Advances in the Astronautical Sciences, Vol. 108, December 2001, pp. 397–408

AAS-01-126

MAN, LIKE THESE ATTITUDES ARE TOTALLY RANDOM! II. OTHER REPRESENTATIONS*

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Knowing the uniform probability density function for the attitude quaternion we derive expressions for the other attitude representations. We show that given our knowledge of the quaternion uniform probability density function there are three different methods by which the uniform probability density function of these other representations may be generated. Numerical methods are presented for generating uniformly distributed random samples of each of the attitude representations as a function of random samples having a uniform probability density function on the interval $[0, 1]$.

GENERATING UNIFORM PROBABILITY DENSITY FUNCTIONS

We saw in the previous paper that the uniform probability density function of any attitude representation can be derived from

$$p_\xi(\xi') = p_\xi(\iota) \left| \frac{\partial \boldsymbol{\alpha}}{\partial(\boldsymbol{\alpha} \circ \xi')} \right|_{\alpha=\iota} = p_\xi(\iota) \left| \frac{\partial \boldsymbol{\alpha}}{\partial(\xi' \circ \boldsymbol{\alpha})} \right|_{\alpha=\iota} . \tag{I-34}$$

Here the notation (I-34) denotes Eq. (34) of Part I (Ref. 1). We call this method the *group method*, because it relies only on the knowledge of the particular form of the group operation. This equation is not always of practical use. For one thing, it may happen that $p_\xi(\iota)$ vanishes, in which case the Jacobian determinant is infinite at $\alpha = \iota$, and the expression is undefined there. Also, if the composition rule for the representation, as for the Euler angles, is very complicated, then the Jacobian determinant will be extremely difficult to evaluate. Hence, we must find another way for most representations.

One method is to use our knowledge of the uniform pdf of the vector components of the quaternion, namely

$$p_\eta(\boldsymbol{\eta}') = \frac{1}{\pi^2 \sqrt{1 - |\boldsymbol{\eta}'|^2}} . \tag{I-43}$$

*Presented at the 11th AAS/AIAA Space Flight Mechanics Meeting, Santa Barbara, California, February 11–14, 2001.

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It follows then from Eq. (I-26) and the transitive properties of the Jacobian determinant that for any three-dimensional attitude representation $\boldsymbol{\xi}$ we can write the uniform pdf as

$$p_{\xi}(\boldsymbol{\xi}') = p_{\eta}(\boldsymbol{\eta}'(\boldsymbol{\xi}')) \left| \frac{\partial \boldsymbol{\eta}(\boldsymbol{\xi}')}{\partial \boldsymbol{\xi}'} \right| . \tag{1}$$

In general, it is much easier to evaluate the Jacobian determinant in Eq. (1) than in Eq. (I-43). We call this method the *transformation method*.

There is also a third way, which uses the result for the uniform pdf for the quaternion with the four components treated as independent, as given by Eq. (I-65). The advantage here is that the pdf of the 4-component quaternion is so simple. We derive this expression now.

We examine a change of variables from (q_1, q_2, q_3, q_4) to (q, ξ_1, ξ_2, ξ_3), where q is the magnitude of the quaternion and (ξ_1, ξ_2, ξ_3) are the three components of the three-dimensional representation of the attitude. Generally, we know the Euler-Rodrigues symmetric parameters as a function of the three-dimensional representation $\boldsymbol{\xi}$. That said, the functional form of the change of variables can obviously be written as

$$\bar{q}_{\pm} = \pm q\, \bar{\eta}(\boldsymbol{\xi}) , \tag{2}$$

and we must consider both signs explicitly, because $\bar{\eta}(\boldsymbol{\xi})$ is, supposedly, a single-valued function of $\boldsymbol{\xi}$. Clearly, the pdf of q and $\boldsymbol{\xi}$ can be written as

$$p_{q,\xi}(q', \boldsymbol{\xi}') = p_{\xi}(\boldsymbol{\xi}')\, \delta(q' - 1) . \tag{3}$$

From Eq. (1) we must also have that

$$p_{q,\xi}(q', \boldsymbol{\xi}') = \sum_{\pm} p_{\bar{q}}(\bar{q}_{\pm}(q', \boldsymbol{\xi}')) \left| \frac{\partial \bar{q}_{\pm}(q', \boldsymbol{\xi}')}{\partial (q', \boldsymbol{\xi}')} \right| . \tag{4}$$

Substituting the uniform pdf for the 4-component quaternion yields

$$\begin{aligned} p_{\xi}(\boldsymbol{\xi}')\, \delta(q' - 1) &= \sum_{\pm} \frac{1}{\pi^2} \delta(\bar{q}_{\pm}'^T \bar{q}_{\pm}' - 1) \left| \frac{\partial \bar{q}_{\pm}(q', \boldsymbol{\xi}')}{\partial (q', \boldsymbol{\xi}')} \right| \\ &= \sum_{\pm} \frac{1}{2\pi^2} \delta(q_{\pm}' - 1) \left| \frac{\partial \bar{q}_{\pm}(q', \boldsymbol{\xi}')}{\partial (q', \boldsymbol{\xi}')} \right| . \end{aligned} \tag{5}$$

