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# Attitude-Independent Magnetometer-Bias Determination: A Survey

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**Abstract**

The currently known algorithms for inflight magnetometer-bias determination without knowledge of the attitude are examined. The majority of these are shown to be limited either by poor convergence properties, significant statistical or analytical approximations, or the discarding of important data. The most robust and accurate of these algorithms is TWOSTEP, an algorithm recently developed which combines the best properties of the existing algorithms. Comparisons of algorithm performance are made both for spinning and inertially stabilized spacecraft. While TWOSTEP performs well in all cases, many of the other algorithms do not converge to the globally optimal estimate of the magnetometer-bias vector or even diverge.

## Introduction

A number of algorithms have been proposed for estimating the magnetometer bias when attitude information is not available. The simplest is to solve for the bias vector by minimizing the weighted sum of the squares of residuals which are the differences in the squares of the magnitudes of the measured and modeled magnetic fields [1]. This approach has the disadvantage that the cost function is quartic in the magnetometer bias and therefore admits multiple minima. Typically, one initiates the least-squares procedure by assuming that the initial magnetometer bias vector vanishes, which may lead to slow convergence or convergence only to a local minimum if the magnetometer bias is large compared to the ambient magnetic field.

Gambhir [1, 2] advocated centering the data to remove the quartic dependence. This leads to a cost function which is quadratic is the bias and, therefore, has a unique solution. The algorithm embodying this centering is called RESIDG (supposedly, "G" for Gambhir) and has been employed for nearly two decades. The centering, however, necessarily discards part of the data, and the effect of this loss

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of data on the accuracy of the algorithm was not studied. In addition, RESIDG does not make any attempt to treat the statistics correctly, so that it is not possible to assess the accuracy of the estimation adequately.

A second approach has been put forth by Thompson *et al.* [3], who preferred to construct a fixed-point algorithm, which was called, with obvious reference, RESIDT. Fixed-point algorithms have the advantage of often converging quickly when one is far from the solution, but can become intolerably slow as one approaches the solution. Thompson's algorithm was successfully employed in support of the AMPTE spacecraft.

Davenport *et al.* [4] have proposed another approach to solving the quartic cost function by computing first an approximate solution for the magnetometer bias. The approximate solution produced by this algorithm, unfortunately, is not consistent. Hence, the approximate solution cannot approach the true solution as the number of data becomes infinite. However, the inconsistency seems to be no worse than about ten per cent for biases as large as one third of the ambient field. Higher accuracy can then be obtained by an iterative procedure, using the approximate estimate as a starting value for minimizing the quartic cost function. Davenport's approximation, however, has no advantage over the centering method of Gambhir, which, at least, *is* consistent and which may serve equally well as a starting point for an iterative solution using the quartic cost function. This algorithm has been applied to the magnetometers of the Hubble Space Telescope.

Acuña [5] has proposed an ingenious method which does not require a geomagnetic field model at all. This model has been used to calibrate the magnetometer in studies of magnetic fields far from the Earth. The model is certainly adequate for most applications, even for spacecraft in low-Earth orbit. However, the fact that it does not take advantage of a field model, limits its accuracy and introduces systematic errors as well. Like Davenport's approximation, Acuña's method may be used to initiate a search for the minimum of the quartic cost function.

The present authors [6] recently proposed a new algorithm which is based on the centering method of Gambhir [1, 2] but which treats the statistics more correctly and provides an efficient method of restoring the error introduced by the centering approximation. Centering necessarily discards part of the data. It is shown, however, that this discarded data can be incorporated as a single effective measurement which is uncorrelated with the centered estimate of the bias. This permits the data discarded by the centering approximation to be included in the final result as an update of the centered estimate. In most cases, the algorithm converges in two iterations. This approach has been applied successfully not only to the estimation of magnetometer biases but also the estimation of scale factors and the nonorthogonality of the magnetometer axes. The authors do not call this new algorithm RESIDA or RESIDS, but rather have chosen the name TWOSTEP. The algorithm was prepared to support SAC-B, the first Argentine spacecraft.

## The Measurement Model

All treatments begin with the model presented in [6] given as

$$\mathbf{B}_k = A_k\mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k, \quad k = 1, \ldots, N \tag{1}$$

where $\mathbf{B}_k$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time $t_k$; $\mathbf{H}_k$ is the corresponding value of the geomagnetic field with respect to an Earth-fixed coordinate system; $A_k$ is the attitude of the magnetometer with respect to the Earth-fixed coordinates; $\mathbf{b}$ is the magnetometer bias; and $\boldsymbol{\varepsilon}_k$ is the measurement noise. The measurement noise, which includes both sensor errors and geomagnetic field model uncertainties, is assumed to be white and Gaussian.

The dependence on the attitude is eliminated by considering the square of the magnitude of the magnetometer readings as an effective measurement. Thus, we define effective measurements and measurement noise as

$$z_k \equiv |\mathbf{B}_k|^2 - |\mathbf{H}_k|^2 \tag{2a}$$

$$\nu_k \equiv 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k - |\boldsymbol{\varepsilon}_k|^2 \tag{2b}$$

whence,

$$z_k = 2\mathbf{B}_k \cdot \mathbf{b} - |\mathbf{b}|^2 + \nu_k, \quad k = 1, \ldots, N \tag{3}$$

This is the starting point for the derivation of all of the algorithms. Assuming that the measurement noise $\boldsymbol{\varepsilon}_k$ on the magnetometer readings is white and Gaussian with $\boldsymbol{\varepsilon}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_k)$, it follows to very good approximation that the effective scalar noise satisfies

$$\nu_k \sim \mathcal{N}(\mu_k, \sigma_k^2) \tag{4}$$

with

$$\mu_k = -\mathrm{tr}(\Sigma_k) \tag{5a}$$

$$\sigma_k^2 = 4(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}}\Sigma_k(\mathbf{B}_k - \mathbf{b}) + 2(\mathrm{tr}\,\Sigma_k^2) \tag{5b}$$

The model is discussed in somewhat more detail in reference [6].

## Maximum Likelihood Estimate of the Bias and Scoring

Given the statistical model above, the negative-log-likelihood function [7] for the magnetometer bias is given by

$$J(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{\sigma_k^2}(z_k - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - \mu_k)^2 + \log \sigma_k^2 + \log 2\pi\right] \tag{6}$$

which is quartic in $\mathbf{b}$. The maximum-likelihood estimate maximizes the likelihood of the estimate, which is the probability density of the measurements (evaluated at their sampled values) given as a function of the magnetometer bias. Hence, it minimizes the negative logarithm of the likelihood (equation (6)), which thus provides a cost function.

Since the domain of $J$ has no boundaries, the maximum-likelihood estimate for $\mathbf{b}$, which we denote by $\mathbf{b}^*$, must satisfy

$$\left.\frac{\partial J}{\partial \mathbf{b}}\right|_{\mathbf{b}^*} = \mathbf{0} \tag{7}$$

Note that only the first of the three terms under the summation depends on the magnetometer bias. Unless one wishes to estimate parameters of the measurement noise, there is no reason to retain the remaining two terms.[3] This quartic dependence can be avoided if complete three-axis attitude information is available, since

[3]See the remarks in a corresponding footnote in reference [6].

the bias term then enters linearly into the measurement model (q.v. equation (1)), as in the work of Lerner and Shuster [8].

The most direct solution is obtained by scoring, which in this case is the Newton–Raphson approximation. We consider the sequence[4]

$$\mathbf{b}_0^{\mathrm{NR}} = \mathbf{0} \tag{8a}$$

$$\mathbf{b}_{i+1}^{\mathrm{NR}} = \mathbf{b}_i^{\mathrm{NR}} - \left[\frac{\partial^2 J}{\partial \mathbf{b}\, \partial \mathbf{b}^{\mathrm{T}}}(\mathbf{b}_i^{\mathrm{NR}})\right]^{-1} \frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}_i^{\mathrm{NR}}) \tag{8b}$$

This series is obtained by expanding $J(\mathbf{b})$ to quadratic order in $(\mathbf{b} - \mathbf{b}_i^{\mathrm{NR}})$, setting the gradient of the truncated series to zero, and solving for $\mathbf{b}_{i+1}$. If for some value of $i$ we are sufficiently close to the maximum-likelihood estimate, then it will be true that

$$\lim_{i\to\infty} \mathbf{b}_i^{\mathrm{NR}} \to \mathbf{b}^* \tag{9}$$

We have adopted the convention here that the partial derivatives of a scalar function with respect to a column vector is again a column vector. The gradient vector $\partial J/\partial \mathbf{b}$ is the $3 \times 1$ matrix

$$\frac{\partial J}{\partial \mathbf{b}} = -\sum_{k=1}^{N} \frac{1}{\sigma_k^2}(z_k - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - \mu_k)2(\mathbf{B}_k - \mathbf{b}) \tag{10}$$

and the Hessian matrix $\partial^2 J/\partial \mathbf{b}\, \partial \mathbf{b}^{\mathrm{T}}$ is given by the $3 \times 3$ matrix

$$\frac{\partial^2 J}{\partial \mathbf{b}\, \partial \mathbf{b}^{\mathrm{T}}} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2}[4(\mathbf{B}_k - \mathbf{b})(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}} + 2(z_k - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - \mu_k)I_{3\times 3}] \tag{11}$$

Generally, the second term in the brackets will be much smaller than the first and can be discarded.

