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Complete Linear Attitude-Independent Magnetometer Calibration

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Abstract

The TWOSTEP algorithm, which has been applied successfully to the attitude independent estimation of magnetometer biases is extended to estimate scale factors and nonorthogonality corrections. For the case of a spinning spacecraft, the algorithm performs well. This goes against the common wisdom that not much more than biases can be determined without knowledge of the attitude.

Introduction

The TWOSTEP algorithm [1] was developed to determine magnetometer biases inflight without knowledge of the attitude under any possible conditions. Exhaustive studies [2] have shown it to be more robust and more efficient than other existing methods. In fact, the authors have been unable up to now to create a non-trivial scenario in which the algorithm will not perform well. In the past, only the determination of magnetometer biases was attempted from such data, it being widely believed that additional parameters were not observable. The present work seeks to reverse that opinion.

The TWOSTEP algorithm relies on a centering procedure for its first step. Such a procedure was first applied to magnetometer bias determination by Gambhir [3, 4]. To understand the need for such a procedure we examine the model for the magnetometer measurement, which we write as

$$\mathbf{B}_k = A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k, \quad k = 1, \dots, N \tag{1}$$

where $\mathbf{B}_k$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time t_k; $\mathbf{H}_k$ is the corresponding value of the geomagnetic field with respect to an Earth-fixed coordinate system; A_k is the attitude of the

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magnetometer with respect to the Earth-fixed coordinates; $\mathbf{b}$ is the magnetometer bias; and $\boldsymbol{\varepsilon}_k$ is the measurement noise. The measurement noise, which includes both sensor errors and geomagnetic field model uncertainties, is generally assumed to be white and Gaussian.

If the attitude matrix A_k is not known, we can work instead with derived scalar measurements and scalar measurement noise defined by

$$z_k \equiv |\mathbf{B}_k|^2 - |\mathbf{H}_k|^2 \tag{2a}$$

$$v_k \equiv 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k - |\boldsymbol{\varepsilon}_k|^2 \tag{2b}$$

Then we can write

$$z_k = 2\mathbf{B}_k \cdot \mathbf{b} - |\mathbf{b}|^2 + v_k, \quad k = 1, \ldots, N \tag{3}$$

If we assume that $\boldsymbol{\varepsilon}_k$ is white and Gaussian

$$\boldsymbol{\varepsilon}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_k) \tag{4}$$

then $v_k \sim \mathcal{N}(\mu_k, \sigma_k^2)$

$$\mu_k \equiv E\{v_k\} = -\mathrm{tr}(\Sigma_k) \tag{5a}$$

$$\sigma_k^2 \equiv E\{v_k^2\} - \mu_k^2 = 4(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}}\Sigma_k(\mathbf{B}_k - \mathbf{b}) + 2(\mathrm{tr}\,\Sigma_k^2) \tag{5b}$$

This effective measurement model is presented in greater detail in [1].

The estimation of $\mathbf{b}$ according to the criterion of maximum likelihood leads us to find the value which minimizes the negative-log-likelihood function [5]

$$J(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{\sigma_k^2}(z_k - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - \mu_k)^2 + \log \sigma_k^2 + \log 2\pi\right] \tag{6}$$

of which only the first term under the summation depends on the magnetometer bias. The minimization of equation (6) is complicated by the fact the negative-log-likelihood function is quartic in the magnetometer bias and, therefore, admits multiple minima and maxima. A search for the global minimum by infinite processes starting at $\mathbf{b}_0 = \mathbf{0}$ is not guaranteed success as shown by the examples in reference [2]. Therefore, it is imperative that any infinite process start with a good estimate of the bias.

Such an estimate is provided by the centering approximation. Given a sequence of variables X_k, $k = 1, \ldots, N$, the centering method defines *center* values of the sequence according to the prescription

$$\overline{X} \equiv \overline{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} X_k \tag{7}$$

where

$$\frac{1}{\overline{\sigma}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{8}$$

and *centered* values according to

$$\tilde{X}_k \equiv X_k - \overline{X}, \quad k = 1, \ldots, N \tag{9}$$

On the basis of these centered and center quantities, it can be shown [1] that the quartic cost function of equation (6) can be written exactly as

$$J(\mathbf{b}) = \tilde{J}(\mathbf{b}) + \bar{J}(\mathbf{b}) \tag{10}$$

where

$$\tilde{J}(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(\tilde{z}_k - 2\tilde{\mathbf{B}}_k \cdot \mathbf{b} - \tilde{\mu}_k)^2 + \text{terms independent of } \mathbf{b} \tag{11a}$$

and

$$\bar{J}(\mathbf{b}) = \frac{1}{2}\frac{1}{\bar{\sigma}^2}(\bar{z} - 2\bar{\mathbf{B}} \cdot \mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2 + \text{terms independent of } \mathbf{b} \tag{11b}$$

Equations (11a) and (11b) give the negative-log-likelihood functions of the centered and center data respectfully with respect to the magnetometer bias vector. The TWOSTEP algorithm consists first of finding the value of $\mathbf{b}$ which minimizes $\tilde{J}(\mathbf{b})$. Because $\tilde{J}(\mathbf{b})$ is a quadratic function of its argument, the estimate of the bias from this first step, the *centered estimate* $\tilde{\mathbf{b}}^*$, is unambiguous, and has the solution

$$\tilde{\mathbf{b}}^* = \tilde{P}_{bb}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(\tilde{z}_k - \tilde{\mu}_k)2\tilde{\mathbf{B}}_k \tag{12a}$$

and the estimate error covariance of the centered estimate is given by the inverse of the Fisher information matrix

$$\tilde{P}_{bb} = \tilde{F}_{bb}^{-1} = \left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}4\tilde{\mathbf{B}}_k\tilde{\mathbf{B}}_k^{\mathrm{T}}\right]^{-1} \tag{12b}$$

The centered estimate $\tilde{\mathbf{b}}^*$ and the center measurement $\bar{z}$ have been shown to be independent sufficient statistics for the magnetometer bias vector under the assumption that the measurement noise on the magnetometer readings is white and Gaussian.