Integrating now over q' from 0 to ∞ leads to the very simple formula

$$p_{\xi}(\boldsymbol{\xi}') = \frac{1}{\pi^2} \left| \frac{\partial \bar{q}(q', \boldsymbol{\xi}')}{\partial (q', \boldsymbol{\xi}')} \right|_{q'=1} . \tag{6}$$

Equation (6) is certainly the "royal road" to calculating uniform pdf's for three-dimensional attitude representations. Note that the Jacobian determinant has the same value for both $\bar{q}_+$ and $\bar{q}_-$, so we need write only one term. Equation (6) provides a ready way to calculate the uniform pdf for any three-dimensional attitude representation. We call this the *quaternion method* for deriving the uniform pdf of a three-dimensional attitude representation.

We now commence the systematic derivation of the uniform pdf for the common three-dimensional attitude representations. We will use principally the first two methods, especially the second, since it is often easier than calculating the determinant of a 4×4 matrix. The reader may use Eq. (6) to check our results.

The Rodrigues Parameters

The 3×1 matrix of Rodrigues parameters (the Rodrigues vector) is related to the quaternion and the axis and angle of rotation by[2]

$$\boldsymbol{\rho} = \frac{\boldsymbol{\eta}}{\eta_4} = \tan(\theta/2)\,\hat{\mathbf{n}}\,, \tag{7}$$

The composition rule is[2]

$$\boldsymbol{\alpha} \circ \boldsymbol{\rho}' = \frac{\boldsymbol{\alpha} + \boldsymbol{\rho}' - \boldsymbol{\alpha} \times \boldsymbol{\rho}'}{1 - \boldsymbol{\alpha} \cdot \boldsymbol{\rho}'}\,. \tag{8}$$

whence

$$\left| \frac{\partial(\boldsymbol{\alpha} \circ \boldsymbol{\rho}')}{\partial \boldsymbol{\alpha}} \right|_{\boldsymbol{\alpha}=\mathbf{0}} = \left| \det\Big(I_{3\times 3} - [[\,\boldsymbol{\rho}'\,]] + \boldsymbol{\rho}'\,\boldsymbol{\rho}'^{T} \Big) \right| = \left(1 + |\boldsymbol{\rho}'|^2\right)^2, \tag{9}$$

where[2]

$$[[\,\mathbf{v}\,]] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}. \tag{10}$$

It follows again from Eq. (I-34), after computing $p_{\boldsymbol{\rho}}(\mathbf{0})$ from Eq. (I-38), that

$$p_{\boldsymbol{\rho}}(\boldsymbol{\rho}') = \frac{1}{\pi^2 (1 + |\boldsymbol{\rho}'|^2)^2}\,. \tag{11}$$

The calculation of the determinant in Eq. (9) is tedious. A simpler method, therefore, is to apply the implicit function theorem directly to Eq. (I-43), calculating the Jacobian determinant $|\partial(\eta_1, \eta_2, \eta_3)/\partial(\rho_1, \rho_2, \rho_3)|$ instead of the result of Eq. (9), and then calculating the pdf for the Rodrigues parameters from the pdf already calculated for the vector components of the quaternion using the relations

$$p_{\rho}(\boldsymbol{\rho}') = p_{\boldsymbol{\eta}}\big(\boldsymbol{\eta}(\boldsymbol{\rho}')\big) \left| \frac{\partial(\eta_1', \eta_2', \eta_3')}{\partial(\rho_1', \rho_2', \rho_3')} \right|, \qquad \text{and} \qquad \boldsymbol{\eta}(\boldsymbol{\rho}) = \frac{\boldsymbol{\rho}}{\sqrt{1 + |\boldsymbol{\rho}|^2}}\,. \tag{12ab}$$

This calculation would be equally tedious except that the vector components of the unit quaternion and the Rodrigues parameters are both proportional to the axis of rotation $\hat{\mathbf{n}}$. Hence, the only interdependence of these two representations that should be of interest is that of $|\boldsymbol{\eta}|$, the magnitude of $\boldsymbol{\eta}$, on $|\boldsymbol{\rho}|$, the magnitude of $\boldsymbol{\rho}$. If we write in the usual short-hand

$$p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')\, d^3\boldsymbol{\eta}' = p_{\boldsymbol{\rho}}(\boldsymbol{\rho}')\, d^3\boldsymbol{\rho}' \tag{13}$$

or

$$p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')\, |\boldsymbol{\eta}'|^2\, d|\boldsymbol{\eta}'|\, d^2\Omega_{\hat{\mathbf{n}}'} = p_{\boldsymbol{\rho}}(\boldsymbol{\rho}')\, |\boldsymbol{\rho}'|^2\, d|\boldsymbol{\rho}'|\, d^2\Omega_{\hat{\mathbf{n}}'}\,. \tag{14}$$

then it is evident that we must have

$$\left| \frac{\partial \boldsymbol{\eta}'}{\partial \boldsymbol{\rho}'} \right| = \frac{|\boldsymbol{\eta}'|^2}{|\boldsymbol{\rho}'|^2} \frac{\partial |\boldsymbol{\eta}'|}{\partial |\boldsymbol{\rho}'|} = \frac{1}{(1 + |\boldsymbol{\rho}'|^2)^{5/2}}\,. \tag{15}$$

The same result can be derived using the transitivity of the Jacobian determinant and the sequence of transformations $(\eta_1, \eta_2, \eta_3) \to (\eta, \theta, \phi) \to (\rho, \theta, \phi) \to (\rho_1, \rho_2, \rho_3)$, where the second and third sets of coordinates are the spherical coordinate representations of $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$, respectively. Note that $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ are parallel.