A second approach to scoring is the Gauss–Newton approximation [9]. In this case, we replace the Hessian matrix by its expectation, the Fisher information matrix $F$. Since

$$E\{(z_k - 2\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - \mu_k)\} = 0 \tag{12}$$

this amounts to discarding the second term in equation (6). According to the law of large numbers, as the number of independent identically distributed (i.i.d.) samples of a random variable becomes infinite (the asymptotic limit), the average of these samples approaches the expectation value of the random variable. Our measurements are not identically distributed because of the dependence on $\mathbf{B}_k$. However, if the distribution of the values of $A_k\mathbf{H}_k$ is regularly repeated, then we may regard the measurements as being i.i.d. for each value of $A_k\mathbf{H}_k$. Therefore, as $N \to \infty$

$$\frac{\partial^2 J}{\partial \mathbf{b}\, \partial \mathbf{b}^{\mathrm{T}}} \to F_{bb} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} 4(\mathbf{B}_k - \mathbf{b})(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}} \tag{13}$$

and the scoring procedure becomes now

$$\mathbf{b}_0^{\mathrm{GN}} = \mathbf{0} \tag{14a}$$

$$\mathbf{b}_{i+1}^{\mathrm{GN}} = \mathbf{b}_i^{\mathrm{GN}} - F_{bb}^{-1} \frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}_i^{\mathrm{GN}}) \tag{14b}$$

[4]Throughout this work we shall use $k$ as the time index and $i$ as the iteration index.

Again, if for some value of $i$ we are sufficiently close to the maximum-likelihood estimate, it will be true that

$$\lim_{i\to\infty} \mathbf{b}_i^{GN} \rightarrow \mathbf{b}^* \tag{15}$$

Rigorously, the Fisher information must be evaluated at the true value of the magnetometer bias. In practice, however, there is little disadvantage to evaluating it at the current value of the estimate. For both the Newton–Raphson and the Gauss–Newton method, the estimate error covariance matrix is given in the limit of infinitely large data samples by

$$P_{bb} \rightarrow F_{bb}^{-1} = \left[ \sum_{k=1}^{N} \frac{1}{\sigma_k^2} 4(\mathbf{B}_k - \mathbf{b})(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}} \right]^{-1} \tag{16}$$

If the measurement noise is Gaussian, then the asymptotic limit is true, in fact, for finite data samples. In most cases, the Fisher information matrix is simpler to evaluate than the Hessian matrix of the negative-log-likelihood function, and often can be evaluated independently of the data.

The earliest estimates of the magnetometer bias were accomplished by the method culminating in equations (6) through (8), although often the weights were not chosen according to a statistical criterion.

## Convergence of the Scoring Approximation

Let $\mathbf{b}_0^{NR}, \mathbf{b}_1^{NR}, \ldots, \mathbf{b}_{i-1}^{NR}, \mathbf{b}_i^{NR}, \ldots,$ be a sequence of scoring estimates of the bias, generated by the Newton–Raphson method, and let us define the correction by

$$\boldsymbol{\varepsilon}_i^{NR} \equiv \mathbf{b}_i^{NR} - \mathbf{b}_{i-1}^{NR} \tag{17}$$

By definition of the Newton–Raphson method

$$\frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}_{i-1}^{NR}) + \left[ \frac{\partial^2 J}{\partial \mathbf{b}\, \partial \mathbf{b}^{\mathrm{T}}}(\mathbf{b}_{i-1}^{NR}) \right] \boldsymbol{\varepsilon}_i^{NR} = \mathbf{0} \tag{18a}$$

$$\frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}_{i}^{NR}) + \left[ \frac{\partial^2 J}{\partial \mathbf{b}\, \partial \mathbf{b}^{\mathrm{T}}}(\mathbf{b}_{i}^{NR}) \right] \boldsymbol{\varepsilon}_{i+1}^{NR} = \mathbf{0} \tag{18b}$$

Writing the gradient vector as

$$\mathbf{g}(\mathbf{b}) \equiv \frac{\partial J}{\partial \mathbf{b}}(\mathbf{b}) \tag{19}$$

we can rewrite equations (18) as

$$\mathbf{g}(\mathbf{b}_{i-1}^{NR}) + (\boldsymbol{\varepsilon}_i^{NR} \cdot \nabla)\mathbf{g}(\mathbf{b}_{i-1}^{NR}) = \mathbf{0} \tag{20a}$$

$$\mathbf{g}(\mathbf{b}_{i}^{NR}) + (\boldsymbol{\varepsilon}_{i+1}^{NR} \cdot \nabla)\mathbf{g}(\mathbf{b}_{i}^{NR}) = \mathbf{0} \tag{20b}$$

and $\nabla$ is the gradient operator with respect to $\mathbf{b}$. If we substitute equation (17) for $\mathbf{b}_i^{NR}$ in equation (20b), we obtain

$$\mathbf{g}(\mathbf{b}_{i-1}^{NR} + \boldsymbol{\varepsilon}_i^{NR}) + (\boldsymbol{\varepsilon}_{i+1}^{NR} \cdot \nabla)\mathbf{g}(\mathbf{b}_{i-1}^{NR} + \boldsymbol{\varepsilon}_i^{NR}) = \mathbf{0} \tag{21}$$

Expanding the function $\mathbf{g}$ in a Taylor series in $\boldsymbol{\varepsilon}_i^{NR}$ and discarding terms higher than quadratic leads to

$$\begin{aligned} \mathbf{g}(\mathbf{b}_{i-1}^{NR}) + (\boldsymbol{\varepsilon}_i^{NR} \cdot \nabla)\mathbf{g}(\mathbf{b}_{i-1}^{NR}) + (1/2)(\boldsymbol{\varepsilon}_i^{NR} \cdot \nabla)^2\mathbf{g}(\mathbf{b}_{i-1}^{NR}) + \cdots \\ + (\boldsymbol{\varepsilon}_{i+1}^{NR} \cdot \nabla)[\mathbf{g}(\mathbf{b}_{i-1}^{NR}) + (\boldsymbol{\varepsilon}_i^{NR} \cdot \nabla)\mathbf{g}(\mathbf{b}_{i-1}^{NR}) + \cdots] = \mathbf{0} \end{aligned} \tag{22}$$

The sum of th first two terms vanishes identically because of equation (20a). Solving for $\boldsymbol{\varepsilon}_{i+1}^{NR}$ and keeping only lowest-order terms leads to

$$\boldsymbol{\varepsilon}_{i+1}^{NR} = -\frac{1}{2}\left[\frac{\partial \mathbf{g}}{\partial \mathbf{b}^T}(\mathbf{b}_{i-1}^{NR})\right]^{-1}(\boldsymbol{\varepsilon}_{i+1}^{NR} \cdot \nabla)^2 \mathbf{g}(\mathbf{b}_{i-1}^{NR}) \tag{23}$$

Thus, when the quadratic approximation in equation (22) is justified, the convergence of the corrections is also quadratic. The Newton–Raphson method generally converges quickly, once the solution is suitably close to the minimum. The same is true for the Gauss–Newton method, as can be shown by arguments identical to those leading to equation (23). (For the Gauss–Newton method, the matrix $\partial \mathbf{g}/\partial \mathbf{b}^T$ is replaced by the Fisher information matrix.) For quartic functions, however, one may never become close enough to the true minimum because of the influence of the other (local) minima. This is a serious drawback of the Newton–Raphson method.

