The Journal of the Astronautical Sciences, Vol. 51, No. 2, April–June 2003, pp. 133–141

Centering and Observability in Attitude-Independent Magnetometer-Bias Determination

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Abstract

The TWOSTEP algorithm is examined for the case where the centered portion of the negative-log-likelihood function provides incomplete observability of the magnetometer-bias vector. In those cases where the full negative-log-likelihood function provides a complete estimate, the TWOSTEP algorithm can be modified to provide an estimate of all three components of the magnetometer bias vector. However, the procedure leads to a discrete degeneracy of the estimate which can be resolved only by explicit evaluation of the negative-log-likelihood.

Introduction

The TWOSTEP algorithm [1] is a very efficient and robust algorithm for the estimation inflight of the magnetometer-bias vector without knowledge of the attitude. Numerous comparisons [2] have shown that TWOSTEP is superior to all other attitude-independent algorithms providing an estimate of the magnetometer-bias vector in hundreds of simulations, in many of which the other algorithms failed completely. Only Acuña's algorithm [2, 3] was seen to be superior to TWOSTEP in some cases, namely, those in which a reference magnetic field was not available, in which case TWOSTEP cannot be applied, or when the measurement noise levels had been severely mismodelled, in which case Acuña's algorithm was sometimes marginally better. Acuña's algorithm, however, requires that the spacecraft be made to spin rapidly about two different axes in succession, which limits its range of applicability. Thus, the TWOSTEP algorithm is the clear algorithm of choice for near-Earth missions, while Acuña's algorithm is unchallenged for interplanetary missions.

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TWOSTEP has been extended also to estimate linear parameters of the magnetometer calibration [4].

The TWOSTEP algorithm depends on the separation of the negative-log-likelihood function into two pieces. Therefore, there is always the possibility that neither piece will have complete information on the magnetometer-bias vector even if the estimation is possible with the full negative-log-likelihood function. These cases will occur mostly when the bias vector is barely observable, since the first step of TWOSTEP works quite well in general. It is these cases of poor observability which we treat in the present work.

The TWOSTEP algorithm [1] assumes a measurement model of the form

$$\mathbf{B}_k = A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k \,, \quad k = 1, \ldots, N \tag{1}$$

where $\mathbf{B}_k$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time t_k; $\mathbf{H}_k$ is the corresponding value of the geomagnetic field with respect to an Earth-fixed coordinate system; A_k is the attitude of the magnetometer with respect to the Earth-fixed coordinates; $\mathbf{b}$ is the magnetometer bias; and $\boldsymbol{\varepsilon}_k$ is the measurement noise. The measurement noise, which includes both sensor errors and geomagnetic field model uncertainties, is assumed to be white and Gaussian.

The dependence on the attitude is eliminated by considering the square of the magnitude of the magnetometer readings as an effective measurement. Thus, we define effective measurements and measurement noise as

$$z_k \equiv |\mathbf{B}_k|^2 - |\mathbf{H}_k|^2 \tag{2a}$$

$$v_k \equiv 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k - |\boldsymbol{\varepsilon}_k|^2 \tag{2b}$$

whence,

$$z_k = 2\,\mathbf{B}_k \cdot \mathbf{b} - |\mathbf{b}|^2 + v_k \,, \quad k = 1, \ldots, N \tag{3}$$

To arrive at a linear quadratic cost function we define center values of the different time series according to

$$\bar{z} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} z_k \,, \qquad \overline{\mathbf{B}} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mathbf{B}_k \tag{4ab}$$

$$\bar{v} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} v_k \,, \qquad \bar{\mu} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mu_k \tag{4cd}$$

where

$$\frac{1}{\bar{\sigma}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{5}$$

Then it follows that

$$\bar{z} = 2\,\overline{\mathbf{B}} \cdot \mathbf{b} - |\mathbf{b}|^2 + \bar{v} \tag{6}$$

If we define now

$$\tilde{z}_k \equiv z_k - \bar{z} \,, \qquad \widetilde{\mathbf{B}}_k \equiv \mathbf{B}_k - \overline{\mathbf{B}} \tag{7ab}$$

$$\tilde{v}_k \equiv v_k - \bar{v} \,, \qquad \tilde{\mu}_k \equiv \mu_k - \bar{\mu} \tag{7cd}$$

then subtracting equation (6) from equation (3) leads to

$$\tilde{z}_k = 2\,\widetilde{\mathbf{B}}_k \cdot \mathbf{b} + \tilde{v}_k\,, \quad k = 1, \ldots, N \tag{8}$$

The derived measurements are linear in the magnetometer-bias vector. This operation is called *centering*. We call $\bar{z}$ the center measurement and $\tilde{z}_k$ the centered measurement at time t_k.

