi)R(\hat{\mathbf{1}}, \vartheta - \pi/2)R(\hat{\mathbf{3}}, \varphi) \end{aligned} \tag{A34}$$

or

$$R_{312}(\varphi, \vartheta, \psi) = R(\hat{\mathbf{1}}, \pi/2)R_{313}(\varphi, \vartheta - \pi/2, \psi) \tag{A35}$$

From the invariance property of the pdf, it follows that the probability density function of $(\varphi, \vartheta, \psi)_{312}$ will be the same as the probability density function of $(\varphi, \vartheta - \pi/2, \psi)_{313}$. To see this, note that equation (A35) can be written as

$$(\varphi', \vartheta', \psi')_{312} = (0, \pi/2, 0)_{313} \circ (\varphi', \pi/2 - \vartheta', \psi')_{313} \tag{A36}$$

from which it follows that

$$p_{(\varphi, \vartheta, \psi)_{312}}(\varphi', \vartheta', \psi') = p_{(0, \pi/2, 0)_{313} \circ (\varphi, \vartheta, \psi)_{313}}(\varphi', \pi/2 - \vartheta', \psi') \tag{A37}$$

and by equation (8)

$$p_{(0, \pi/2, 0)_{313} \circ (\varphi, \vartheta, \psi)_{313}}(\varphi', \pi/2 - \vartheta', \psi') = p_{(\varphi, \vartheta, \psi)_{313}}(\varphi', \pi/2 - \vartheta', \psi') \tag{A38}$$

Hence,

$$p_{312}(\varphi', \vartheta', \psi') = p_{313}(\varphi', \vartheta' - \pi/2, \psi') = \left|\frac{\sin(\pi/2 - \vartheta')}{8\pi^2}\right| = \frac{\cos\vartheta'}{8\pi^2} \tag{A39}$$

which is defined over the region $0 \le \varphi' < 2\pi$, $-\pi/2 \le \vartheta' \le \pi/2$, $0 \le \psi' < 2\pi$.

Similarly to equation (A29) we can write

$$p_{312}(\varphi', \vartheta', \psi') = p_{\varphi}(\varphi')p_{\vartheta}(\vartheta')p_{\psi}(\psi') \tag{A40}$$

with

$$p_{\varphi}(\varphi') = \frac{1}{2\pi}, \qquad p_{\vartheta}(\vartheta') = \frac{\cos\vartheta'}{2}, \qquad p_{\psi}(\psi') = \frac{1}{2\pi} \tag{A41abc}$$

While the pdf for the 3-1-2 Euler angles does not vanish at $\boldsymbol{\iota} = (0, 0, 0)$, the application of the composition rule for the 3-1-2 Euler angles would be hampered by the fact that no closed-form expression is currently known for it, [10] and an intermediate representation would need to be used.