Let us examine these statements in more detail for a specific example. Suppose that the actual measurements were obtained from a random function

$$\mathbf{B}_k = A_k \mathbf{H}_k + \mathbf{b}^{true} + \boldsymbol{\varepsilon}_k, \qquad k = 1, \ldots, N \tag{24}$$

where $\mathbf{b}^{true}$ is the true value of the bias. We assume also that the $A_k\mathbf{H}_k$, $k = 1, \ldots, N$, are uniformly distributed over all possible directions and that $\boldsymbol{\varepsilon}_k$ is isotropic, that is, it has a covariance matrix proportional to the identity matrix. In that case $\sigma_k^2$ is a constant, which we denote simply by $\sigma^2$, and the negative-log-likelihood function will have the form.

$$J = \frac{1}{2\sigma^2}\sum_{k=1}^{N}\big[|A_k\mathbf{H}_k + \mathbf{b}^{true} + \boldsymbol{\varepsilon}_k|^2 - |A_k\mathbf{H}_k|^2$$
$$-2(A_k\mathbf{H}_k + \mathbf{b}^{true} + \boldsymbol{\varepsilon}_k)\cdot \mathbf{b} + |\mathbf{b}|^2 - \mu\big]^2 + \text{terms independent of } \mathbf{b} \tag{25}$$

We will assume that $N$ is very large, so that we can replace $J$ by its expectation value, and we can replace the sum over $N$ by $N$ times the average over the directions of $A_k\mathbf{H}_k$. This leads to

$$J(\mathbf{b}) = \frac{N}{2\sigma^2}\left[\frac{4}{3}H^2|\mathbf{b}^{true} - \mathbf{b}|^2 + |\mathbf{b}^{true} - \mathbf{b}|^4\right] \tag{26}$$

Here $H$ is the magnitude of $\mathbf{H}_k$, which by our initial assumption is a constant. In this special case, $J$ has a unique minimum at $\mathbf{b}^{true}$. If we differentiate with respect to $\mathbf{b}$, we obtain

$$\mathbf{g}(\mathbf{b}) = \frac{\partial J}{\partial \mathbf{b}} = -\frac{N}{\sigma^2}\left[\frac{4}{3}H^2(\mathbf{b}^{true} - \mathbf{b}) + 2|\mathbf{b}^{true} - \mathbf{b}|^2(\mathbf{b}^{true} - \mathbf{b})\right] \tag{27a}$$

$$\frac{\partial \mathbf{g}(\mathbf{b})}{\partial \mathbf{b}^T} = \frac{N}{\sigma^2}\left[\frac{4}{3}H^2 I_{3\times3} + 2|\mathbf{b}^{true} - \mathbf{b}|^2 I_{3\times3} + 4(\mathbf{b}^{true} - \mathbf{b})(\mathbf{b}^{true} - \mathbf{b})^T\right] \tag{27b}$$

Substituting these expressions into equation (23) leads to

$$\boldsymbol{\varepsilon}_{i+1}^{NR} = \left[\left(\frac{1}{3}H^2 + \frac{1}{2}|\mathbf{b}^{true} - \mathbf{b}_i|^2\right)I_{3\times3} + (\mathbf{b}^{true} - \mathbf{b}_i)(\mathbf{b}^{true} - \mathbf{b}_i)^T\right]^{-1}$$
$$\cdot\left[2(\mathbf{b}^{true} - \mathbf{b}_i)\cdot \boldsymbol{\varepsilon}_i^{NR}\boldsymbol{\varepsilon}_i^{NR} - |\boldsymbol{\varepsilon}_i^{NR}|^2(\mathbf{b}^{true} - \mathbf{b}_i)\right] \tag{28}$$

Close to the solution, the first matrix is simply $(3/H^2)I_{3\times 3}$ and the numerator decreases faster than $|\boldsymbol{\varepsilon}_i^{NR}|^2$ because $|(\mathbf{b}^{true} - \mathbf{b}_i)|$ is also tending to zero. The convergence, therefore, will be very fast. Note that $\mathbf{b}^{true}$ is a double root in this special case and that the minimum of the cost function is unique. Hence, for small perturbations of the conditions, we expect to find two roots very close together. This is, perhaps, the signpost of trouble ahead.

## Fixed-Point Method

Thompson, Neal and Shuster [3] proposed a fixed-point algorithm. Define the quantities

$$G \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2}[4\mathbf{B}_k\mathbf{B}_k^T + 2(z_k - \mu_k)I_{3\times 3}] \tag{29a}$$

$$\mathbf{a} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2}(z_k - \mu_k)2\mathbf{B}_k \tag{29b}$$

$$\mathbf{f}(\mathbf{b}) \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2}[4(\mathbf{B}_k \cdot \mathbf{b})\mathbf{b} + 2|\mathbf{b}|^2(\mathbf{B}_k - \mathbf{b})] \tag{29c}$$

Then the gradient vector defined by equations (10) and (19) becomes

$$\mathbf{g}(\mathbf{b}) = G\mathbf{b} - \mathbf{a} - \mathbf{f}(\mathbf{b}) \tag{30}$$

Since the gradient vector must vanish at the maximum-likelihood estimate, it follows that

$$G\mathbf{b}^* - \mathbf{a} - \mathbf{f}(\mathbf{b}^*) = \mathbf{0} \tag{31}$$

Hence,

$$\mathbf{b}^* = G^{-1}[\mathbf{a} + \mathbf{f}(\mathbf{b}^*)] \tag{32}$$

Thus, we have an implicit solution for the magnetometer bias. Typically, this algorithm is solved iteratively using

$$\mathbf{b}_0^{FP} = \mathbf{0} \tag{33a}$$

$$\mathbf{b}_{i+1}^{FP} = G^{-1}[\mathbf{a} + \mathbf{f}(\mathbf{b}_i^{FP})] \tag{33b}$$

and we expect that once $\mathbf{b}_i^{FP}$ is sufficiently close to the solution that

$$\lim_{i\to\infty} \mathbf{b}_i^{FP} = \mathbf{b}^* \tag{34}$$

Unfortunately, the convergence of fixed-point algorithms is usually poor.

The convergence properties of the fixed-point method can be found in the same manner as those of the Newton–Raphson method. Again we write

$$\mathbf{b}_i^{FP} = \mathbf{b}_{i-1}^{FP} + \boldsymbol{\varepsilon}_i^{FP} \tag{35}$$

and

$$\mathbf{b}_i^{FP} + \boldsymbol{\varepsilon}_{i+1}^{FP} = G^{-1}[\mathbf{a} + \mathbf{f}(\mathbf{b}_{i-1}^{FP} + \boldsymbol{\varepsilon}_i^{FP})] \tag{36a}$$

$$= G^{-1}[\mathbf{a} + \mathbf{f}(\mathbf{b}_i^{FP}) + (\boldsymbol{\varepsilon}_i^{FP} \cdot \nabla)\mathbf{f}(\mathbf{b}_{i-1}^{FP})] \tag{36b}$$

Collecting terms and recalling equation (33b) yields

$$\boldsymbol{\varepsilon}_{i+1}^{FP} = G^{-1}(\boldsymbol{\varepsilon}_i^{FP} \cdot \nabla)\mathbf{f}(\mathbf{b}_{i-1}^{FP}) \tag{37}$$

so that convergence is now only linear.

Let us apply the example that was used for the convergence study of the Newton–Raphson method. In this case

$$G(\mathbf{b}) = \frac{4N}{\sigma^2}\left[\frac{1}{3}H^2 I_{3\times3} + \mathbf{b}^{\text{true}}\mathbf{b}^{\text{true}T}\right] \tag{38a}$$

$$\mathbf{a} = \mathbf{0} \tag{38b}$$

$$\mathbf{f}(\mathbf{b}) = \frac{4N}{\sigma^2}\left[(\mathbf{b}^{\text{true}}\cdot\mathbf{b})\mathbf{b} + \frac{1}{2}|\mathbf{b}|^2(\mathbf{b}^{\text{true}} - \mathbf{b})\right] \tag{38c}$$

This leads to

$$\boldsymbol{\varepsilon}_{i+1}^{\text{FP}} = \left[\frac{1}{3}H^2 I_{3\times3} + \mathbf{b}^{\text{true}}\mathbf{b}^{\text{true}T}\right]^{-1} \cdot\left[(\mathbf{b}^{\text{true}}\cdot\boldsymbol{\varepsilon}_i^{\text{FP}})\mathbf{b}_i + (\mathbf{b}^{\text{true}}\cdot\mathbf{b}_i)\boldsymbol{\varepsilon}_i^{\text{FP}} + (\mathbf{b}\cdot\boldsymbol{\varepsilon}_i^{\text{FP}})(\mathbf{b}^{\text{true}} - \mathbf{b}_i) - \frac{1}{2}|\mathbf{b}_i|^2\boldsymbol{\varepsilon}_i^{\text{FP}}\right] \tag{39}$$

The convergence is only linear. Furthermore, from equation (39) the convergence factor for small values of the magnetometer bias is typically on the order of

$$\frac{15|\mathbf{b}^{\text{true}}|^2}{2H^2}$$

For a bias magnitude of 100 mG, which was observed for the SEASAT spacecraft [8], the convergence factor at the magnetic equation is roughly 0.8, so that convergence would be very poor in this case. For field values at high altitudes, the factor could be greater than one, and the fixed-point algorithm would not converge at all.