The second step consists of using the centered estimate as an initial value and computing the corrected estimate by applying the Gauss-Newton method to the full negative-log-likelihood function, which we write equivalently as

$$J(\mathbf{b}) = \frac{1}{2}(\mathbf{b} - \tilde{\mathbf{b}}^*)^{\mathrm{T}}\tilde{P}_{bb}^{-1}(\mathbf{b} - \tilde{\mathbf{b}}^*) + \frac{1}{2\bar{\sigma}^2}(\bar{z} - 2\bar{\mathbf{B}} \cdot \mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2 + \text{terms independent of } \mathbf{b} \tag{13}$$

Thus

$$\mathbf{b}_1^{\mathrm{GN}} = \tilde{\mathbf{b}}^* \tag{14a}$$

$$\mathbf{b}_{i+1}^{\mathrm{GN}} = \mathbf{b}_i^{\mathrm{GN}} - \left[\tilde{P}_{bb}^{-1} + \frac{4}{\bar{\sigma}^2}(\bar{\mathbf{B}} - \mathbf{b}_i^{\mathrm{GN}})(\bar{\mathbf{B}} - \mathbf{b}_i^{\mathrm{GN}})^{\mathrm{T}}\right]^{-1}\frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}_i^{\mathrm{GN}}) \tag{14b}$$

and it is equation (13) which is differentiated in equation (14).

Convergence of this method has been seen to be very rapid and reliable [1, 2]. The centered estimate by itself, the first step, provides a consistent estimate of the magnetometer bias, which will provide adequate accuracy in most cases. Note that the centered estimate also requires an additional iteration to recompute the weights, as discussed in [1]. Because the TWOSTEP method has been derived rigorously, it admits various statistical tests as figures of merit and special techniques for treating cases of poor observability.

The purpose of this contribution is to extend this method to the estimation of other parameters than simply the magnetometer bias. We may write the most general linear functional relationship between the measured magnetic field in magnetometer coordinates, the model magnetic field in our chosen reference coordinate system (typically Earth-fixed) and Gaussian measurement noise as

$$\mathbf{B}_k = T^{-1}[A_k \mathbf{H}_k + \mathbf{b}' + \boldsymbol{\varepsilon}_k'], \quad k = 1, \ldots, N \tag{15}$$

where for later convenience we have written the general 3×3 matrix as T^{-1} and placed primes on the symbols. Our task is to estimate $\mathbf{b}'$ and the parameters of T.

By the polar decomposition theorem [6] we may write

$$T = \mathcal{O}(I + D) \tag{16}$$

where $\mathcal{O}$ is orthogonal and D is symmetric. Then, it follows that equation (16) is equivalent to

$$(I + D)\mathbf{B}_k = \mathcal{O}^T A_k \mathbf{H}_k + \mathcal{O}^T \mathbf{b}' + \mathcal{O}^T \boldsymbol{\varepsilon}_k' \tag{17a}$$

$$= \mathcal{O}^T A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k \tag{17b}$$

and we have defined

$$\mathbf{b} = \mathcal{O}^T \mathbf{b}', \qquad \boldsymbol{\varepsilon}_k = \mathcal{O}^T \boldsymbol{\varepsilon}_k' \tag{18}$$

Clearly, if A_k is not known, one cannot estimate the parameters of $\mathcal{O}$. In particular, with only magnitude data, we have in analogy with the development in reference [1]

$$|\mathbf{H}_k|^2 = |(I + D)\mathbf{B}_k - \mathbf{b} - \boldsymbol{\varepsilon}_k|^2 \tag{19}$$

in which $\mathcal{O}$ no longer appears. Therefore, we can at best estimate only the parameters of $\mathbf{b}$ and the symmetric matrix D.[3] The diagonal elements of D are the scale factor corrections and the off-diagonal elements the nonorthogonality corrections. Misalignments would correspond to the antisymmetric part of D and are, as we have seen, unobservable from magnitude data derived from the magnetometer readings.

Note that we can only estimate $\mathbf{b}$ and not $\mathbf{b}'$. However, if the magnetometer is indeed misaligned by a rotation $\mathcal{O}$, then it is $\mathbf{b}$ and not $\mathbf{b}'$ which is physically meaningful, because that is the bias vector in the magnetometer frame, rather than in the *a priori* frame determined from A_k.

The task of the present report, then, is to develop an algorithm for determining $\mathbf{b}$ and D. We consider two cases: (1) the special case that D is diagonal, so that we estimate only the three scale-factor corrections; and (2) the general case that D is symmetric and fully populated, and we estimate the full set of six parameters characterizing D, both scale-factor and nonorthogonality corrections.