The centered measurements are all mutually correlated through the common term $\bar{z}$. Nonetheless, it is possible to write the full negative-log-likelihood function $J(\mathbf{b})$

$$J(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{\sigma_k^2}(z_k - 2\,\mathbf{B}_k\cdot\mathbf{b} + |\mathbf{b}|^2 - \mu_k)^2 + \log\sigma_k^2 + \log 2\pi\right] \tag{9}$$

as the sum of two statistically independent terms, one depending only on the centered measurements and the other only on the center measurement. Thus,

$$J(\mathbf{b}) = \widetilde{J}(\mathbf{b}) + \overline{J}(\mathbf{b}) \tag{10}$$

where

$$\widetilde{J}(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(\tilde{z}_k - 2\widetilde{\mathbf{B}}_k\cdot\mathbf{b} - \tilde{\mu}_k)^2 + \text{terms independent of } \mathbf{b} \tag{11a}$$

and

$$\overline{J}(\mathbf{b}) = \frac{1}{2}\frac{1}{\bar{\sigma}^2}(\bar{z} - 2\,\overline{\mathbf{B}}\cdot\mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2 + \text{terms independent of } \mathbf{b} \tag{11b}$$

Further details of these functions can be found in [1].

The TWOSTEP algorithm consists in first finding the estimate $\widetilde{\mathbf{b}}^*$ of the magnetometer-bias vector based on the centered measurements alone and then using this as a starting value for the minimization of the entire negative-log-likelihood function to find $\mathbf{b}^*$. Except in the cases treated in this work, the centered estimate alone seems to be always adequate, and the treatment of the center negative-log-likelihood function $\overline{J}(\mathbf{b})$ seldom requires more than a single iteration.

The estimation error covariance can be computed from the Fisher information matrix, F_{bb}

$$F_{bb} \equiv E\left\{\frac{\partial^2 J}{\partial\mathbf{b}\partial\mathbf{b}^{\mathrm{T}}}\right\} \tag{12}$$

where $E(\cdot)$ denotes the expectation and which can also be written as the sum of two terms

$$F_{bb} = \widetilde{F}_{bb} + \overline{F}_{bb} \tag{13}$$

with

$$\widetilde{F}_{bb} = \left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\,4\,\widetilde{\mathbf{B}}_k\widetilde{\mathbf{B}}_k^{\mathrm{T}}\right]^{-1} \tag{14a}$$

$$\overline{F}_{bb} = \frac{4}{\bar{\sigma}^2}(\overline{\mathbf{B}} - \mathbf{b})(\overline{\mathbf{B}} - \mathbf{b})^{\mathrm{T}} \tag{14b}$$

and in the limit of infinite data, the asymptotic approximation, the estimate-error covariance is given by

$$P_{bb} = F_{bb}^{-1} \tag{15}$$

We can also write an estimate-error covariance matrix based on the centered measurements alone, which is equal to

$$\tilde{P}_{bb} = \tilde{F}_{bb}^{-1} \tag{16}$$

which is correct also for small samples, because $\tilde{J}(\mathbf{b})$ is a quadratic function of the measurements. Note that there is no $\overline{P}_{bb}$, because $\overline{F}_{bb}$ is only rank 1.

Centering and Observability

The observability of the magnetometer-bias vector is closely related to the centering operation. It may happen that $\tilde{F}_{bb}$ is ill conditioned or even singular. In that case we cannot minimize equation (11a) to obtain $\tilde{\mathbf{b}}^{*\prime}$ because P_{bb} will not exist. This situation will occur when there is insufficient variation in the magnetic field as measured in the frame of the magnetometer. This may be due either to poor orbit geometry (e.g., a spin-stabilized spacecraft in the equatorial plane with spin axis pointing north and one of the magnetometer axes aligned with the spin axis) or simply to a very short data span. This problem is most easily corrected operationally by taking data over a longer time span or by performing a calibration maneuver to insure that there is adequate change in the magnetic field in the magnetometer coordinate frame.