## Davenport's Approximation

Davenport and his collaborators [4] have offered a very clever approximate form for the bias vector estimator at the cost of having an estimator which is inconsistent. He begins by writing an approximate cost function as

$$J_D(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(z_k - 2\mathbf{B}_k\cdot\mathbf{b} + \lambda^2 - \mu_k)^2 \tag{40}$$

where $\lambda$ is a constant. This cost function would agree within constant terms with that of equation (6) when $\lambda = |\mathbf{b}|$. Davenport, however, allows $\lambda$ to be a free parameter, independent of $\mathbf{b}$.

The cost function of equation (40) is only quadratic in $\mathbf{b}$. Differentiating this cost function with respect to $\mathbf{b}$ and setting the gradient equal to zero leads to

$$\left.\frac{\partial}{\partial\mathbf{b}}J_D(\mathbf{b})\right|_{\mathbf{b}_D^*} = -\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(z_k - 2\mathbf{B}_k\cdot\mathbf{b}_D^* + \lambda^2 - \mu_k)2\mathbf{B}_k = \mathbf{0} \tag{41}$$

which has the solution

$$\mathbf{b}_D^* = \mathbf{U} + \lambda^2\mathbf{V} \tag{42}$$

where

$$\mathbf{U} \equiv \left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}4\mathbf{B}_k\mathbf{B}_k^T\right]^{-1}\left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}(z_k - \mu_k)2\mathbf{B}_k\right] \tag{43a}$$

$$\mathbf{V} \equiv \left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}4\mathbf{B}_k\mathbf{B}_k^T\right]^{-1}\left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}2\mathbf{B}_k\right] \tag{43b}$$

One next chooses $\lambda$ to be consistent with

$$|\mathbf{b}_D^*|^2 = \lambda^2 \tag{44}$$

Thus, one computes the square of both members of equation (42), which, noting equation (44), leads to

$$\lambda^2 = |\mathbf{U}|^2 + 2(\mathbf{U} \cdot \mathbf{V})\lambda^2 + |\mathbf{V}|^2\lambda^4 \tag{45}$$

This can be recast as

$$a\lambda^4 + b\lambda^2 + c = 0 \tag{46}$$

with

$$a = |\mathbf{V}|^2, \qquad b = 2\mathbf{U} \cdot \mathbf{V} - 1, \qquad c = |\mathbf{U}|^2 \tag{47}$$

The solutions are given by the quadratic theorem

$$\lambda^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{48}$$

Since $a$ and $c$ are positive and the discriminant must be non-negative to lead to real roots, it is obvious that $b^2 \geq 4ac$, and therefore $|b| \geq \sqrt{b^2 - 4ac}$. It follows that $b$ must be negative, otherwise $\lambda^2$ will not be positive. In the absence of a magnetometer bias and of measurement noise, $c = 0$ and also $\mathbf{b}_D^* = 0$, which corresponds to the negative sign of the square root in equation (48). Therefore, the solution for $\lambda^2$ in that case is given by

$$\lambda^2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{49}$$

Davenport argues that this sign must be chosen for the square root in all cases. We shall see later that this choice of sign may not always be correct.

Unfortunately, there is no good way to assess the error created by replacing $|\mathbf{b}|^2$ by $\lambda^2$ and then choosing $\lambda^2$ to restore some semblance of self-consistency. As a result of this approximation, the algorithm will not yield the exact value for the bias in the limit that the measurement noise vanishes. Mathematically speaking, we say that such an algorithm is inconsistent. To appreciate the magnitude of the inconsistency, consider again our example above for the isotropic field distribution but with vanishing measurement and model noise. We find in this case that Davenport's estimate is given by

$$\mathbf{b}_D^* = \frac{1 - \sqrt{1 - 4x(1 + x)^2}}{2x(1 + 2x)^2} \tag{50}$$

where

$$x \equiv \frac{3|\mathbf{b}^{\text{true}}|^2}{2H^2} \tag{51}$$

for $x \ll 1/4$, equation (50) becomes

$$\mathbf{b}_D^* = (1 - x)^2\mathbf{b}^{\text{true}} \tag{52}$$

For $x = 1/6$, corresponding to a magnetometer bias approximately one third of the ambient field (the SEASAT case at equatorial latitudes), the inconsistency in the estimator as given by equation (50) in this case is approximately 12 percent. For a bias of magnitude 20 mG, the inconsistency becomes only 1.5 percent. Again, we emphasize that these errors are not due to measurement noise but to approximations in the cost function. The errors cannot be decreased by increasing the number of data. Davenport has used this approximation also as the starting point of a complete Newton–Raphson process.

## Acuña's Algorithm

A very intriguing alternative to the preceding algorithms is that of Acuña [5], which does not rely on a field model at all. Thus, Acuña's algorithm is ideally suited to determining the magnetometer bias when the satellite is far from the Earth and the ambient field is not known. The algorithm was developed, in fact, for experimental studies of the magnetic field of celestial bodies other than the Earth.

Acuña defines the derived measurement

$$z_{k,\ell} = |\mathbf{B}_k|^2 - |\mathbf{B}_\ell|^2 \tag{53}$$

where, as before, $\mathbf{B}_k$ is the magnetometer at time $t_k$. Recalling the model of equation (1) we can write this as

$$z_{k,\ell} = 2(\mathbf{B}_k - \mathbf{B}_\ell) \cdot \mathbf{b} + \Delta_{k,\ell} \tag{54}$$

with the effective measurement error $\Delta_{k,\ell}$ given by

$$\Delta_{k,\ell} = |\mathbf{H}_k|^2 - |\mathbf{H}_\ell|^2 + \nu_k - \nu_\ell \tag{55}$$

and $\nu_k$ is as given in equation (2b). There are thus two disturbances which contribute to the noise term, a largely random term $\nu_k - \nu_\ell$ and a systematic model error $|\mathbf{H}_k|^2 - |\mathbf{H}_\ell|^2$. Because Acuña's effective measurement involves a subtraction, it also avoids the quartic dependence in the cost function.

Acuña argues that if the spacecraft is spinning, we can neglect the model error compared to the random noise term. Since the algorithm will be applied to systems in which there is no available model for the effective noise term, Acuña assumes effectively that the random noise term is a white Gaussian identically distributed process and determines the $\mathbf{b}$ which minimizes the cost function

$$J(\mathbf{b}) = \frac{1}{2} \sum_{k,\ell}{}' [z_{k,\ell} - 2(\mathbf{B}_k - \mathbf{B}_\ell) \cdot \mathbf{b}]^2 \tag{56}$$

Here, the prime on the summation symbol serves to remind us that no index occurs more than once, so that the individual magnetometer measurements are not counted double. (That is, $n$ individual magnetometer measurements can yield no more than $n/2$ effective measurements.) Straightforward differentiation of equation (56) leads to

$$\mathbf{b}_A^* = \left[ \sum_{k,\ell}{}' 4(\mathbf{B}_k - \mathbf{B}_\ell)(\mathbf{B}_k - \mathbf{B}_\ell)^T \right]^{-1} \sum_{k,\ell}{}' 2(\mathbf{B}_k - \mathbf{B}_\ell) z_{k\ell} \tag{57}$$

Substituting equation (54) we may rewrite this as

$$\mathbf{b}_A^* = \mathbf{b} + \Delta\mathbf{b}_A \tag{58}$$

where

$$\Delta\mathbf{b}_A = \left[ \sum_{k,\ell}{}' 4(\mathbf{B}_k - \mathbf{B}_\ell)(\mathbf{B}_k - \mathbf{B}_\ell)^T \right]^{-1} \sum_{k,\ell}{}' 2(\mathbf{B}_k - \mathbf{B}_\ell)\, \Delta_{k\ell} \tag{59}$$

From equation (55), we see that the estimation error contains both random and systematic terms

$$\Delta \mathbf{b}_{\mathrm{A}}^{\mathrm{random}} = \left[ \sum_{k,\ell}{}' 4(\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell)(\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell)^{\mathrm{T}} \right]^{-1} \sum_{k,\ell}{}' 2(\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell)\, \Delta_{k\ell}^{\mathrm{ran}} \tag{60a}$$

$$\Delta \mathbf{b}_{\mathrm{A}}^{\mathrm{systematic}} = \left[ \sum_{k,\ell}{}' 4(\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell)(\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell)^{\mathrm{T}} \right]^{-1} \sum_{k,\ell}{}' 2(\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell)\, \Delta_{k\ell}^{\mathrm{sys}} \tag{60b}$$

where

$$\Delta_{k\ell}^{\mathrm{ran}} = \nu_k - \mu_k - \nu_\ell + \mu_\ell \quad \text{and} \quad \Delta_{k\ell}^{\mathrm{sys}} = |\mathbf{H}_{\mathrm{k}}|^2 - |\mathbf{H}_\ell|^2 + \mu_k - \mu_\ell \tag{61}$$

The relative importance of the random and systematic errors determine how one should best construct the effective measurements $z_{k\ell}$. If the random errors dominate, then it is advantageous to choose $k$ and $\ell$ so that $|\mathbf{B}_{\mathrm{k}} - \mathbf{B}_\ell|$ is maximized. Since the spacecraft is assumed to be spinning, this means that the two measured magnetic field components perpendicular to the spacecraft spin axis are nominally equal in magnitude and opposite in sign. If the random errors indeed dominate, it follows that the measurement error variance will be independent of the choice of $k$ and $\ell$, while the sensitivity to $\mathbf{b}$ will be maximized. If, on the other hand it is the systematic model error which dominates, then it will be advantageous to choose $k$ and $\ell$ close together to minimize this.