Estimation of the Magnetometer Bias and Scale Factors

The TWOSTEP algorithm presented above is easily extended to the estimation of both the bias vector and the three magnetometer scale factors. We assume now that the magnetometer measurements can be modeled as

$$\mathbf{B}_k = (I_{3\times3} + D)^{-1}(\mathcal{O}^T A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k), \quad k = 1, \ldots, N \tag{20}$$

[3] If attitude information is available, then we can determine the orthogonal matrix as well, as done, for example by Lerner and Shuster [7].

where $\mathcal{O}$ is proper orthogonal and

$$D \equiv \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \equiv \mathrm{diag}(d_1, d_2, d_3) \tag{21}$$

is the matrix of scale-factor errors. Thus

$$\mathcal{O}^T A_k \mathbf{H}_k = (I_{3\times 3} + D)\mathbf{B}_k - \mathbf{b} - \boldsymbol{\varepsilon}_k \tag{22}$$

and

$$z_k \equiv |\mathbf{B}_k|^2 - |\mathbf{H}_k|^2 \tag{23a}$$

$$= -\mathbf{B}_k^T(2D + D^2)\mathbf{B}_k + 2\mathbf{B}_k^T(I + D)\mathbf{b} - |\mathbf{b}|^2 + v_k \tag{23b}$$

where now

$$v_k \equiv 2[(I + D)\mathbf{B}_k - \mathbf{b}] \cdot \boldsymbol{\varepsilon}_k - |\boldsymbol{\varepsilon}_k|^2 \tag{24}$$

We no longer indicate the 3×3 dimension of the identity matrix explicitly. In general, we expect the elements of D to be very small.

To estimate D and $\mathbf{b}$, define first the quantities

$$E \equiv 2D + D^2 = \mathrm{diag}(e_1, e_2, e_3) \tag{25a}$$

$$\mathbf{e} \equiv [e_1, e_2, e_3]^T \tag{25b}$$

$$\mathbf{c} \equiv (I + D)\mathbf{b} \tag{25c}$$

In terms of these new parameters the measurement equation becomes

$$z_k = -\mathbf{B}_k^T E \mathbf{B}_k + 2\mathbf{B}_k^T \mathbf{c} - |\mathbf{b}(\mathbf{c}, \mathbf{e})|^2 + v_k \tag{26}$$

and the only nonlinear dependence on the parameters is contained in $\mathbf{b}(\mathbf{c}, \mathbf{e})|^2$. Had we not made the change of variables given by equations (25), this would not be the case.

We may write

$$\mathbf{B}_k^T E \mathbf{B}_k = B_{1,k}^2 e_1 + B_{2,k}^2 e_2 + B_{3,k}^2 e_3 = K_k \mathbf{e} \tag{27}$$

with

$$K_k \equiv [B_{1,k}^2, B_{2,k}^2, B_{3,k}^2] \tag{28}$$

Thus,

$$z_k = -K_k \mathbf{e} + 2\mathbf{B}_k^T \mathbf{c} - |\mathbf{b}(\mathbf{c}, \mathbf{e})|^2 + v_k \tag{29a}$$

$$= L_k \boldsymbol{\theta}' - |\mathbf{b}(\boldsymbol{\theta}')|^2 + v_k \tag{29b}$$

with

$$L_k \equiv [2\mathbf{B}_k^T \,|\, -K_k], \quad \boldsymbol{\theta}' \equiv \begin{bmatrix} \mathbf{c} \\ \mathbf{e} \end{bmatrix} \tag{30}$$

Note the prime on $\boldsymbol{\theta}'$. We reserve the notation $\boldsymbol{\theta}$ to denote the original parameters

$$\boldsymbol{\theta} \equiv \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix} \tag{31}$$

with

$$\mathbf{d} \equiv [d_1, d_2, d_3]^{\mathrm{T}} \tag{32}$$

Defining

$$\overline{L} \equiv \overline{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} L_k, \quad \tilde{L}_k \equiv L_k - \overline{L} \tag{33}$$

The centered and center measurements become

$$\tilde{z}_k = \tilde{L}_k \boldsymbol{\theta}' + \tilde{v}_k, \quad k = 1, \ldots, N \tag{34a}$$

$$\overline{z} = \overline{L}\boldsymbol{\theta}' - |\mathbf{b}(\boldsymbol{\theta}')|^2 + \overline{v} \tag{34b}$$

Note that $\sigma_k^2 = E\{v_k^2\} - \mu_k^2$, as in equation (5b). However, v_k is somewhat different in value than in reference [1] owing to the dependence on the scale factors (equation (24)).

The principal reason for defining the intermediate parameters **c** and **e** was to place the entire nonlinear dependence of the measurements on the parameters into the term $|\mathbf{b}(\theta')|^2$. Had we not made this substitution, then the centered measurements, $\tilde{z}_k$, would not have been linear in the parameters, which was the whole point of the centering operation.

The calculation proceeds as before. Note that because D and E are diagonal, the implicit matrix inverse contained in the parameterization of $|\mathbf{b}(\boldsymbol{\theta}')|^2$, is, in fact, quite simple and can be written as

$$|\mathbf{b}(\boldsymbol{\theta}')|^2 = \sum_{j=1}^{3} c_j^2/(1 + d_j)^2 = \sum_{j=1}^{3} c_j^2/(1 + e_j) \tag{35}$$

The calculation of the centered estimate is identical to the earlier calculation [1] *mutatis mutandis* leading to

$$\tilde{J} = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}_k - \tilde{L}_k \boldsymbol{\theta}' - \tilde{\mu}_k)^2 + \text{terms independent of } \boldsymbol{\theta}' \tag{36}$$

whence

$$\tilde{\boldsymbol{\theta}}'^* = \tilde{P}_{\theta'\theta'} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}_k - \tilde{\mu}_k) \tilde{L}_k^{\mathrm{T}} \tag{37a}$$

$$\tilde{P}_{\theta'\theta'}^{-1} = \tilde{F}_{\theta'\theta'} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tilde{L}_k^{\mathrm{T}} \tilde{L}_k \tag{37b}$$

and the center cost function is

$$\overline{J}(\boldsymbol{\theta}') = \frac{1}{2\overline{\sigma}^2} (\overline{z} - \overline{L}\boldsymbol{\theta}' + |\mathbf{b}(\boldsymbol{\theta}')|^2 - \overline{\mu})^2 + \text{terms independent of } \boldsymbol{\theta}' \tag{38}$$