From Eq. (15) and

$$p_{\boldsymbol{\eta}}\big(\boldsymbol{\eta}(\boldsymbol{\rho}')\big) = \frac{\sqrt{1+|\boldsymbol{\rho}'|^2}}{\pi^2} \tag{16}$$

we arrive again at Eq. (11). It is left as an exercise for the reader to show that Eq. (6) leads to this same result.

The Modified Rodrigues Parameters

For the modified Rodrigues parameters[2] we have (for the positive form)

$$\mathbf{p} = \frac{\boldsymbol{\eta}}{1+\eta_4} = \tan(\theta/4)\,\hat{\mathbf{n}}\,, \qquad \text{and} \qquad \boldsymbol{\rho}(\mathbf{p}) = 2\,\mathbf{p}/(1-|\mathbf{p}|^2)\,. \tag{17ab}$$

The pdf can be most easily calculated from the pdf of the Rodrigues vector, and again the Jacobian determinant can be obtained from the calculation of a single radial derivative with the results

$$p_{\boldsymbol{\rho}}\big(\boldsymbol{\rho}(\mathbf{p}')\big) = \frac{(1-|\mathbf{p}'|^2)^4}{\pi^4(1+|\mathbf{p}'|^2)^4}\,, \qquad \left|\frac{\partial\boldsymbol{\rho}'}{\partial\mathbf{p}'}\right| = \frac{8\,(1+|\mathbf{p}'|^2)}{(1-|\mathbf{p}'|^2)^4}\,. \tag{18ab}$$

Hence,

$$p_{\mathbf{p}}(\mathbf{p}') = \frac{8}{\pi^2(1+|\mathbf{p}'|^2)^3}\,, \tag{19}$$

where we restrict $|\mathbf{p}'|$ to the region $|\mathbf{p}'| \le 1$. One could have extended the domain of $\mathbf{p}$ to all space, but we chose to avoid infinite values of the representation, nor do we wish to have a representation which is non-unique.

For the negative form of the vector

$$\mathbf{m} = \frac{\boldsymbol{\eta}}{1-\eta_4} = \cot(\theta/2)\,\hat{\mathbf{n}} \tag{20}$$

we find equally easily

$$p_{\mathbf{m}}(\mathbf{m}') = \frac{8}{\pi^2(1+|\mathbf{m}'|^2)^3}\,, \tag{21}$$

but we restrict $\mathbf{m}$ to the region $|\mathbf{m}'| \ge 1$.

The Rotation Vector

It is easiest to compute the pdf for the rotation vector[2] from that of the vector components of the quaternion. We note that

$$\boldsymbol{\theta} = \theta\,\hat{\mathbf{n}}\,, \qquad \text{and} \qquad \boldsymbol{\eta}(\boldsymbol{\theta}) = \sin(|\boldsymbol{\theta}|/2)\,\hat{\mathbf{n}}\,, \tag{22ab}$$

from which it follows that

$$p_{\boldsymbol{\eta}}\big(\boldsymbol{\eta}(\boldsymbol{\theta}')\big) = \frac{1}{\pi^2\,\cos(|\boldsymbol{\theta}'|/2)}\,, \quad \text{and} \quad \left|\frac{\partial\boldsymbol{\eta}'}{\partial\boldsymbol{\theta}'}\right| = \frac{\sin^2(|\boldsymbol{\theta}'|/2)\,\cos(|\boldsymbol{\theta}'|/2)}{2\,|\boldsymbol{\theta}'|^2}\,, \tag{22cd}$$

and finally

$$p_{\boldsymbol{\theta}}(\boldsymbol{\theta}') = \frac{\sin^2(|\boldsymbol{\theta}'|/2)}{2\,\pi^2\,|\boldsymbol{\theta}'|^2} = \frac{1-\cos(|\boldsymbol{\theta}'|)}{4\,\pi^2\,|\boldsymbol{\theta}'|^2}\,, \tag{23}$$

defined in the region $0 \le |\boldsymbol{\theta}'| \le \pi$,

The Axis and Angle of Rotation

We see immediately that the result for the rotation vector can be factored as

$$p_{\theta}(\boldsymbol{\theta}')\, d^3\boldsymbol{\theta}' = \left(\frac{1-\cos(|\boldsymbol{\theta}'|)}{\pi\,|\boldsymbol{\theta}'|^2}\right)\left(\frac{1}{4\pi}\right)|\boldsymbol{\theta}'|^2\, d\theta'\, d\Omega_{\hat{\mathbf{n}}'} = p_{\theta}(\theta')\, d\theta'\, p_{\hat{\mathbf{n}}}(\hat{\mathbf{n}}')\, d\Omega_{\hat{\mathbf{n}}'}\,. \tag{24}$$