To judge the relative importance of these two contributions to the error in low Earth polar orbit, we note that the geomagnetic field has a magnitude of approximately 300 mG at the equator and 600 mG at the poles. Since the orbital period of a near Earth spacecraft is approximately 100 minutes, the time to travel from the equator to the pole is a quarter orbit or 1500 seconds. It follows that a typical derivative of $|\mathbf{H}|^2$ with respect to time is given by

$$\left| \frac{d|\mathbf{H}|^2}{dt} \right| \approx \frac{(600)^2 - (300)^2}{1500} \,(\mathrm{mG})^2/\mathrm{sec} \approx 180\,(\mathrm{mG})^2/\mathrm{sec} \tag{62}$$

For a typical time between measurements of 10 seconds, this amounts to

$$|\Delta_{k\ell}^{\mathrm{model}}| \approx 1800\,(\mathrm{mG})^2 \tag{63}$$

We assume that the contribution of the $\mu$'s is much less. The typical random error, on the other hand, is approximately

$$|\Delta_{k\ell}^{\mathrm{random}}| \approx 4|\mathbf{H}|\sigma_\varepsilon \tag{64}$$

where $\sigma_\varepsilon$ is the standard deviation of the individual magnetometer measurements (per axis). Taking $\sigma_\varepsilon \approx 2$ mG and an average value of 450 mG for the geomagnetic field leads to

$$\Delta_{k\ell}^{\mathrm{random}} \approx 1800\,(\mathrm{mG})^2 \tag{65}$$

The two contributions are roughly equal. The above discussion considered only linear terms in the model errors. When we take account of quadratic and higher-order errors, then clearly the systematic errors will be rather more important. Note also that we have chosen a sampling interval of 10 seconds. Had we chosen a sampling interval of 30 seconds, then the systematic error would have tripled while the random error remained the same. Clearly, to optimize Acuña's algorithm, we wish to take the sampling interval to be no larger than 10 seconds.

For small $\Delta t$ we can obtain a very approximate expression for the typical systematic error in the estimate of the $j$-th component of $\mathbf{b}$ using Acuña's algorithm, which is

$$|\Delta b_j^{\text{systematic}}| \approx \frac{1}{2\omega|\mathbf{H}|}\left|\frac{d|\mathbf{H}|^2}{dt}\right| \approx 0.4 \text{ mG} \tag{66}$$

independent of $\Delta t$. Here, $\omega$ is the magnitude of the angular velocity, which we have chosen to be 5 rpm. At a sampling interval of 10 seconds, the random error will be approximately the same amount.

Acuña's algorithm is the only choice for magnetometer bias determination when there is no reliable magnetic field model. However, if a reliable field model is available, then one can obtain greater accuracy by accounting explicitly for the systematic error term, which is the case for the other algorithms in this study. Also, differencing the magnetometer measurements necessarily discards half of the data. Thus, Acuña's algorithm is not the ideal algorithm for spacecraft in low Earth orbit, though it is certainly the preferred algorithm when the magnetic field in space is no longer dominated by the main field of the Earth.

## The TWOSTEP Method

The TWOSTEP algorithm [6] is based on the RESIDG algorithm of Gambhir [1, 2], but has been considerably improved and extended.

Given a sequence of variables $X_k$, $k = 1, \ldots, N$, the centering method defines *center* values of the sequence according to the prescription

$$\bar{X} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} X_k \tag{67}$$

where

$$\frac{1}{\bar{\sigma}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{68}$$

and *centered* values according to

$$\tilde{X}_k \equiv X_k - \bar{X}, \qquad k = 1, \ldots, N \tag{69}$$

On the basis of these center and centered quantities, it can be shown [6] that the quartic cost function of equation (6) can be written exactly as

$$J(\mathbf{b}) = \tilde{J}(\mathbf{b}) + \bar{J}(\mathbf{b}) \tag{70}$$

where

$$\tilde{J}(\mathbf{b}) = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}_k - 2\tilde{\mathbf{B}}_k \cdot \mathbf{b} - \tilde{\mu}_k)^2 + \text{terms independent of } \mathbf{b} \tag{71a}$$

and

$$\bar{J}(\mathbf{b}) = \frac{1}{2} \frac{1}{\bar{\sigma}^2} (\bar{z} - 2\bar{\mathbf{B}} \cdot \mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2 + \text{terms independent of } \mathbf{b} \tag{71b}$$

Equations (71) give the negative-log-likelihood functions of the centered and center data respectfully with respect to the magnetometer bias vector. The TWOSTEP algorithm consists first of finding the value of $\mathbf{b}$ which minimizes $\tilde{J}(\mathbf{b})$. Because

$\tilde{J}(\mathbf{b})$ is a quadratic function of its argument, the estimate of the bias from this first step, the *centered estimate* $\tilde{\mathbf{b}}^*$, is unambiguous and has the solution

$$\tilde{\mathbf{b}}^* = \tilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}_k - \tilde{\mu}_k) 2\tilde{\mathbf{B}}_k \tag{72a}$$

and the estimate error covariance of the centered estimate is given by

$$\tilde{P}_{bb} = \tilde{F}_{bb}^{-1} = \left[ \sum_{k=1}^{N} \frac{1}{\sigma_k^2} 4\tilde{\mathbf{B}}_k \tilde{\mathbf{B}}_k^{\mathrm{T}} \right]^{-1} \tag{72b}$$

The centered estimate $\tilde{\mathbf{b}}^*$ and the center measurement $\bar{z}$ have been shown to be independent sufficient statistics for the magnetometer bias vector under the assumption that the measurement noise on the magnetometer readings is white and Gaussian.

The second step consists of using the centered estimate as an initial value and computing the corrected estimate by applying the Gauss-Newton method to the full negative-log-likelihood function, which we write equivalently as

$$J(\mathbf{b}) = \frac{1}{2}(\mathbf{b} - \tilde{\mathbf{b}}^*)^{\mathrm{T}} \tilde{P}_{bb}^{-1} (\mathbf{b} - \tilde{\mathbf{b}}^*) + \frac{1}{2\bar{\sigma}^2} (\bar{z} - 2\bar{\mathbf{B}} \cdot \mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2$$
$$+ \text{ terms independent of } \mathbf{b} \tag{73}$$

for which the minimization is straightforward and rapid. The centered estimate by itself, the first step, provides a consistent estimate of the magnetometer bias, which will provide adequate accuracy in most cases. Note that the centered estimate also requires some iteration to recompute the weights, as discussed in [6]. Because the TWOSTEP method has been derived rigorously, it admits various statistical tests as figures of merit and special techniques for treating cases of poor observability.

## Intermezzo

We have presented six methods above for the estimation of magnetometer biases without information about the attitude. Three of these are single-step methods:

- Davenport's approximation
- Acuña's algorithm
- The centered estimate of TWOSTEP.

The other three methods are infinite processes:

- Naive quartic scoring
- The fixed-point method
- TWOSTEP (the complete algorithm).

One may also consider an iterative process initiated with Davenport's approximation, a process studied by Davenport [4], or with Acuña's algorithm [5]. These iterative processes will certainty not be as efficient as TWOSTEP because the centered approximation is a much better approximation of the global minimum. We consider these possible algorithms also but present the results in less detail.

Davenport's approximation and the centered estimate, originally proposed by Gambhir but considerably refined here, are very different in character. The centered estimate achieves its simplicity by discarding one linear combination of the measurements. No approximations are made, however, except for the initial universal assumption that the measurement noise on the magnetometer readings is white and Gaussian. For this reason, the centered estimate is consistent. If the amount of data becomes infinite or if the measurement noise becomes vanishingly small, then with probability 1.0 the centered estimate will yield the correct value of the bias. Davenport's approximation, however, achieves a solution by changing the dependence of the measurements on the magnetometer bias. Thus, it cannot yield the correct value of the bias except in the special case that the bias vanishes identically.