It may seem possible to compute the magnetometer bias from this poor data, if the full information matrix F_{bb} is non-singular. However, we must evaluate this matrix at the true value of the magnetometer-bias vector and, if $\tilde{F}_{bb}$ is singular, we no longer have the centered estimate as a good approximation of the true value. Thus, from the operational standpoint, we require a somewhat different approach in this case.

What we can do in this case is perform an eigenvalue decomposition of $\tilde{F}_{bb}$ so that we can compute those components of $\tilde{\mathbf{b}}^{*\prime}$ which *are* observable, and then attempt to use the center correction to compute the remaining component. Thus, we write the eigenvalue decomposition of $\tilde{F}_{bb}$ as

$$\tilde{F}_{bb} = \mathcal{O}\,\mathcal{F}\,\mathcal{O}^{\mathrm{T}} \tag{17}$$

where $\mathcal{O}$ is orthogonal and $\mathcal{F}$ is diagonal.

$$\mathcal{F} = \mathrm{diag}(f_1,\ f_2,\ f_3) \tag{18}$$

Because $\tilde{F}_{bb}$ is positive semi-definite, we may choose $\mathcal{F}$ so that

$$f_1 \geq f_2 \geq f_3 \geq 0 \tag{19}$$

In actual practice, this may not be the case, because $\tilde{F}_{bb}$ may not be positive semi-definite due to numerical roundoff error, and f_3 may be slightly negative. In that case, clearly, f_3 will not be numerically significant and should be replaced by zero. The Fisher information matrix for the center term, presented in equation (14b), is necessarily of rank 1. Hence, in order to be able to compute a solution at all, $\tilde{F}_{bb}$ must be at least of rank 2. From the practical standpoint, an absolute requirement is that

$$f_2 \geq \frac{1}{\sigma_{\max}^2} \tag{20}$$

where $\sigma_{\max}$ is the maximum standard deviation that we will tolerate for the magnetometer-bias estimation. If equation (20) not satisfied, then we must abandon the estimation of the magnetometer bias.

Let us suppose that equation (20) is satisfied, but that the Fisher information matrix is nearly singular. Then we can write

$$\mathcal{O} = [\,\hat{\mathbf{u}}_1 \quad \hat{\mathbf{u}}_2 \quad \hat{\mathbf{u}}_3\,] \tag{21}$$

where $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$, and $\hat{\mathbf{u}}_3$ are a right-hand orthonormal triad of column vectors, and

$$\widetilde{F}_{bb} = f_1\,\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^{\mathrm{T}} + f_2\,\hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^{\mathrm{T}} \tag{22}$$

We have discarded the term in f_3 because it is not numerically significant. The pseudo-inverse solution for $\widetilde{\mathbf{b}}$, which we write as $\widetilde{\mathbf{b}}^{\#}$ is then given by

$$\widetilde{\mathbf{b}}^{\#} = \widetilde{F}_{bb}^{\#} \sum_{k=1}^{N} \frac{1}{\sigma_k^2}\,(\tilde{z}_k - \tilde{\mu}_k)\,2\widetilde{\mathbf{B}}_k \tag{23}$$

where

$$\widetilde{F}_{bb}^{\#} = f_1^{-1}\,\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^{\mathrm{T}} + f_2^{-1}\,\hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^{\mathrm{T}} \tag{24}$$

is the pseudo-inverse of $\widetilde{F}_{bb}$.

We could now hope to estimate $\mathbf{b}$ as before provided we make the substitution

$$\left(\mathbf{b} - \widetilde{\mathbf{b}}^{*\prime}\right)^{\mathrm{T}} \widetilde{P}_{bb}^{-1} \left(\mathbf{b} - \widetilde{\mathbf{b}}^{*\prime}\right) \rightarrow \left(\mathbf{b} - \widetilde{\mathbf{b}}^{\#\prime}\right)^{\mathrm{T}} \widetilde{F}_{bb} \left(\mathbf{b} - \widetilde{\mathbf{b}}^{\#\prime}\right) \tag{25}$$

in our center correction procedure. However, we no longer have a good initial estimate for all three components of $\mathbf{b}$, and, consequently, convergence of the Newton–Raphson process may be erratic.