We note that pdf does not depend on $\hat{\mathbf{n}}'$ at all, hence, we must have

$$p_{\theta}(\theta') = \frac{1-\cos(\theta')}{\pi}\,, \qquad p_{\hat{\mathbf{n}}}(\hat{\mathbf{n}}') = \frac{1}{4\pi} \tag{25ab}$$

To understand Eq. (25) we note that the unit axis vector $\hat{\mathbf{n}}$ is parameterized in terms of spherical angles as

$$\hat{\mathbf{n}} = \begin{bmatrix} \sin\alpha\,\cos\beta \\ \sin\alpha\,\sin\beta \\ \cos\alpha \end{bmatrix}, \qquad 0 \le \alpha \le \pi\,, \quad 0 \le \beta < 2\pi\,. \tag{26}$$

Here, α is the angle between the z-axis and $\hat{\mathbf{n}}$, and β is the dihedral angle about the positive z-axis from the xz-plane to the plane containing the z-axis and $\hat{\mathbf{n}}$. Thus,

$$d^3\boldsymbol{\theta}' = (\theta')^2\, d\theta'\, d^2\hat{\mathbf{n}}' \quad \text{with} \quad d^2\hat{\mathbf{n}}' \equiv \sin\alpha'\, d\alpha'\, d\beta' \equiv d^2\Omega_{\hat{\mathbf{n}}'}\,, \tag{27}$$

and the integral of $d^2\Omega_{\hat{\mathbf{n}}'}$ over all directions is 4π. The range of θ' is $0 \le \theta' \le \pi$. Otherwise, the representation will not be unique almost everywhere.

Note that for $|\boldsymbol{\theta}'| \ll 1$ that $p_{\theta}(\boldsymbol{\theta}')$ is nearly constant, as would be expected from the fact that infinitesimal angles add by approximately componentwise addition.

The Symmetric Sequence of Euler Angles

For the 3-1-3 Euler angles[2] we begin again with the pdf for the vector components of the quaternion. Defining first, in the notation of Ref. 5,

$$R_{313}(\varphi,\, \vartheta,\, \psi) \equiv R(\hat{\mathbf{3}},\, \psi)\, R(\hat{\mathbf{1}},\, \vartheta)\, R(\hat{\mathbf{3}},\, \varphi)\,, \tag{28}$$

the vector components of the quaternion are then given by

$$\boldsymbol{\eta} = \begin{bmatrix} s(\vartheta)\, c(\varphi-\psi) \\ s(\vartheta)\, s(\varphi-\psi) \\ c(\vartheta)\, s(\varphi+\psi) \end{bmatrix}, \tag{29}$$

where

$$s(x) \equiv \sin(x/2)\,, \qquad c(x) \equiv \cos(x/2)\,. \tag{30}$$

Hence, from $\eta_4' = c(\vartheta')\, c(\varphi+\psi)$ it follows that

$$p_{\boldsymbol{\eta}}\big(\boldsymbol{\eta}(\varphi',\, \vartheta',\, \psi')\big) = \frac{1}{\pi^2\,|c(\vartheta')\,\cos(\varphi'+\psi')|}\,, \tag{31}$$

and

$$\left|\frac{\partial(\boldsymbol{\eta}')}{\partial(\varphi',\, \vartheta',\, \psi')}\right| = s(\vartheta)\, c^2(\vartheta')\, |c(\varphi'+\psi')|/4\,, \tag{32}$$

whence

$$p_{313}(\varphi',\, \vartheta',\, \psi') = \frac{\sin\vartheta'}{8\,\pi^2}\,, \tag{33}$$

which is defined on the intervals $0 \le \varphi' < 2\pi$, $0 \le \vartheta' \le \pi$, $0 \le \psi' < 2\pi$. The identical result holds for the other five symmetric sets of Euler angles.

We may write this result equivalently as

$$p_{313}(\varphi', \vartheta', \psi') = p_\varphi(\varphi')\, p_\vartheta(\vartheta')\, p_\psi(\psi'), \tag{34}$$

with

$$p_\varphi(\varphi') = \frac{1}{2\pi}, \qquad p_\vartheta(\vartheta') = \frac{\sin\vartheta'}{2}, \qquad p_\psi(\psi') = \frac{1}{2\pi}. \tag{35abc}$$

Again, the same pdf will be obtained for any symmetric sequence of Euler angles.

Note that Eq. (33) would be difficult to obtain from Eq. (I-34) because the pdf vanishes at $\boldsymbol{\iota} = (0, 0, 0)$. Either L'Hôpital's rule would need to be invoked when taking the limit of Eq. (I-36), or Eq. (I-36) would need to be evaluated at a different value of α. In addition, the application of the composition rule for the 3-1-3 Euler angles[3] would not be simple.

The Asymmetric Sequence of Euler Angles

For the 3-1-2 Euler angles the calculation of the Jacobian determinant for the transformation from the vector components of the unit quaternion to the 3-1-2 Euler angles is an ordeal. Therefore, we seek a method to avoid this calculation and rely instead on the invariance principle set forth in Eq. (I-32).