Davenport's approximate method has other disadvantages. Because the centered estimate yields the maximum likelihood estimate of the bias given the reduced subset of the data, this estimate may be used as a sufficient statistic, to be combined with the remaining datum. No information is required from the centered data other than the centered estimate, the associated estimate error covariance matrix, and the cross-covariance of the centered estimate with the remaining datum (which we have shown to vanish identically). There is no need to recompute any quantity with the centered data. Davenport's approximation, because it is simply an *ad hoc* functional approximation, does not provide a maximum-likelihood estimate for any subset of the data. For our present purposes, therefore, it provides at best an initial value for naive quartic scoring.

Acuña's algorithm is also not consistent and in low Earth orbit provides no more than a good starting point for naive quartic scoring. However, one advantage it has over Davenport's approximation is that the approximation is not arbitrary. The central assumption of Acuña is that the change in the measured magnetic field due to the rotation of a spinning spacecraft far exceeds that due to the orbital motion of the spacecraft. This assumption is largely justified. In cases where one does not possess a field model, for example, when one is very far from the Earth, Acuña's algorithm is the only available means for estimating the magnetometer bias.

Of the three infinite processes, TWOSTEP is the most appealing. It is the only method which has a consistent estimate as a starting value, and therefore, asymptotically will have an initial value closest to the global maximum of the likelihood. If we consider naive quartic scoring initialized by Davenport's approximation or Acuña's algorithm as fourth and fifth infinite processes, then it is surely the best method of five.

An additional advantageous property of TWOSTEP is that the center correction contains little information when the data is isotropic. This further increases the likelihood that TWOSTEP will converge to the global maximum of the likelihood. It is likely that in many cases, the center correction will be insignificant.

## Numerical Examples

The algorithms treated in this work have been examined for two typical scenarios: a spacecraft spinning at 15 rpm and an inertially stabilized spacecraft. The spacecraft orbit has been chosen to be circular with an altitude of 560 km and an inclination of 38 deg. The geomagnetic field in our studies has been simulated using the International Geomagnetic Reference Field (IGRF (1985)) [10–13], which has been extrapolated to 1994. These are the simulation parameters that we used in reference [6].

For purposes of simulation we have assumed as in reference [6] an effective white Gaussian magnetometer measurement error with isotropic error distribution and a standard deviation per axis of 2.0 mG, corresponding to an angular error of approximately 0.5 deg at the equator. We have assumed also that the $x$-axis of the magnetometer is parallel to the spacecraft spin axis, which always points toward the

**TABLE 1. Comparison of Single-Step Algorithms for Noise-Free Data**

| Model Bias (mG) | Centered Estimate | Davenport | Acuña |
|---|---|---|---|
| [ 10., 20., 30.] | [ 10., 20., 30.] | [ 10., 20., 30.] | [ 20., 20., 30.] |
| [ 30., 60., 90.] | [ 30., 60., 90.] | [ 30., 50., 90.] | [ 40., 60., 90.] |
| [ 60., 120., 180.] | [ 60., 120., 180.] | [ 60., 120., 180.] | [ 70., 120., 180.] |
| [100., 200., 300.] | [100., 200., 300.] | [ 99., 196., 263.] | [110., 200., 300.] |
| [200., 400., 600.] | [200., 400., 600.] | [198., 397., 561.] | [210., 400., 600.] |

Sun. The Sun direction makes an angle of approximately 40 degrees with the orbit plane. Thus, for a spinning spacecraft we expect the estimation errors for the magnetometer bias to be largest for the $x$-component. The magnetometer data were sampled every eight seconds. All entries in the tables for the estimated magnetometer bias and the associated standard deviations are in mG.

We examine first the three single-step methods, that is, the centered estimate, Davenport's approximation, and Acuña's algorithm. To highlight the inconsistency of the last two methods, we examine the behavior of both methods for noise-free data. The results for two orbits of data for the spinning spacecraft are shown in Table 1. The equivalent results for noisy data are presented in Table 2. (Note that confidence intervals are provided only when the algorithm itself can provide them. Heuristic algorithms, like those of Davenport and Acuña, do not provide a ready means for computing the confidence interval.)

**TABLE 2. Comparison of Single-Step Algorithms for Noisy Data**

| Model Bias (mG) | Centered Estimate |
|---|---|
| [ 10., 20., 30.] | [ 9.98 ± 0.09, 19.87 ± 0.10, 29.97 ± 0.10] |
| [ 30., 60., 90.] | [ 30.00 ± 0.09, 60.00 ± 0.10, 90.16 ± 0.10] |
| [ 60., 120., 180.] | [ 60.02 ± 0.09, 119.98 ± 0.10, 180.05 ± 0.10] |
| [100., 200., 300.] | [100.00 ± 0.09, 199.82 ± 0.10, 300.01 ± 0.10] |
| [200., 400., 600.] | [200.04 ± 0.09, 400.05 ± 0.10, 600.12 ± 0.10] |
| **Model Bias (mG)** | **Davenport's Approximation** |
| [ 10., 20., 30.] | [ 9.95, 19.89, 29.74] |
| [ 30., 60., 90.] | [ 29.99, 60.04, 90.17] |
| [ 60., 120., 180.] | [ 60.00, 119.98, 180.05] |
| [100., 200., 300.] | [ 83.05, 170.57, 253.67] |
| [200., 400., 600.] | [190.04, 380.19, 570.39] |
| **Model Bias (mG)** | **Acuña's Approximation** |
| [ 10., 20., 30.] | [ 8.92, 20.07, 29.57] |
| [ 30., 60., 90.] | [115.89, 59.84, 90.56] |
| [ 60., 120., 180.] | [143.65, 120.50, 180.10] |
| [100., 200., 300.] | [190.15, 199.69, 300.54] |
| [200., 400., 600.] | [303.90, 400.10, 600.21] |

For small values of the magnetometer bias, Davenport's approximation yields quite acceptable results. For values of the magnetometer bias comparable to or greater than the magnitude of the ambient magnetic field, the errors in Davenport's approximation become unacceptably large. These statements hold both for the noise-free and the noisy data. This behavior is to be expected since for small values of the bias the contribution of the $|\mathbf{b}|^2$ term in the measurement model will be smallest. Note that within numerical roundoff error the centered estimate yields the exact value of the bias for noise-free data, as it must. For small values of the magnetometer bias vector, Davenport's approximation shows small errors which do not appear in the first few significant digits, and hence are lost in Table 1. To within numerical roundoff error, the centered estimate yields the exact result.

Tables 1 and 2 show also that Acuña's algorithm has a very large error owing to the small variation of the magnetic field as measured along the magnetometer $x$-axis. Because of this, the matrix which is inverted in equation (57) is nearly singular and the modeling errors of the estimator are greatly magnified. Performance would have improved dramatically if the spacecraft attitude motion had not been confined to rotation about one axis. This is the procedure recommended by Acuña [5].

We can gain a greater appreciation of the behavior of these three algorithms if we examine the normalized errors defined by

$$\tilde{\eta} \equiv \frac{1}{\sqrt{6}}[(\mathbf{b}^{\text{true}} - \tilde{\mathbf{b}}^*)^{\text{T}}\tilde{P}_{bb}^{-1}(\mathbf{b}^{\text{true}} - \tilde{\mathbf{b}}^*) - 3] \tag{74}$$

which should have mean zero and standard deviation unity. We can define likewise

$$\eta_D \equiv \frac{1}{\sqrt{6}}[(\mathbf{b}^{\text{true}} - \mathbf{b}_{\text{D}}^*)^{\text{T}}\tilde{P}_{bb}^{-1}(\mathbf{b}^{\text{true}} - \mathbf{b}_{\text{D}}^*) - 3] \tag{75a}$$

$$\eta_A \equiv \frac{1}{\sqrt{6}}[(\mathbf{b}^{\text{true}} - \mathbf{b}_{\text{A}}^*)^{\text{T}}\tilde{P}_{bb}^{-1}(\mathbf{b}^{\text{true}} - \mathbf{b}_{\text{A}}^*) - 3] \tag{75b}$$

These functions are normalized to the covariance matrix of the random estimation error for the centered approximation in order that these three figures of merit can have a common scale. Davenport's approximation and the Acuña algorithm do not provide a ready measure of their estimation errors. A comparison of the quantities is given in Table 3. Note that the samples of $\tilde{\eta}$ behave in fact like a random variable with mean zero and variance one, which is not always the case for $\eta_D$ and $\eta_A$. For small to moderate values of the magnetometer bias vector, Davenport's approxi-