An alternate method is to write the centered cost function approximately as

$$\widetilde{J}(b_1, b_2, b_3) = \frac{1}{2}\left[f_1\,b_1^2 + f_2\,b_2^2 - 2g_1 b_1 - 2g_2 b_2\right]^2 + \text{terms independent of } \mathbf{b} \tag{26}$$

where we have defined

$$b_j \equiv \hat{\mathbf{u}}_j \cdot \mathbf{b}, \quad \text{and} \quad g_j = \sum_{k=1}^{N} \frac{1}{\sigma_k^2}\,(\tilde{z}_k' - \tilde{\mu}_k)\,2(\hat{\mathbf{u}}_j \cdot \widetilde{\mathbf{B}}_k) \quad j = 1, 2, 3 \tag{27}$$

Because $\widetilde{J}(\mathbf{b})$ is positive semi-definite and the quadratic term in b_3 vanishes, it follows that the term linear in b_3 also vanishes, so that $g_3 = 0$, and $\widetilde{J}(\mathbf{b})$, therefore, depends only on b_1 and b_2. A centered estimate is possible, clearly only for these two components. The centered estimate for b_1 and b_2 is just

$$\begin{bmatrix} \tilde{b}_1^* \\ \tilde{b}_2^* \end{bmatrix} = \begin{bmatrix} g_1/f_1 \\ g_2/f_2 \end{bmatrix} \tag{28}$$

We now try to use the center estimate and the center term to estimate b_3. To do this, we write

$$\begin{aligned} \overline{J}(\mathbf{b}) &= \overline{J}(b_1, b_2, b_3) \\ &= \frac{1}{2\overline{\sigma}^2}\left[\overline{z}' - 2(\hat{\mathbf{u}}_1 \cdot \overline{\mathbf{B}})b_1 - 2(\hat{\mathbf{u}}_2 \cdot \overline{\mathbf{B}})b_2 - 2(\hat{\mathbf{u}}_3 \cdot \overline{\mathbf{B}})b_3 + b_1^2 + b_2^2 + b_3^2 - \overline{\mu}\right]^2 \\ &\qquad + \text{terms independent of } \mathbf{b} \end{aligned} \tag{29}$$

and minimize $\overline{J}(\tilde{b}_1^{*\prime}, \tilde{b}_2^{*\prime}, b_3)$ over b_3. Since the values $b_1 = \tilde{b}_1^{*\prime}$ and $b_2 = \tilde{b}_2^{*\prime}$ minimize $\widetilde{J}(\mathbf{b})$, this procedure provides an approximate minimization of $J(\mathbf{b})$. The minimization would be exact if $\overline{\mathbf{B}}$ were parallel to $\hat{\mathbf{u}}_3$.

Since the problem is now one-dimensional, one can solve for b_3 analytically. Define

$$\overline{y}' \equiv \overline{z}' - 2(\hat{\mathbf{u}}_1 \cdot \overline{\mathbf{B}})\tilde{b}_1^{*\prime} - 2(\hat{\mathbf{u}}_2 \cdot \overline{\mathbf{B}})\tilde{b}_2^{*\prime} + \tilde{b}_1^{*\,2} + \tilde{b}_2^{*\,2} - \overline{\mu} \tag{30}$$

Then

$$\overline{J}(\tilde{b}_1^{*\prime}, \tilde{b}_2^{*\prime}, b_3) = \frac{1}{2\overline{\sigma}^2}\left[\,\overline{y}' - 2(\hat{\mathbf{u}}_3 \cdot \overline{\mathbf{B}})b_3 + b_3^2\,\right]^2 \tag{31}$$

which for the desired value of b_3, which we denote by $\overline{b}_3^{*\prime}$, vanishes identically. There are two possible solutions, which are

$$\overline{b}_3^{*\prime} = (\hat{\mathbf{u}}_3 \cdot \overline{\mathbf{B}}) \pm \sqrt{(\hat{\mathbf{u}}_3 \cdot \overline{\mathbf{B}})^2 - \overline{y}'} \tag{32}$$

The solution now is seen to be indeterminate, since both $(\tilde{b}_1^*, \tilde{b}_2^*, \overline{b}_{3+}^*)'$ and $(\tilde{b}_1^*, \tilde{b}_2^*, \overline{b}_{3-}^*)'$ minimize $\overline{J}(\tilde{b}_1^*, \tilde{b}_2^*, b_3)$. If $(\hat{\mathbf{u}}_3 \cdot \overline{\mathbf{B}})$ turns out to be a very large number, then we expect that the smaller of $(\tilde{b}_1^*, \tilde{b}_2^*, \overline{b}_{3\pm}^*)'$ will be the more likely choice for the bias, for instance, if the two solutions for $\overline{b}_3$ are 6,000 mG and 20 mG. In general, however, the answer will be ambiguous.