Consider the direction-cosine matrix generated by a 3-1-2 sequence of Euler angles:

$$R_{312}(\varphi, \vartheta, \psi) = R(\hat{\mathbf{2}}, \psi)\, R(\hat{\mathbf{1}}, \vartheta)\, R(\hat{\mathbf{3}}, \varphi). \tag{36}$$

We note that

$$\hat{\mathbf{2}} = R(\hat{\mathbf{1}}, \pi/2)\, \hat{\mathbf{3}}, \tag{37}$$

from which it follows that[2]

$$\begin{aligned} R(\hat{\mathbf{2}}, \psi) &= R\big(R(\hat{\mathbf{1}}, \pi/2)\, \hat{\mathbf{3}}, \psi)\big) \\ &= R(\hat{\mathbf{1}}, \pi/2)\, R(\hat{\mathbf{3}}, \psi)\, R^T(\hat{\mathbf{1}}, \pi/2). \end{aligned} \tag{38}$$

Thus,

$$\begin{aligned} R_{312}(\varphi, \vartheta, \psi) &= R(\hat{\mathbf{1}}, \pi/2)\, R(\hat{\mathbf{3}}, \psi)\, R^T(\hat{\mathbf{1}}, \pi/2)\, R(\hat{\mathbf{1}}, \vartheta)\, R(\hat{\mathbf{3}}, \varphi) \\ &= R(\hat{\mathbf{1}}, \pi/2)\, R(\hat{\mathbf{3}}, \varphi) R(\hat{\mathbf{1}}, \vartheta - \pi/2)\, R(\hat{\mathbf{3}}, \varphi), \end{aligned} \tag{39}$$

or

$$R_{312}(\varphi, \vartheta, \psi) = R(\hat{\mathbf{1}}, \pi/2)\, R_{313}(\varphi, \vartheta - \pi/2, \psi). \tag{40}$$

From the invariance property of the pdf, it follows that the probability density function of $(\varphi, \vartheta, \psi)_{312}$ will be the same as the probability density function of $(\varphi, \vartheta - \pi/2, \psi)_{313}$. To see this, note that Eq. (40) can be written as

$$(\varphi', \vartheta', \psi')_{312} = (0, \pi/2, 0)_{313} \circ (\varphi', \pi/2 - \vartheta', \psi')_{313}, \tag{41}$$

from which it follows that

$$p_{(\varphi,\vartheta,\psi)_{312}}(\varphi', \vartheta', \psi') = p_{(0,\pi/2,0)_{313} \circ (\varphi,\vartheta,\psi)_{313}}(\varphi', \pi/2 - \vartheta', \psi'), \tag{42}$$

and by Eq. (I-32)

$$p_{(0,\pi/2,0)_{313} \circ (\varphi,\vartheta,\psi)_{313}}(\varphi', \pi/2 - \vartheta', \psi') = p_{(\varphi,\vartheta,\psi)_{313}}(\varphi', \pi/2 - \vartheta', \psi'). \tag{43}$$

Hence,

$$p_{312}(\varphi', \vartheta', \psi') = p_{313}(\varphi', \vartheta' - \pi/2, \psi') = \left| \frac{\sin(\vartheta' - \pi/2)}{8\pi^2} \right| = \frac{\cos\vartheta'}{8\pi^2}\,, \tag{44}$$

which is defined over the region $0 \leq \varphi' < 2\pi$, $-\pi/2 \leq \vartheta' \leq \pi/2$, $0 \leq \psi' < 2\pi$.

Similarly to Eq. (34) we can write

$$p_{312}(\varphi', \vartheta', \psi') = p_\varphi(\varphi')\,p_\vartheta(\vartheta')\,p_\psi(\psi')\,, \tag{45}$$

with

$$p_\varphi(\varphi') = \frac{1}{2\pi}\,, \qquad p_\vartheta(\vartheta') = \frac{\cos\vartheta'}{2}\,, \qquad p_\psi(\psi') = \frac{1}{2\pi}\,. \tag{46abc}$$

While the pdf for the 3-1-2 Euler angles does not vanish at $\boldsymbol{\iota} = (0, 0, 0)$, the application of the composition rule for the 3-1-2 Euler angles would be hampered by the fact that no closed-form expression is currently known for it,[3] and an intermediate representation would need to be used.