**TABLE 3. Comparison of Normalized Errors for the Single-Step Methods Applied to Noisy Data**

| Model Bias (mG) | $\tilde{\eta}$ | $\eta_D$ | $\eta_A$ |
|---|---|---|---|
| [ 10., 20., 30.] | −0.50 | −0.67 | $3.22 \times 10^5$ |
| [ 30., 60., 90.] | −0.70 | −0.01 | $3.78 \times 10^5$ |
| [ 60., 120., 180.] | −0.54 | −1.09 | $3.59 \times 10^5$ |
| [100., 200., 300.] | 0.09 | $1.43 \times 10^5$ | $4.17 \times 10^5$ |
| [200., 400., 600.] | −0.67 | $5.91 \times 10^4$ | $5.54 \times 10^4$ |

**TABLE 4. Comparison of Single-Step Algorithms for Noisy Data and Two-Axis Slews**

| Model Bias (mG) | Centered Estimate |
|---|---|
| [ 10., 20., 30.] | [ 10.00 ± 0.10, 19.91 ± 0.11, 30.16 ± 0.11] |
| [ 30., 60., 90.] | [ 29.91 ± 0.10, 59.99 ± 0.11, 89.99 ± 0.11] |
| [ 60., 120., 180.] | [ 59.95 ± 0.10, 119.81 ± 0.11, 179.58 ± 0.11] |
| [100., 200., 300.] | [ 99.83 ± 0.10, 199.76 ± 0.11, 299.58 ± 0.11] |
| [200., 400., 600.] | [199.96 ± 0.10, 400.03 ± 0.11, 600.12 ± 0.11] |
| **Model Bias (mG)** | **Davenport's Approximation** |
| [ 10., 20., 30.] | [ 10.02, 19.89, 30.16] |
| [ 30., 60., 90.] | [ 29.87, 60.03, 90.03] |
| [ 60., 120., 180.] | [ 59.81, 119.80, 179.52] |
| [100., 200., 300.] | [ 83.86, 179.35, 250.32] |
| [200., 400., 600.] | [169.66, 347.30, 495.07] |
| **Model Bias (mG)** | **Acuña's Approximation** |
| [ 10., 20., 30.] | [ 10.65, 19.94, 30.51] |
| [ 30., 60., 90.] | [ 29.39, 60.27, 90.01] |
| [ 60., 120., 180.] | [ 60.27, 119.72, 179.53] |
| [100., 200., 300.] | [ 99.75, 200.07, 300.07] |
| [200., 400., 600.] | [199.39, 400.27, 600.01] |

mation of the magnetometer bias performs well. For values of the magnetometer bias vector comparable to the ambient magnetic field, however, we see from Table 3 that the errors in Davenport's approximation are on the order of 50 $\tilde{\sigma}$ and are therefore useful only as an initial value for minimization of the full quartic cost function. Acuña's algorithm should not be criticized too harshly since it has been applied to a case where it cannot be expected to give reasonable results. Nonetheless, the results about two axes are quite good, leading us to believe that it would provide a useful result if all three components of the bias were made observable.

To obtain the true measure of the performance of Acuña's algorithm compared to the others, we have considered the case of a spacecraft which is alternately spinning about the magnetometer $x$-axis at 1 rpm for one orbit, and then for one orbit about the magnetometer $y$-axis at one 1 rpm. The data has been sampled at a rate of one sample every ten seconds. The results are shown in Tables 4 and 5. We see

**TABLE 5. Comparison of Normalized Errors for the Single-Step Methods Applied to Noisy Data and Two-Axis Slews**

| Model Bias (mG) | $\tilde{\eta}$ | $\eta_D$ | $\eta_A$ |
|---|---|---|---|
| [ 10., 20., 30.] | 0.07 | 0.10 | 24.00 |
| [ 30., 60., 90.] | −0.85 | −0.59 | 15.59 |
| [ 60., 120., 180.] | 0.16 | 9.66 | 11.47 |
| [100., 200., 300.] | 1.57 | $1.09 \times 10^5$ | 1.45 |
| [200., 400., 600.] | −0.84 | $5.09 \times 10^5$ | 15.59 |

that in these cases Acuña's algorithm performs quite well, though not as well as the centered algorithm. Note that Davenport's approximation is superior to Acuña's algorithm for small values of the bias vector, where the quadratic term is negligible and has little effect on Davenport's method, while the model errors are relatively large in Acuña's algorithm. When the bias vector is large compared to the ambient field, however, then the reverse is seen to be true, since the quadratic terms now dominate and Davenport's approximation is no longer justified, while the large bias swamps the model errors in Acuña's method, which yields very useful results. Note that if errors in the field model were large compared with instrument error, the situation which prevails when one studies the magnetic fields of other celestial bodies than our own, then Acuña's method would be superior to both the centered algorithm and TWOSTEP.

For the same spinning spacecraft that was the subject of Tables 2 and 3, we have examined the convergence of the iterative algorithms. The algorithms tested are: naive quartic scoring; the fixed-point method; and TWOSTEP. The first two methods are initialized at $\mathbf{b} = \mathbf{0}$. This is emphasized in the tables by showing the value (0, 0, 0) as the zeroth iteration for the bias. The centered estimate is a finite procedure and does not require an initial value. Therefore, no zeroth iteration is indicated. However, if the $\sigma_k^2$ must be calculated from an assumed value of the bias, as we have chosen to do in this study, it is wise to recompute these variances at $\tilde{\mathbf{b}}^*$ and then recompute $\tilde{\mathbf{b}}^*$ for consistency. This was the procedure followed in the present study. In the naive quartic scoring algorithm, the sigma's have been recomputed at each iteration. In the table entries for TWOSTEP, the first entry (Iteration 1) is the centered estimate $\tilde{\mathbf{b}}^*$, followed by iterations of the center correction. The results for the spinning spacecraft are shown in Table 6. For all three algorithms tested the infinite process was terminated when convergence had occurred to two decimal places. Table 7 gives the results for an inertially stabilized spacecraft in the same configuration described at the beginning of this section. In both Tables 6 and 7, data was sampled for 25 minutes. The true value of the magnetometer bias was chosen to be [200., 100., −200.] mG.

For the spinning spacecraft, as shown in Table 6, all algorithms performed well. However, naive quartic scoring required five iterations in order to attain convergence to two decimal places, while the fixed-point method required 30 iterations to reach this same degree of convergence. The TWOSTEP algorithm required only a single iteration of the center correction to attain this accuracy. The improvement over the centering approximation, however, was only one part in 10,000, so that the centering approximation by itself was surely adequate.

For the inertially stabilized spacecraft on the other hand, the results are much different. Naive quartic scoring has converged to a false minimum, while the fixed-point method has diverged. This divergence of the fixed-point method was even observed for the spinning spacecraft for a different value of the seed used in the random-number generator for the measurement noise. In contrast to the other algorithms, the TWOSTEP algorithm has again converged to two decimal places in a single iteration of the center correction with the centering approximation again providing adequate accuracy.

The values of the magnetometer bias and the random-number seed in the numerical example of Table 7 were not accidental. It was chosen to demonstrate the dangers of naive quartic scoring and the fixed-point method. It is well to ask, then, what is the frequency of these diseased occurrences? Of 100 different simulations

**TABLE 6. Comparison of Iterative Methods of Magnetometer Bias Determination for a Spinning Spacecraft with $\mathbf{b}^{true}$ = [200., 100., −200.] mG**

| Iteration | Bias Estimate (mG) |
|---|---|
| Naive Quartic Scoring | |
| 0 | [ 0.0, 0.0, 0.0 ] |
| 1 | [ 24.28, 67.61, −153.34] |
| 2 | [ 95.88, 102.08, −201.97] |
| 3 | [ 195.32, 100.31, −200.92] |
| 4 | [ 200.48, 99.92, −220.52] |
| 5 | [ 200.46, 99.93, −200.51] |
| | ±[0.28, 0.24, 0.24] |
| Fixed-Point Method | |
| 0 | [ 0.0, 0.0, 0.0 ] |
| 1 | [−41.62, −49.57, 108.28] |
| 2 | [ 67.14, 155.23, −323.16] |
| 3 | [ 105.37, 146.01, −300.63] |
| 4 | [ 142.43, 135.51, −276.27] |
| 5 | [ 170.51, 125.55, −254.08] |
| | . . . |
| 10 | [ 204.14, 103.66, −208.10] |
| | . . . |
| 20 | [ 200.85, 100.09, −200.86] |
| | . . . |
| 30 | [ 200.46, 99.93, −200.51] |
| | . . . |
| 44 | [ 200.46, 99.93, −200.51] |
| TWOSTEP | |
| 1 | [ 200.48, 99.93, −200.51] |
| | ±[0.28, 0.24, 0.24] |
| 2 | [ 200.46, 99.93, −200.51] |
| | ±[0.28, 0.24, 0.24] |

with the measurement noise parameters and data span as given above, and with the components of the magnetometer bias sampled uniformly on the interval $[-300, 300]$ mG, it was found that naive quartic scoring converged to the wrong value in six cases, while the fixed-point algorithm converged to the wrong value or diverged in 38 cases. For the failures of naive quartic scoring, the magnetometer bias vector was in all six cases of comparable magnitude to the ambient magnetic field. In five of these six cases, using Davenport's approximation as an initial value for naive quartic scoring led to convergence to a local minimum which was far from the true bias vector. In these five cases, however, we found that choosing the positive sign in equation (49) led to a good approximation of the true minimum, after which quartic scoring converged to the correct result. Thus, it would seem that Davenport's arguments [4] for the choice of sign in equation (49) require that the bias be small compared to the ambient field. Acuña's algorithm was tested for these same six cases and found to be a good initial value for locating the minimum of the cost function in only three cases. Davenport's and Acuña's methods were not tested for the other 94 cases. In contrast to this, in no case did the TWOSTEP algorithm require more than two iterations to the center correction in order to converge to the correct answer to within two decimal places.