Hence, the center contribution to the Fisher information matrix (for us, the inverse covariance) is simply

$$\overline{P}_{\theta'\theta'}^{-1} = \overline{F}_{\theta'\theta'} = \frac{1}{\overline{\sigma}^2} \left(\overline{L} - \frac{\partial |\mathbf{b}|^2}{\partial \boldsymbol{\theta}'^{\mathrm{T}}} \right)^{\mathrm{T}} \left(\overline{L} - \frac{\partial |\mathbf{b}|^2}{\partial \boldsymbol{\theta}'^{\mathrm{T}}} \right) \tag{39}$$

and the partial derivatives of $|\mathbf{b}|^2$ with respect to the components of $\mathbf{c}$ and $\mathbf{e}$ are readily calculated from equation (35)

$$\frac{\partial|\mathbf{b}|^2}{\partial\boldsymbol{\theta}'} = \left[\frac{2c_1}{1+e_1}, \frac{2c_2}{1+e_2}, \frac{2c_3}{1+e_3}, -\frac{c_1^2}{(1+e_1)^2}, -\frac{c_2^2}{(1+e_2)^2}, -\frac{c_3^2}{(1+e_3)^2}\right]^{\mathrm{T}} \tag{40}$$

The complete cost function is given by

$$J(\theta') = \frac{1}{2}(\boldsymbol{\theta}' - \boldsymbol{\theta}'^*)^{\mathrm{T}}\tilde{P}_{\theta'\theta'}^{-1}(\boldsymbol{\theta}' - \boldsymbol{\theta}'^*) + \bar{J}(\boldsymbol{\theta}') \tag{41}$$

from which the estimates $\mathbf{c}^*$ and $\mathbf{e}^*$ are obtained by the same iterative procedure presented as for $\mathbf{b}^*$ above.

Following the calculation of $\mathbf{c}^*$ and $\mathbf{e}^*$, we compute $\mathbf{b}^*$ and $\mathbf{d}^*$ according to

$$d_j^* = -1 + \sqrt{1 + e_k^*}, \quad j = 1, 2, 3 \tag{42a}$$

$$b_j^* = c_j^*/\sqrt{1 + e_k^*}, \quad j = 1, 2, 3 \tag{42b}$$

The estimate error covariance associated with $\boldsymbol{\theta}$ is obtained from that computed for $\boldsymbol{\theta}'$ by applying the transformation

$$P_{\theta\theta} = \left(\frac{\partial(\mathbf{b}, \mathbf{d})}{\partial(\mathbf{c}, \mathbf{e})}\right)P_{\theta'\theta'}\left(\frac{\partial(\mathbf{b}, \mathbf{d})}{\partial(\mathbf{c}, \mathbf{e})}\right)^{\mathrm{T}} \tag{43}$$

with

$$\left(\frac{\partial(\mathbf{b}, \mathbf{d})}{\partial(\mathbf{c}, \mathbf{e})}\right) = \left(\frac{\partial(\mathbf{c}, \mathbf{e})}{\partial(\mathbf{b}, \mathbf{d})}\right)^{-1} = \begin{bmatrix} (I + D) & \mathrm{diag}(\mathbf{b}) \\ 0 & (I + 2D) \end{bmatrix}^{-1} \tag{44a}$$

$$= \frac{1}{\det(I + D)\det(I + 2D)}\begin{bmatrix} (I + 2D) & -\mathrm{diag}(\mathbf{b}) \\ 0 & (I + D) \end{bmatrix} \tag{44b}$$

Estimation of the Magnetometer Bias, Scale Factors, and Nonorthogonality Corrections

The algorithm presented above is easily extended to include also the estimation of nonorthogonality corrections. Like the scale factor corrections, nonorthogonality corrections have their origin solely in the magnetometer, and occur because the individual magnetometer axes are not orthonormal, due typically to thermal gradients within the magnetometer or to mechanical stresses from the spacecraft.

Following the discussion in the introduction, we assume now that the magnetometer measurements can be modeled as

$$\mathbf{B}_k = (I + D)^{-1}(\mathcal{O}^T A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k) \tag{45}$$

where again $\mathcal{O}$ is proper orthogonal but now D is a fully-populated symmetric matrix and, therefore, depends on six parameters, which we may take to be the upper triangular elements of D. Equations (20) through (24) otherwise remain unchanged.

To estimate D and $\mathbf{b}$ define again the quantities

$$E \equiv 2D + D^2 \tag{46a}$$

$$\mathbf{c} = (I + D)\mathbf{b} \tag{46b}$$

The matrix E is symmetric but not diagonal. Thus

$$z_k = -\mathbf{B}_k^{\mathrm{T}} E \mathbf{B}_k + 2\mathbf{B}_k^{\mathrm{T}}\mathbf{c} - |\mathbf{b}(\mathbf{c}, \mathbf{E})|^2 + v_k \tag{47}$$

We may write

$$\mathbf{B}_k^{\mathrm{T}} E \mathbf{B}_k = K_k \mathbf{E} \tag{48}$$

with

$$K_k = [B_{1,k}^2, B_{2,k}^2, B_{3,k}^2, 2B_{1,k}B_{2,k}, 2B_{1,k}B_{3,k}, 2b_{2,k}B_{3,k}] \tag{49a}$$

$$\mathbf{E} \equiv [E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}]^{\mathrm{T}} \tag{49b}$$