Marginal Uniform PDF's for the DCM and Other Representations

A formulation of the joint probability density function for the elements of the direction-cosine matrix (DCM) is of uncertain value, since the variables would be subject to six constraints, which would need to be implicit in the functional form of the joint pdf of all nine elements. While a complete description of this joint pdf lies outside our capacities and our interest, a partial description is still possible and enlightening. For the symmetric sets of Euler angles, we have seen that $\cos\vartheta$ is uniform on the interval $[-1, 1]$, while for the symmetric sets of Euler angles it is $\sin\vartheta$ which has a uniform distribution on this interval. If we examine the formulae for the DCM as a function of the twelve sets of Euler angles, we find that

$$\begin{aligned}
&R_{121}(1, 1) = \cos\vartheta\,, &&R_{231}(1, 2) = \sin\vartheta\,, &&R_{321}(1, 3) = -\sin\vartheta\,,\\
&R_{132}(2, 1) = -\sin\vartheta\,, &&R_{232}(2, 2) = \cos\vartheta\,, &&R_{312}(2, 3) = \sin\vartheta\,,\\
&R_{123}(3, 1) = \sin\vartheta\,, &&R_{213}(3, 2) = -\sin\vartheta\,, &&R_{313}(3, 3) = \cos\vartheta\,,
\end{aligned} \tag{47}$$

where the subscript labels the Euler-angle sequence, and the arguments label the matrix element of the DCM. Note that the diagonal elements of R are associated above with a symmetric sequence of Euler angles and the matrix element is $\cos\vartheta$, while the off-diagonal elements of R are associated with an asymmetric sequence of Euler angles, and the matrix element is $\pm\sin\vartheta$. Thus, it follows that each matrix element of a uniformly distributed random DCM, must be uniformly distributed on $[-1, 1]$. It is obvious also from any table of the DCM as a function of the twelve sets of Euler angles that the correlation between any two different elements of the DCM must vanish. In summary,

$$R_{ij} \sim U(-1, 1)\,, \qquad i, j = 1, 2, 3\,, \tag{48}$$

$$E\{R_{ij}\,R_{k\ell}\} = \frac{1}{3}\,\delta_{ik}\,\delta_{j\ell}\,, \qquad i, j, k, \ell = 1, 2, 3\,. \tag{49}$$

Where $E\{\cdot\}$ denotes the expectation. Note that although the nine elements are uncorrelated, they are certainly not independent.

The same is not true for the four elements of the quaternion. If we integrate Eq. (I-43) over any two of the components of $\boldsymbol{\eta}'$, we will find for the remaining component

$$p_{\eta_i}(\eta_i') = \frac{2}{\pi}\sqrt{1 - (\eta_i')^2}\,, \quad i = 1, 2, 3, 4\,. \tag{50}$$

That this is true also for η_4 has not been proved but will become obvious in the next section. Thus, the components of the uniformly distributed random unit quaternion are not distributed uniformly. However, the four components of the uniformly distributed random quaternion are uncorrelated

$$E\{\eta_i \, \eta_j\} = \frac{1}{4}\,\delta_{ij}\,, \qquad i,\ j = i,\ 2,\ 3,\ 4\,. \tag{51}$$

The four components of the uniformly distributed random quaternion are uncorrelated but not independent.

The three *Cartesian* components of the uniformly distributed random Rodrigues vector, modified Rodrigues vector and rotation vector are also uncorrelated, but not independent. (The lack of correlation among the Cartesian components of the Rodrigues vector is not very interesting, since the variances of these components are infinite.) The *spherical* components of these three representations and of the vector components of the quaternion when uniformly distributed random vectors are independent. The uniformly distributed random Euler angles, for all twelve sets, are independent, hence uncorrelated. Thus, it would seem to be characteristic of the known uniformly random attitude representations that the elements be uncorrelated.

GENERATING A UNIFORMLY DISTRIBUTED RANDOM ATTITUDE SEQUENCE

High-level computer languages generally have in their function libraries routines for computing samples of a random variable uniformly distributed on the interval [0, 1]. If x_i is the i-th sample of this random variable, then samples of an equivalent random variable y, distributed uniformly on the interval $[a,\ b]$ can be generated according to

$$y_i = a + (b - a)\,x_i\,. \tag{52}$$

In the more general case, if we wish to transform samples of a random variable y uniformly distributed on $[a,\ b]$ to samples of a random variable $z(y)$ with pdf $p_z(z')$, then we first note that under the assumption that $z(y)$ is a monotonically increasing function of y

$$\int_{z(a)}^{z(y')} p_z(z'')\,dz'' = \int_a^{y'} p_y(y'')\,dy'' = \frac{1}{b-a}\int_a^{y'} dy''\,. \tag{53}$$

Thus, the function $z(y)$ is obtained by solving the equation

$$\mathsf{P}_z(z') \equiv \int_{z(a)}^{z'} p_z(z'')\,dz'' = \frac{y' - a}{b - a} \tag{54}$$

for z'. Our ability to find a closed-form expression for $z(y)$, therefore, depends on our ability to find a closed-form expression for the probability function $\mathsf{P}_z(z')$ which can be inverted.

Most frequently it is a uniformly random sequence of quaternions which are desired, or a uniformly random sequence of direction-cosine matrices, for which the computation of the quaternion is generally an efficient intermediate step. Thus, we will focus in this section on the generation of uniformly random sequences of quaternions.