**TABLE 7. Comparison of Iterative Methods of Magnetometer Bias Determination for an Inertially Stabilized Spacecraft with $\mathbf{b}^{\text{true}}$ = [200., 100., −200.] mG**

| Iteration | Bias Estimate (mG) |
|---|---|
| Naive Quartic Scoring | |
| 0 | [ 0.0, 0.0, 0.0 ] |
| 1 | [ 422.89, −482.10, 1564.72] |
| 2 | [ 401.53, −210.59, 763.69] |
| 3 | [ 320.48, −85.56, 376.61] |
| 4 | [ 281.75, −28.67, 197.82] |
| 5 | [ 266.71, −8.40, 131.05] |
| 6 | [ 263.37, −4.93, 117.73] |
| 7 | [ 263.07, −4.86, 116.97] |
| 8 | [ 263.05, −4.86, 116.95] |
| Fixed-Point Method | |
| 0 | [ 0.0, 0.0, 0.0 ] |
| 1 | [ 0.48, −8.45, −16.47] |
| 2 | [ 7.14, −16.50, −20.00] |
| 3 | [ 10.77, −19.67, −20.64] |
| 10 | [ 12.54, −33.46, −23.00] |
| | . . . |
| 20 | [ 13.85, −55.20, −27.65] |
| | . . . |
| 30 | [ 20.73, −93.34, −38.63] |
| | . . . |
| 40 | [ 1555.84, 775.42 −34.25] |
| 41 | [−5102.70, −9605.79 −3786.40] |
| 42 | $[4.36, 9.62, 6.60,] \times 10^{8}$ |
| 43 | $[-0.56, -0.97, -1.41] \times 10^{18}$ |
| 44 | $[2.11, 1.31, 7.02] \times 10^{45}$ |
| 45 | $[-1.57, 0.55, -6.41] \times 10^{128}$ |
| | . . . |
| TWOSTEP | |
| 1 | [ 200.45, 98.92, −197.04] |
| | ±[0.28, 0.46, 1.30] |
| 2 | [ 199.90, 99.86, −199.82] |
| 3 | [ 199.91, 99.85, −199.80] |
| | ±[0.13, 0.13, 0.17] |

As a final test we have compared TWOSTEP with Acuña's algorithm under optimal conditions for the latter. Two orbits of data were generated, one with the spacecraft spinning about the $x$-axis at 1 rpm and the other with the spacecraft spinning about the $y$-axis at this same rate. In this example, Acuña's algorithm will benefit from the most advantageous conditions. The great advantage of Acuña's algorithm is that it doesn't require a model for the magnetic field. Its disadvantage is that it makes approximations which introduce systematic errors. We saw in Tables 4 and 5 that the TWOSTEP algorithm was the best algorithm when the statistical model was known to be Gaussian. The question is whether the uncertainties in the statistical model assumed by TWOSTEP are more important than the systematic errors inherent in Acuña's algorithm. To test this we have allowed the noise model to have mismodeling errors and have compared the TWOSTEP algorithm with

Acuña's algorithm for the colored noise model and "realistic" noise models of reference [6]. In both cases we have assumed a moderate magnetometer bias vector (30., 60., 90.) mG. The results are shown in Table 8. The absolute error is the magnitude of the difference of the estimate from the true value.

In the above examples the performance of the two algorithms is very similar. More extensive studies have shown that for both models one algorithm is sometimes better than the other by as much as a factor of two, with either method having equal probability of being better. This confirms again the excellent performance of TWOSTEP and the very good performance of Acuña's algorithm in this situation.

## Discussion

After exhaustive testing, the new algorithm, TWOSTEP, is seen to outperform overall all of the other algorithms. It is susceptible to none of the diseases which can cause these algorithms to give erroneous results. Since it begins with a very good (and consistent) initial estimate for the bias, it is more likely to converge to the correct minimum than does naive quartic scoring [1] or the fixed-point method of Thompson *et al.* [3], which begin at $\mathbf{b} = \mathbf{0}$. Unlike the centered algorithm of RESIDG fame [2], it does not discard data ultimately and does the centering in a statistically correct way, apart from the approximation that the measurement errors on the attitude-independent derived measurement are Gaussian and uncorrelated, which is almost certainly not the case. However, TWOSTEP has been shown to perform well even in cases of severe mismodeling of the measurement noise [6].

In general, the centered estimate for the magnetometer bias has been observed to be a better approximation than ignoring the quadratic behavior of $|\mathbf{b}|^2$ as in the work of Davenport *et al.* [4], particularly for moderate to large magnetometer biases. In many cases, Davenport's approximation provided a good initial value for iteratively computing the optimal estimate of the bias from naive quartic scoring. However, as we have seen, it led to an incorrect final result in at least five percent of the cases studied.[5] In those cases, our numerical experiments seem to indicate that Davenport's prescription for choosing the sign in equation (49) is not correct. Thus, it

**TABLE 8. Comparison of TWOSTEP and Acuña's Algorithm for Mismodeled Measurement Noise and Two-Axis Slews. $\mathbf{b}^{\text{true}}$ = [30., 60., 90.] mG**

| Algorithm | Bias Estimate (mG) | Absolute Error (mG) |
|---|---|---|
| Colored Noise Model | | |
| Centering Approximation | [30.13, 60.24, 89.97] | .27 |
| TWOSTEP | [30.11, 60.27, 89.97] | .30 |
| Acuña | [29.86, 60.07, 90.02] | .16 |
| "Realistic" Noise Model | | |
| Centering Approximation | [30.30, 60.65, 90.57] | .92 |
| TWOSTEP | [30.14, 60.96, 90.57] | .92 |
| Acuña | [30.92, 60.16, 90.72] | 1.18 |

[5]It would seem that reference [4] simply did not examine a sufficient number of cases before reaching its conclusions. A similar criticism may be made about reference [3], for which the senior author of the present work bears some culpability.

seems that greater reliability would be obtained by calculating equation (49) for both values of the sign and computing the cost function at each final value to obtain the better result. We have not tested this hypothesis extensively, however, and cannot guarantee that one of the two signs of equation (49) will yield a good approximation of the magnetometer bias vector when this quantity is large.

Similar comments can be made for Acuña's algorithm, whose performance suffers noticeably when two orthogonal calibration slews cannot be accommodated in the mission profile. As we have seen, Acuña's algorithm did not provide a useful initial value for finding the true minimum of the cost function in at least three percent of such cases. However, it should be borne in mind that Acuña's algorithm is the only algorithm which functions not only in the absence of knowledge of the attitude but also in the total absence of *a priori* knowledge even of the magnetic field. Under these very ungenerous circumstances, for which it was designed, it performs extraordinarily well, provided that there is sufficient independent variation in the magnetic field measurements along all three axes of the magnetometer. The results in Table 8 show, in fact, that Acuña's algorithm behaved equally to or in individual cases sometimes even better than TWOSTEP for two-axis slews and mismodeled measurement noise. However, the performance of TWOSTEP on average performed equally well in this case, and it has the advantage of performing very well in less opportune situations where Acuña's algorithm is at a serious disadvantage.

A characteristic of the centered estimate, the first step in TWOSTEP, is that it is often good enough. The Fisher information associated with $\tilde{\mathbf{b}}$ genuinely characterizes the quality of the centered estimate. A comparison of this and the Fisher information associated with the center term can be used to decide whether it is worthwhile to carry out the center correction. This was demonstrated explicitly for the case of the SAC-B spacecraft in the previous work [6]. We see that a careful statistical treatment of the magnetometer bias gives us many more insights into the behavior of the estimator.

In summary: we have tested the currently known algorithms for attitude-independent magnetometer bias determination for a number of scenarios. We have found the TWOSTEP algorithm to perform well in all cases, while the other iterative algorithms are susceptible to consistency or divergence problems. Among the single-step algorithms, we have found that the centering approximation, the first step in TWOSTEP, to be by far the most capable performer overall.

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