Alternatively, given these two possible estimates for the magnetometer-bias vector, we may try to gain greater determinacy by minimizing the entire cost function, which we now write (to within terms linear in $\mathbf{b}$) as

$$J(\mathbf{b}) = \frac{1}{2}\left(\mathbf{b} - \widetilde{\mathbf{b}}^{\#\prime}\right)^{\mathrm{T}} \widetilde{F}_{bb}\left(\mathbf{b} - \widetilde{\mathbf{b}}^{\#\prime}\right) + \overline{J}(\mathbf{b}) \tag{33}$$

using the two approximate minima as starting values. Such trials, however, are not guaranteed to remove the ambiguity. We emphasize that these difficulties arise from inadequacies in the data, and not in the underlying methodology of the TWOSTEP algorithm. In cases where the observability of b_3 is poor, the method of the present section will likely lead a far better result because it will not prejudice the weights in the centered estimate using a possibly unreliable estimate of b_3. After b_3 has been estimated, the entire process can be repeated again, with a now complete initial estimate.

For carrying out refinements of the two solutions, we note the following approximate form for the estimate error covariance

$$P_{bb} = f_1^{-1}\,\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^{\mathrm{T}} + f_2^{-1}\,\hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^{\mathrm{T}} + \frac{\overline{\sigma}^2}{4(\hat{\mathbf{u}}_3 \cdot \overline{\mathbf{B}} - b_3^{*\prime})^2}\,\hat{\mathbf{u}}_3\hat{\mathbf{u}}_3^{\mathrm{T}} \tag{34}$$

which can be used to initialize the search for the global minimum.

We see an important result from this discussion. The observability of the magnetometer-bias vector depends, in general, only on the centered data. Therefore, apart from not being able to compute the σ_k^2 accurately, to which the centered estimate is not very sensitive, we can determine whether the data permit an unambiguous estimate of the magnetometer bias by simply computing the eigenvalues of the centered Fisher information matrix $\widetilde{F}_{bb}$. We see also that we require at least four measurements of the magnetic field vector in order to achieve an unambiguous solution. If we had knowledge of the attitude, then a single measurement of the magnetic field vector would suffice. The requirement of at least four magnetic field measurements applies, obviously, to any algorithm for computing the magnetometer bias from measurements of the magnetic field magnitude.

Numerical Examples

The new algorithm developed in this work has been examined for a typical scenario where we can expect observability to be a problem: an inertially stabilized spacecraft. The spacecraft orbit has been chosen to be circular with an altitude of 560 km and an inclination of 38 deg. The geomagnetic field in our studies has been simulated using the International Geomagnetic Reference Field (IGRF (1985)) [5], which has been extrapolated to 1994. More recent field models are available, but IGRF (1985) is adequate for our simulation needs.

For purposes of simulation we have assumed an effective white Gaussian magnetometer measurement error with isotropic error distribution and a standard deviation per axis of 2.0 mG, corresponding to an angular error of approximately 0.5 deg at the equator. We have assumed also that the x-axis of the magnetometer always points toward the Sun. The Sun direction makes an angle of approximately 40 degrees with the orbit plane. The magnetometer data were sampled every eight seconds. All entries in the tables for the estimated magnetometer bias and the associated standard deviations are in mG.

Table 1 displays the results. Nearly 200 different cases were simulated in testing the algorithm. The above cases were typical except that we have modified the field model slightly so that the third component of the bias would be less observable from the centered data alone. This was done to illustrate more acutely the possible importance of the center correction and the performance of the special algorithm developed for cases of poor observability.

We found for this case

$$\tilde{F}_{bb} = \begin{bmatrix} 76.23 & -7.30 & -7.86 \\ -7.30 & 45.66 & 14.25 \\ -7.86 & 14.25 & 5.89 \end{bmatrix} (\mathrm{mG})^{-2} \tag{35a}$$

while the Fisher information for the center correction alone $\bar{F}_{bb}$ has the value

$$\bar{F}_{bb} = \begin{bmatrix} 4.43 & 10.84 & 28.12 \\ 10.84 & 26.58 & 68.92 \\ 28.12 & 68.92 & 178.70 \end{bmatrix} (\mathrm{mG})^{-2} \tag{35b}$$

Clearly the center data provides most of the information on b_1 and b_2, while the center term provides most of the information on b_3.