For the vector components of the unit quaternion the inversion of Eq. (54) is not possible in closed form. To see this we note that the pdf for the magnitude of the vector components of the unit quaternion is given by

$$p_\eta(\eta') \equiv \int p_{\boldsymbol{\eta}}(\boldsymbol{\eta}')\,(\eta')^2\,d^2\hat{\boldsymbol{\eta}} = \frac{4(\eta')^2}{\pi\sqrt{1-(\eta')^2}}\,, \tag{55}$$

from which we obtain

$$\mathsf{P}_\eta(\eta') = \frac{2}{\pi}\,(\sin^{-1}\eta' - \eta'\sqrt{1-(\eta')^2})\,, \tag{56}$$

This is expressed equivalently and more conveniently in terms of the probability function of the angle of rotation (Eq. (25a))

$$\mathsf{P}_\theta(\theta') = \frac{1}{\pi}\,(\theta' - \sin\theta')\,. \tag{57}$$

The right member of neither Eq. (56) nor Eq. (57) can be inverted in closed form, but only by an infinite process.

Thus, the magnitude of the vector components of the unit quaternion or the angle of rotation does not provide a convenient vehicle for generating uniformly distributed random samples of the attitude representations. A similarly non-invertible function will, in fact, appear equivalently in the probability function of all the other attitude representations except for the Euler angles.

Let us consider next the 3-1-3 Euler angles. Clearly, both ϕ and ψ are uniformly distributed on the interval $[0, 2\pi)$. The probability function for ϑ is trivial to determine, namely

$$\mathsf{P}_\vartheta(\vartheta') = \frac{1-\cos\vartheta'}{2} \equiv \mu(\vartheta')\,. \tag{58}$$

The random variable μ is uniformly distributed on the interval $[0, 1]$, and

$$\vartheta' = \cos^{-1}(1-2\mu')\,. \tag{59}$$

Thus, we need only compute random samples of ϕ, ψ, and μ, after which the quaternion may be calculated from the formula[2]

$$\bar{\eta} = \begin{bmatrix} s(\vartheta)c(\phi-\psi) \\ s(\vartheta)s(\phi-\psi) \\ c(\vartheta)s(\phi+\psi) \\ c(\vartheta)c(\phi+\psi) \end{bmatrix}\,, \tag{60}$$

with $s(x)$ and $c(x)$ as in Eq. (30).

In fact it is unnecessary to determine ϑ' as an intermediate variable since

$$s(\vartheta') = \sqrt{\mu'}\,, \qquad c(\vartheta') = \sqrt{1-\mu'} \tag{61}$$

with the positive sign always chosen for the square roots. From the standard formulas and the uniformly distributed random sequence of quaternions or Euler angles it is then simple to generate a uniformly distributed random sequence of the other attitude representations, including the direction-cosine matrix.

Note that if φ and ψ are strictly uniform on $[0, 2\pi]$, then so are $\varphi \pm \psi$ equivalently when one treats the addition of angles as being modulo 2π. Thus, we may rewrite Eq. (60) in the form

$$\bar{\eta} = \begin{bmatrix} s(\vartheta)c(\sigma) \\ s(\vartheta)s(\sigma) \\ c(\vartheta)s(\tau) \\ c(\vartheta)c(\tau) \end{bmatrix}\,, \tag{60'}$$

and a uniform distribution of $\bar{\eta}$ is obtained by assuming the identical pdf for ϑ as before and with σ and τ distributed uniformly on $[0, 2\pi]$.

Thus, we may propose the following algorithm for the generation of uniformly random sequences of quaternions:

The Euler-Angle Method

- Compute samples σ' and τ' of σ and τ, which are independent and are distributed uniformly on $[0, 2\pi]$.
- Compute a sample μ' of μ, which is distributed uniformly on $[-1, 1]$.
- Compute the sample $\bar{\eta}'$ of $\bar{\eta}$ according to Eqs. (60') and (61).

It is amusing to note that the Euler angles, generally shunned in attitude studies because of their singularity and the cumbersome trigonometric functions, seem to be the superior representation for generating uniformly distributed random sequences of attitude in simulation. This said, we present in passing a very simple algorithm for generating a uniformly randomly distributed sequence of quaternions directly. This algorithm is based on the result, proved in the previous sections, that the uniformly distributed random quaternion has constant pdf over the entire hypersphere S^3.

The Ball in the Box Algorithm

Let x_i, $i = 1, 2, 3, 4$, be independent random variables distributed uniformly on the interval $[-1, 1]$. Then $[x_1, x_2, x_3, x_4]$ is uniformly distributed in the interior of a hypercube of side 2. Consequently, the samples are distributed uniformly (and uniformly) on any hyperspherical surface entirely contained in the hypercube. Hence, the algorithm for generating uniformly distributed random unit quaternions is

- Compute four independent samples, x_i', $i = 1, \ldots, 4$, uniformly distributed on the interval $[-1, 1]$.
- If $a \equiv (x_1')^2 + (x_2')^2 + (x_3')^2 + (x_4')^2 > 1$, then discard the samples; else $\bar{\eta}' = [x_1', x_2', x_3', x_4']^T/\sqrt{a}$.
- Repeat this process until the desired number of sampled unit quaternions has been generated.

This algorithm requires one more evaluation of a uniformly distributed random sample but fewer evaluations of special functions than the algorithm for generating uniformly distributed Euler angles. However, a large fraction of the computations are wasted in the discarded cases. To see this, note that the volume of a unit ball in four dimensions is $\pi^2/2$, and the volume of a cube of side 2 in four dimensions is 16. Thus, the probability of a successful simulation of a point in the interior of the unit ball is $\pi^2/32 = .308$, so that nearly 70% of the samples will be discarded on average.