Thus, as before

$$z_k = -K_k\mathbf{E} + 2\mathbf{B}_k^{\mathrm{T}}\mathbf{c} - |\mathbf{b}(\mathbf{c}, \mathbf{E})|^2 + v_k \tag{50a}$$

$$= L_k\boldsymbol{\theta}' - |\mathbf{b}(\boldsymbol{\theta}')|^2 + v_k \tag{50b}$$

with

$$L_k \equiv [2\mathbf{B}_k^{\mathrm{T}} | -K_k], \quad \boldsymbol{\theta}' \equiv \begin{bmatrix} \mathbf{c} \\ \mathbf{E} \end{bmatrix} \tag{51}$$

But $\boldsymbol{\theta}'$ is now 9×1 and L_k is now 1×9. Defining as before

$$\bar{L} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} L_k, \quad \tilde{L}_k \equiv L_k - \bar{L} \tag{52}$$

The centered and center measurements become

$$\tilde{z}_k = \tilde{L}_k \cdot \boldsymbol{\theta}' + \tilde{v}_k, \quad k = 1, \ldots, N \tag{53a}$$

$$\bar{z} = \bar{L}\boldsymbol{\theta}' - |\mathbf{b}(\boldsymbol{\theta}')|^2 + \bar{v} \tag{53b}$$

The calculation proceeds as before. Note that because D is no longer diagonal, we no longer have quite as simple an expression for $|\mathbf{b}(\boldsymbol{\theta}')|^2$, which now is

$$|\mathbf{b}(\boldsymbol{\theta}')|^2 = \mathbf{c}^{\mathrm{T}}(I + D)^{-2}\mathbf{c} = \mathbf{c}^{\mathrm{T}}(I + E)^{-1}\mathbf{c} \tag{54}$$

The partial derivatives of $|\mathbf{b}(\boldsymbol{\theta}')|^2$ are given by

$$\frac{\partial}{\partial c_m}|\mathbf{b}(\boldsymbol{\theta}')|^2 = 2((I + E)^{-1}\mathbf{c})_m \tag{55a}$$

$$\frac{\partial}{\partial E_{m,n}}|\mathbf{b}(\boldsymbol{\theta}')|^2 = -(2 - \delta_{mn})((I + E)^{-1}\mathbf{c})_m((I + E)^{-1}\mathbf{c})_n \tag{55b}$$

where $((I + E)^{-1}\mathbf{c})_m$ denotes the mth element of $((I + E)^{-1}\mathbf{c})$, and δ_{mn} is the Kronecker symbol. Again we note that the intermediate parameters $\mathbf{c}$ and $\mathbf{E}$ have been introduced in order that the only nonlinear dependence will be found in $|\mathbf{b}(\boldsymbol{\theta}')|^2$.

The calculation of the centered estimate is identical to the calculation of the previous section with L_k, $\boldsymbol{\theta}'$, and related quantities appropriately redefined leading to

$$\tilde{J} = \frac{1}{2}\sum_{k=1}^{N} \frac{1}{\sigma_k^2}(\tilde{z}_k - \tilde{L}_k\boldsymbol{\theta}' - \tilde{\mu}_k)^2 + \text{terms independent of } \boldsymbol{\theta}' \tag{56}$$

whence

$$\tilde{\boldsymbol{\theta}}'^* = \tilde{P}_{\theta'\theta'} \sum_{k=1}^{N} \frac{1}{\sigma_k^2}(\tilde{z}_k - \tilde{\mu}_k)\tilde{L}_k^{\mathrm{T}} \tag{57a}$$

$$\tilde{P}_{\theta'\theta'}^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2}\tilde{L}_k\tilde{L}_k^{\mathrm{T}} \tag{57b}$$

and the center cost function is once more

$$\bar{J}(\boldsymbol{\theta}') = \frac{1}{2\bar{\sigma}^2}(\bar{z} - \bar{L}\boldsymbol{\theta}' + |\mathbf{b}(\boldsymbol{\theta}')|^2 - \bar{\mu})^2 \tag{58}$$

and the center contribution to the Fisher information matrix (for us the inverse covariance) is again simply

$$\bar{P}^{-1}_{\theta'\theta'} = \frac{1}{\bar{\sigma}^2}\left(\bar{L} - \frac{\partial|\mathbf{b}|^2}{\partial\boldsymbol{\theta}'^{\mathrm{T}}}\right)^{\mathrm{T}}\left(\bar{L} - \frac{\partial|\mathbf{b}|^2}{\partial\boldsymbol{\theta}'^{\mathrm{T}}}\right) \tag{59}$$

Following the calculation of $\mathbf{c}^*$ and $\mathbf{E}^*$, we must compute D^* and $\mathbf{b}^*$. To compute D^* we write

$$E^* = USU^{\mathrm{T}} \tag{60}$$

where U is orthogonal and S is diagonal

$$S = \operatorname{diag}(s_1, s_2, s_3) \tag{61}$$

We define W to be the diagonal matrix $\operatorname{diag}(w_1, w_2, w_3)$ satisfying

$$S = 2W + W^2 \tag{62}$$

In general, the elements of S are much less than unity so that a solution will exist. In analogy to equations (42), the diagonal elements of W have the solution

$$w_j = -1 + \sqrt{1 + s_j}, \quad j = 1, 2, 3 \tag{63}$$

The maximum likelihood estimate of the scale-factor-nonorthogonality matrix D is then given by

$$D^* = UWU^{\mathrm{T}} \tag{64}$$

with U the orthogonal matrix of equation (60). The maximum likelihood estimate of the magnetometer bias vector is then given finally by