Table 1. Performance of TWOSTEP for an Inertially Stabilized Spacecraft.

$\mathbf{b}^{\text{true}} = [\,10., 20., 30.\,]$ mG.

step	bias estimate (mG)
centering approximation	[9.92, 20.00, 29.68] ±[0.14, 0.33, 0.98]
with center correction	[9.94, 19.94, 29.92] ±[0.11, 0.17, 0.11]

To test the algorithms developed for conditions of poor observability, we have reconsidered the data of Table 1 and have carried out the regime for ill-conditioned statistics, although obviously the TWOSTEP algorithm yield excellent results in this case without such intervention. An eigenvalue decomposition of the Fisher information matrix of equation (35a) yielded the eigenvalues 79.57, 47.28, and 0.95 $(\mathrm{mG})^{-2}$, of which we have rejected the smallest (even though it provides an equivalent accuracy of 1.03 mG for the bias vector along $\hat{\mathbf{u}}_3$.

Calculating the two solutions by the method outlined above yields

$$\mathbf{b}_1^{*\,\text{singular method}} = (9.94,\ 19.94,\ 29.92)\ \text{mG} \tag{36a}$$

$$\mathbf{b}_2^{*\,\text{singular method}} = (-23.03,\ 155.24,\ -411.54)\ \text{mG} \tag{36b}$$

We recognize the first solution as close to the TWOSTEP result of Table 1, correct to two decimal places. The second solution is the spurious solution. If we calculate the value of the cost function according to equation (33), we arrive at the values

$$J(\mathbf{b}_1^{*\,\text{singular method}}) = 0.03\,, \quad \text{and} \quad J(\mathbf{b}_2^{*\,\text{singular method}}) = 100,250. \tag{37}$$

so that in this example, there is no doubt as to which solution to choose. The uncommonly large value of b_3 for the second solution should have made us immediately wary. Calculating the local minimum of the full negative-log-likelihood function, without treating $\widetilde{F}_{bb}$ as singular (i.e., we do not discard the term $f_3\hat{\mathbf{u}}_3\hat{\mathbf{u}}_3^{\mathrm{T}}$) yields a very similar result for the position of the spurious minimum and the corresponding value of the cost function. The singular method would seem to work very well.

Discussion

TWOSTEP provides insights into the nature of ill-conditioned cases. It is very clear from our discussion that observability of the magnetometer bias is tantamount to observability from the centered data alone. Thus, in order to measure the three components of the bias one requires at least four magnetometer measurements. Otherwise, the quadratic dependence of the measurement on the bias will lead to a two-fold ambiguity. In some cases the ambiguity can be eliminated, in others, however, the solution may remain indeterminate. This is a problem not of the method but of the data. Other methods will fail to produce a result with even greater frequency, and provide less understanding of the reasons for failure. We saw examples of this in [2].

Acknowledgments

The authors are grateful to M. Acuña, M. Challa, J. M. Hanson, E. Ketchem, R. V. F. Lopes, F. L. Markley, and J. Thienel (formerly Deutschmann) for interesting discussions and helpful criticisms. Dr. Markley is to be thanked especially for his superb powers of conservation. One of the authors (R. A.) wishes to thank in addition Dr.'Henry Hoffman and the Guidance and Control Branch of NASA Goddard Space Flight Center for their hospitality during the course of this work. Likewise, he is grateful to the Comisión Nacional de Actividades Espaciales (CONAE) and the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) of Argentina for their support during the period of this work.

References

[1] ALONSO, R. and SHUSTER, M. D. "TWOSTEP: A Fast Robust Algorithm for Attitude-Independent Magnetometer-Bias Determination," to appear in the *Journal of the Astronautical Sciences*.

[2] ALONSO, R. and SHUSTER, M. D. "Attitude-Independent Magnetometer-Bias Determination: A Survey," to appear in the *Journal of the Astronautical Sciences*.

[3] ACUÑA, M. (private communication).

[4] ALONSO, R. and SHUSTER, M. D. "Complete Linear Attitude-Independent Magnetometer Calibration," to appear in the *Journal of the Astronautical Sciences*.

[5] LANGEL, R. A. "The Main Field," in JACOBS, J. A. (ed.) *Geomagnetism,* Academic Press, Orlando, 1987, pp. 348–349.