An algorithm which discards no sample is the following:

Normal Random Unit Quaternion Generator

Let x_i', $i = 1, 2, 3, 4$, be sampled from a normal random distribution with mean 0 and standard deviation σ. Then the joint pdf $p_{\mathbf{x}}(\mathbf{x}')$, with $\mathbf{x} \equiv [x_1', x_2', x_3', x_4']^T$, is simply

$$p_{\mathbf{x}}(\mathbf{x}') = \frac{1}{(2\pi\sigma^2)^2} \exp\left(-\frac{\mathbf{x}^T\mathbf{x}}{2\sigma^2}\right) , \tag{62}$$

which depends only on the length of $\mathbf{x}$ and not on its direction in 4-space. Hence, we are led to the following generator of uniformly distributed random unit quaternions:

- Compute four samples, x'_i, $i = 1, \ldots, 4$, sampled independently from a normal distribution with mean 0 and standard deviation 1.
- Compute $a \equiv (x'_1)^2 + (x'_2)^2 + (x'_3)^2 + (x'_4)^2$.
- The unit quaternion is given by $\bar{\eta}' = [x'_1, x'_2, x'_3, x'_4]^T / \sqrt{a}$.

This algorithm has an advantage over the previous one in that it discards no data. However, to compute an approximate sample from a distribution $\mathcal{N}(0, 1)$ most computers simply sample y_i, $i = 1, \ldots, 12$, from the distribution $U(0, 1)$ and set

$$x' = y'_1 + y'_2 + \ldots + y'_{12} - 6\,, \tag{63}$$

so that every sampled uniformly distributed random unit quaternion requires 48 samples from $U(0, 1)$. This, however is probably a smaller computational burden than computing two square roots and four trigonometric functions for each quaternion.

DISCUSSION

There are many uses for the simulation of a uniformly distributed random attitude. First, it allows one to study the performance of a control system or attitude determination system when the attitude may not be known at all *a priori*. This may be particularly useful in testing "lost in Space" attitude determination and control scenarios. If one believes that the spacecraft may be tumbling at some point of the mission and wishes to predict the rate of solar power acquisition, then the simple simulation of a random attitude is certainly much simpler than a full-scale simulation of the dynamics. Most important, perhaps, is that this study is the first step in an attempt to provide a general probabilistic description of attitude which takes direct account of the group properties.

Equation (6) provides a simple and direct method for calculating the uniform pdf for any three-component representation of the attitude. However, this equation requires the computation of the determinant of a 4×4 matrix, while the earlier examples of pdf's for three-component needed only the determinant of a 3×3 matrix and frequently only the determinant of a 1×1 matrix, which pose a much smaller burden.

We have presented three different methods for deriving the expression for the uniform pdf for a three-component representation of the attitude:

- The group method, Eqs. (I-34) and (I-38),
- The transformation method, Eq. (1), and
- The quaternion method, Eq. (6).

The group-theoretic method used to obtain the pdf for the 3-1-2 Euler angles provides a fourth method. However, it is unlikely that that method will find further application.

At this point we may make a remark about the initialization of the Kalman filter. There are generally two approaches to this. One is to process a suitable number of the initial measurements in a batch processor and so obtain an initial estimate and initial error covariance matrix. If the data consists of directions, as from a star tracker, then this step can be accomplished using the QUEST algorithm.[4]

Some workers, however, initialize the Kalman filter with a state vector whose components all vanish and a diagonal state estimate covariance matrix whose variances are exceeding large, say, even 10^5 rad. This is unnecessary. Unless one chooses the attitude representation in the filter to be the Rodrigues vector, the variance of any of the three components will be finite and of magnitude surely smaller than $(2\pi)^2$. The covariance matrices are easy enough to calculate from the pdf's above. To use a very large covariance is generally uncalled for. Of course, one should avoid using the quaternion (or, worse, the attitude matrix) for anything except the prediction step for the state vector, because it is difficult otherwise to maintain the singularity of the estimate error covariance matrix.

It is surely remarkable that so many interesting results should be derivable from only two simple concepts: the group invariance of the uniform pdf, and the implicit function theorem. While a little knowledge of the attitude may be complicated and unesthetic to describe, the description of total ignorance of the attitude seems a thing of beauty and wonder.

ACKNOWLEDGMENT

The author is grateful to Dr. F. Landis Markley for making him aware of the ball-in-the-box algorithm and for posing seven years ago the casual question which started the author on this work. The author is very grateful to Dr. Hanspeter Schaub, Mr. Reid Reynolds and Dr. Panagiotis Tsiotras for critical readings of the manuscript and for numerous helpful suggestions for its improvement. Mr. Robert Bauer has obtained some of the results of this work independently, relying on results from Measure Theory and Differential Geometry. He is also the originator of the "Ball in the Box Algorithm," which he has kindly allowed me to reproduce in this report. Mr. Bauer's approach to this work is challenging, and because of his very different approach offers insights which are different from the present work. Hopefully his work will be published in the near future and become available to a much wider audience.

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