$$\mathbf{b}^* = (I + D^*)^{-1}\mathbf{c}^* \tag{65}$$

To transform the covariance matrix of $\boldsymbol{\theta}'$ to the covariance matrix of $\boldsymbol{\theta}$ we perform a transformation similar to that of equation (43)

$$P_{\theta\theta} = \left(\frac{\partial(\mathbf{b}, \mathbf{D})}{\partial(\mathbf{c}, \mathbf{E})}\right) P_{\theta'\theta'} \left(\frac{\partial(\mathbf{b}, \mathbf{D})}{\partial(\mathbf{c}, \mathbf{E})}\right)^{\mathrm{T}} \tag{66}$$

where, in analogy to equation (49b), we have defined

$$\mathbf{D} \equiv [D_{11}, D_{22}, D_{33}, D_{12}, D_{13}, D_{23}]^{\mathrm{T}} \tag{67}$$

Then

$$\left(\frac{\partial(\mathbf{b}, \mathbf{D})}{\partial(\mathbf{c}, \mathbf{E})}\right) = \left(\frac{\partial(\mathbf{c}, \mathbf{E})}{\partial(\mathbf{b}, \mathbf{D})}\right)^{-1} \tag{68a}$$

$$= \begin{bmatrix} (I + D) & M_{cD}(\mathbf{b}) \\ O_{6\times3} & 2I_{6\times6} + M_{ED}(\mathbf{D}) \end{bmatrix}^{-1} \tag{68b}$$

with

$$M_{cD}(\mathbf{b}) = \begin{bmatrix} b_1 & 0 & 0 & b_2 & b_3 & 0 \\ 0 & b_2 & 0 & b_1 & 0 & b_3 \\ 0 & 0 & b_3 & 0 & b_1 & b_2 \end{bmatrix} \tag{69a}$$

and

$$M_{ED}(\mathbf{D}) = \begin{bmatrix} 2D_1 & 0 & 0 & 2D_4 & 2D_5 & 0 \\ 0 & 2D_2 & 0 & 2D_4 & 0 & 2D_6 \\ 0 & 0 & 2D_3 & 0 & 2D_5 & 2D_6 \\ D_4 & D_4 & 0 & D_1 + D_2 & D_6 & D_5 \\ D_5 & 0 & D_5 & D_6 & D_1 + D_3 & D_4 \\ 0 & D_6 & D_6 & D_5 & D_4 & D_2 + D_3 \end{bmatrix} \tag{69b}$$

Quadratic Calibration of Magnetometers

The techniques developed here thus far and in early publications [1, 2] for determining the magnetometer calibration parameters were limited to constant and linear corrections. Quadratic corrections, unfortunately, can be handled much less easily. If we consider quadratic corrections to the magnetometer outputs, then we are led by arguments similar to those above to the relation

$$(I + D)\mathbf{B}_k = \mathcal{O}^T A_k \mathbf{H}_k + \mathbf{b} + C{:}(A_k\mathbf{H}_k) \otimes (A_k\mathbf{H}_k) + \boldsymbol{\varepsilon}_k \tag{70}$$

where $C{:}(A_k\mathbf{H}_k) \otimes (A_k\mathbf{H}_k)$ denotes the vector whose ℓth component is given by

$$\sum_{m=1}^{3} \sum_{n=1}^{3} C_{\ell mn}(A_k\mathbf{H}_k)_m(A_k\mathbf{H}_k)_n \quad \ell = 1, 2, 3$$

Thus, we are led to

$$|\mathbf{H}_k|^2 = |(I + D)\mathbf{B}_k - \mathbf{b} - C{:}(A_k\mathbf{H}_k) \otimes (A_k\mathbf{H}_k) - \boldsymbol{\varepsilon}_k|^2 \tag{71}$$

There is now an explicit dependence on the attitude. Since the quadratic corrections are supposedly small, we can eliminate the attitude by making the replacement in the quadratic term of

$$A_k\mathbf{H}_k \approx \mathcal{O}[(I + D)\mathbf{B}_k - \mathbf{b}] \tag{72}$$

which leads to

$$|\mathbf{H}_k|^2 = |(I + D)\mathbf{B}_k - \mathbf{b} - C{:}\{\mathcal{O}[(I + D)\mathbf{B}_k - \mathbf{b}]\} \otimes \{\mathcal{O}[(I + D)\mathbf{B}_k - \mathbf{b}]\} - \boldsymbol{\varepsilon}_k|^2 \tag{73}$$

If we make the approximation now that $\mathcal{O}$ is close to the identity matrix, so that we can ignore terms on the order of $C\delta$, where δ is the typical size of the misalignments, then we are led to effective measurements of the form

$$z_k = |\mathbf{B}_k^2 - |\mathbf{H}_k|^2 \tag{74a}$$

$$\approx -\mathbf{B}_k^T(2D + D^2)\mathbf{B}_k + 2\mathbf{B}_k^T(I + D)\mathbf{b} - |\mathbf{b}|^2$$
$$+ 2\sum_{\ell mn} C_{\ell mn}[(I + D)\mathbf{B}_k - \mathbf{b}]_\ell[(I + D)\mathbf{B}_k - \mathbf{b}]_m[(I + D)\mathbf{B}_k - \mathbf{b}]_n \tag{74b}$$

which should be compared with equations (26) and (47). Unfortunately, it is obvious from equation (74) that, because of the quadratic term, there is no simple way to redefine the parameters in such a way that all of the nonlinear dependence is in a single data-independent term. Therefore, the centering operation will not lead to any simplification of the estimation procedure in this case. At best we can use the TWOSTEP algorithm to compute the constant and linear terms and use these estimates and the values $C = 0$ as in initial estimate in a Gauss-Newton minimization of the full cost function. Note that this cost function will now be of sixth degree in the magnetometer bias and therefore of eighth degree in all parameters. No attempt has been made to study the possibility of estimating the 36 "observable" constant, linear, and quadratic calibration parameters.

Numerical Examples

The algorithms treated in this work have been examined for a subset of cases selected from [1] and [2]. In general, these involve two typical scenarios: a spacecraft spinning at 15 rpm and an inertially stabilized spacecraft. The spacecraft orbit has been chosen to be circular with an altitude of 560 km and an inclination of 38 deg. The geomagnetic field in our studies has been simulated using the International Geomagnetic Reference Field (IGRF (1985)) [8], which has been extrapolated to 1994. Magnetometer data were sampled every 8 seconds.

Tables 1 and 2 show the results for estimating the magnetometer calibration parameters for the spinning spacecraft. The magnetometer measurement noise was assumed to be white and Gaussian, the assumption used by the estimator, and the covariance was taken to be isotropic with a standard deviation per axis of 2.0 mG. These were the parameters of the spinning spacecraft in Table 1 of reference [1]. We have chosen an intermediate set of values for the magnetometer bias, which we believe are typical [7], and suitably small values for the nonvanishing elements of D. The agreement is uniformly good in all cases. As expected, there is little improvement from the center correction in this case.

To test the estimator more rigorously, we have used also the very unforgiving case of the inertially stabilized spacecraft in Table 7 of reference [2]. In reference [2] it was seen that the values of the magnetometer bias and the random number seed of the measurement noise for this case and the short data span of 25 minutes created conditions of observability that were so poor and nearby local minima that were so disadvantageously placed that naive quartic scoring converged to the wrong minimum and the fixed-point method diverged completely. The TWOSTEP algo-

TABLE 1. Estimation of Magnetometer Biases and Scale Factors for a Spinning Spacecraft Using TWOSTEP. θ Is the Parameter, $\tilde{\theta}^*$ Is the Centered Estimate, and θ^* Is the Centered Estimate with Center Correction

	θ	$\tilde{\theta}^*$	θ^*
b_1	30. mG	29.53 ± .33	29.67 ± .30
b_2	60. mG	59.80 ± .10	59.81 ± .10
b_3	90. mG	98.98 ± .26	99.42 ± .24
d_1	.05	.04874 ± .0016	.04911 ± .0013
d_2	.10	.09993 ± .0086	.10008 ± .0008
d_3	.05	.04864 ± .0024	.04920 ± .0022

TABLE 2. Estimation of Magnetometer Biases, Scale Factors, and Nonorthogonality Corrections for a Spinning Spacecraft Using TWOSTEP. θ Is the Parameter, $\tilde{\theta}^*$ Is the Centered Estimate, and θ^* Is the Centered Estimate with Center Correction

	θ	$\tilde{\theta}^*$	θ^*
b_1	30. mG	29.45 ± .37	29.51 ± .33
b_2	60. mG	59.95 ± .21	59.96 ± .19
b_3	90. mG	90.10 ± .36	90.13 ± .32
D_{11}	.05	.0485 ± .0016	.0487 ± .0014
D_{22}	.10	.1004 ± .0012	.1005 ± .0011
D_{33}	.05	.0489 ± .0028	.0492 ± .0023
D_{12}	.05	.0502 ± .0007	.0504 ± .0011
D_{13}	.05	.0503 ± .0011	.0504 ± .0011
D_{23}	.05	.0501 ± .0013	.0503 ± .0012

rithm, however, converged to the correct solution with customary rapidity. We have simulated these results for the case of the fully populated matrix D. These results are shown in Table 3.

In Tables 4 and 5 we have examined the behavior of the algorithm when the measurement noise has been mismodeled. Table 4 used the colored noise model of reference [1] in simulating the measurements. Table 5 used the "realistic" noise model. In both cases two orbits of data were used with the spacecraft spinning at 15 rpm about the magnetometer x-axis. The confidence intervals were calculated on the basis of the Gaussian statistics by the estimator. Clearly, despite the fact that the estimator continues to assume Gaussian white noise, the agreement is quite good. As expected, the actual errors are typically much larger than would be expected from the computed confidence intervals.

Discussion

We see that one can estimate not only magnetometer biases from scalar magnetometer data but magnetometer scale factors and nonorthogonality corrections as well. Misalignments are not observable from magnitude data, as we have shown formally. The quality of the estimates is quite good, even in the presence of mod-

TABLE 3. Estimation of Magnetometer Biases, Scale Factors, and Nonorthogonality Corrections for an Inertially Stabilized Spacecraft Using TWOSTEP. θ Is the Parameter, $\tilde{\theta}^*$ Is the Centered Estimate, and θ^* Is the Centered Estimate with Center Correction

	θ	$\tilde{\theta}^*$	θ^*
b_1	200. mG	196.97 ± 2.7	191.00 ± 2.5
b_2	100. mG	87.93 ± 1.1	98.95 ± 1.0
b_3	−200. mG	−166.71 ± 4.0	−204.40 ± 3.0
D_{11}	.05	.032 ± .018	.022 ± .018
D_{22}	.10	.110 ± .014	.093 ± .010
D_{33}	.05	.219 ± .210	−.073 ± .040
D_{12}	.05	.037 ± .014	.056 ± .005
D_{13}	.05	.070 ± .050	.063 ± .023
D_{23}	.05	−.018 ± .056	.063 ± .007

TABLE 4. Estimation of Magnetometer Biases, Scale Factors, and Nonorthogonality Corrections for an Inertially Stabilized Spacecraft and Colored Measurement Noise Using TWOSTEP. θ Is the Parameter, $\tilde{\theta}^*$ Is the Centered Estimate, and θ^* Is the Centered Estimate with Center Correction

	θ	$\tilde{\theta}^*$	θ^*
b_1	30. mG	30.42 ± .23	30.61 ± .17
b_2	60. mG	60.17 ± .16	60.22 ± .14
b_3	90. mG	90.14 ± .18	90.21 ± .14
D_{11}	.05	.0505 ± .0016	.0514 ± .0010
D_{22}	.10	.0990 ± .0016	.0999 ± .0010
D_{33}	.05	.0494 ± .0015	.0503 ± .0010
D_{12}	.05	.0508 ± .0007	.0509 ± .0007
D_{13}	.05	.0509 ± .0007	.0509 ± .0007
D_{23}	.05	.0497 ± .0008	.0498 ± .0008

TABLE 5. Estimation of Magnetometer Biases, Scale Factors, and Nonorthogonality Corrections for an Inertially Stabilized Spacecraft and "Realistic" Measurement Noise Using TWOSTEP. θ Is the Parameter, $\tilde{\theta}^*$ Is the Centered Estimate, and θ^* Is the Centered Estimate with Center Correction

	θ	$\tilde{\theta}^*$	θ^*
b_1	30. mG	30.58 ± .08	30.73 ± .06
b_2	60. mG	50.76 ± .06	60.81 ± .05
b_3	90. mG	90.86 ± .06	90.92 ± .05
D_{11}	.05	.052 ± .0006	.053 ± .0003
D_{22}	.10	.102 ± .0006	.103 ± .0004
D_{33}	.05	.053 ± .0005	.053 ± .0003
D_{12}	.05	.050 ± .0002	.050 ± .0002
D_{13}	.05	.050 ± .0002	.050 ± .0002
D_{23}	.05	.050 ± .0003	.050 ± .0003

eling errors in the measurement noise. This goes very much against the common wisdom that only biases are truly observable from magnitude data. While an accurate attitude reference will clearly be advantageous to determining the calibration parameters of a magnetometer, it is obviously also true that a great deal can be learned from magnitude data alone.

The results for the very marginally observable case shown in Table 3 are consistent with the computed confidence intervals but obviously not as good as one might have if the spacecraft were spinning. Still, they show the power of the TWOSTEP method under very severe conditions. Note the importance of the center correction in this case. Clearly, if one wishes to estimate more than just magnetometer biases, one should include a calibration maneuver, which ideally would cause the spacecraft to rotate or librate about two axes. The mediocrity of these results for an inertially stabilized spacecraft should not be of serious concern, since these spacecraft are invariably provided with sensors much more accurate than the magnetometer for three-axis attitude. Thus, it is unlikely that one would ever want to estimate linear corrections to the magnetometer calibration for an inertially stabilized spacecraft during mission mode using TWOSTEP. (SAC-B here is a notable exception.) At orbit insertion, when such a spacecraft might be spinning, one clearly has little need of the linear corrections. It would appear, in fact, that to date no mission has attempted to estimate the linear magnetometer calibration terms for a spinning spacecraft.

When the noise is severely mismodeled, as in Tables 4 and 5, which show the results using the colored noise and "realistic" noise models of reference [1] while TWOSTEP assumes that the noise is white and Gaussian, the results are still quite good, although the confidence bounds are obviously optimistic.

The efficacy and robustness of the TWOSTEP algorithm has been amply demonstrated in several scenarios for estimates of the bias only [1], of the bias and linear corrections (this work), and in comparison with other algorithms [2]. Until reference [2], none of the other algorithms have been subjected to stringent testing. With the sole exception of TWOSTEP, all of the algorithms have been shown to be either inconsistent or, in the case of the competing iterative algorithms, to harbor serious diseases. The TWOSTEP algorithm has provided rapid and consistent estimates of the magnetometer biases and other parameters in all of the nearly 200 simulations which have been performed in the three studies.

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References

[1] ALONSO, R. and SHUSTER, M. D. "TWOSTEP: A Fast Robust Algorithm for Attitude-Independent Magnetometer-Bias Determination," *The Journal of the Astronautical Sciences,* Vol. 50, No. 4, October–December 2002, pp. 433–451.

[2] ALONSO, R. and SHUSTER, M. D. "Attitude-Independent Magnetometer-Bias Determination: A Survey," *The Journal of the Astronautical Sciences,* Vol. 50, No. 4, October–December 2002, pp. 453–475.

[3] GAMBHIR, B. "Determination of Magnetometer Biases Using Module RESIDG," Computer Sciences Corporation, Report No. 3000-32700-01TN, March 1975.

[4] LERNER, G. M. "Scalar Checking," in Wertz, J. R. (editor) *Spacecraft Attitude Determination and Control,* Kluwer Academic Publishers, Dordrecht, 1978, pp. 328–334.

[5] SORENSON, H. W. *Parameter Estimation,* Marcel Dekker, New York, 1980.

[6] HIGHAM, N. J. "Computing the Polar Decomposition with Applications," *SIAM Journal on Scientific and Statistical Computing,* Vol. 7, No. 4, October 1986, 1160–1174.

[7] LERNER, G. M. and SHUSTER, M. D. "Inflight Magnetometer Calibration and Attitude Determination for Near-Earth Spacecraft," *Journal of Guidance and Control,* Vol. 4, No. 5, 1981, pp. 518–522.

[8] LANGEL, R. A. "The Main Field," in JACOBS, J. A. (ed.), *Geomagnetism,* Academic Press, Orlando, 1987, pp. 348–349.