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A BROAD LOOK AT DETERMINISTIC THREE-AXIS ATTITUDE DETERMINATION

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Abstract

The general problem of determining the attitude deterministically, that is, directly without the optimization of a cost function, from measurements of angles and directions is examined. While there is no continuous ambiguity for this problem, because effectively three data are given, nonetheless, the attitude still has generally a finite degeneracy which can be removed only by the addition of further data. Specific algorithms are developed for all cases, and the nature of the degeneracy is explored in detail.

Introduction

It is customary to divide algorithms for estimating three-axis attitude into two classes. The first class uses a minimal set of data, corresponding to three scalar measurements, and then solves three possibly non-linear equations to obtain the attitude. This class is generally referred to as "deterministic," a name which has been popularized by Wertz.[1]. The other class of algorithms, generally referred to as "optimal," determine the attitude by minimizing an appropriate cost function. Such algorithms are called for when more than three scalar measurements are processed to obtain a more accurate estimate of the attitude. Perhaps the best known deterministic algorithm in current use is the TRIAD algorithm,[2] in use since at least 1964,[3] while the best known optimal algorithm nowadays is certainly the QUEST algorithm,[2] in frequent use for computing spacecraft attitude since 1979.

The overwhelming prevalence of complete vector data for determining spacecraft attitude and the availability of the QUEST algorithm for nearly two decades has largely obviated the need to calculate spacecraft attitude from anything but complete vector data. In fact, even the TRIAD algorithm is in infrequent use nowadays, since the computational burden of the QUEST algorithm and its competitors (for example, the SVD and FOAM algorithms of Markley[4,5] and recently published algorithms by Mortari[6–8]) is not much greater and provides additional enhancements.

Deterministic attitude-determination algorithms, although in infrequent use still find application. A recent example was the need to determine three-attitude for the Oscar-30 spacecraft from the observed geomagnetic field vector from a three-axis magnetometer (TAM) and the observed angle between the geomagnetic field vector and the Sun line from a spinning digital solar aspect detector (SDSAD). One of the methods developed in this paper is, in fact, the one employed for the processing of Oscar-30 attitude data.[9]

With the exception of studies of the TRIAD algorithm,[2] no systematic studies have been taken of deterministic three-axis attitude determination methods. It turns out that apart from the TRIAD algorithm, which, in fact, uses more than three effective scalar measurements for the construction of the direction-cosine matrix,[10] deterministic algorithms do not in general lead to unambiguous results for the attitude. The data may admit two-fold, four-fold and eight-fold degeneracies in the attitude solution. This is the subject of the present work.

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Nature of the Measurements

Attitude measurements are generally of angles. Thus, if $\hat{\mathbf{S}}$ is some direction fixed in the spacecraft, for example, the axis of a focal-plane sensor, and if $\hat{\mathbf{W}}$ is a direction of some object in space (or of a measurable vector field at the spacecraft, such as the geomagnetic field) coordinated in the spacecraft body frame, then the most basic vector measurement is simply

$$d = \hat{\mathbf{S}} \cdot \hat{\mathbf{W}} + \Delta d \tag{1}$$

where d is the measurement and Δd is the measurement noise. In general, we also know the representation of the partially observed direction in our reference coordinate system (typically geocentric inertial, which we will refer to in this work as the *space* frame). We write the space- referenced vector as $\hat{\mathbf{V}}$, and this is connected to the body- referenced vector $\hat{\mathbf{W}}$ by the direction-cosine matrix A,

$$\hat{\mathbf{W}} = A\hat{\mathbf{V}} \tag{2}$$

Thus, our basic measurement model is

$$d = \hat{\mathbf{S}}^T A \hat{\mathbf{V}} + \Delta d \tag{3}$$

In general we assume that Δd is normally distributed with mean zero and variance σ_d^2

$$\Delta d \sim \mathcal{N}(0, \sigma_d^2) \tag{4}$$

The measurement d is of a direction-cosine, from which we may unambiguously obtain the angle (i.e., arc length) between $\hat{\mathbf{W}}$ and $\hat{\mathbf{S}}$, if we restrict its value to lie in the interval $[0, \pi]$. For this reason we will refer to d henceforth as an angle measurement.

For a vector sensor we effectively measure the three direction cosines of $\hat{\mathbf{W}}$, with respect to three orthonormal axes fixed in the spacecraft. For the case of a vector magnetometer, this is accomplished by measuring the three components of the magnetic field directly with respect to three orthogonally oriented scalar magnetometers, each of which measures a single component of the magnetic field. We may then reconstruct $\hat{\mathbf{W}}$ unambiguously given the alignment matrix of the vector magnetometer. For focal-plane sensors, such as star cameras and vector Sun sensors, we measure effectively the ratio of the x- and y-components of the line or lines of sight to the z-component of this same direction. Given the alignment matrix of the sensor, and knowing that the z-component of the line-of-sight is positive, we may again reconstruct $\hat{\mathbf{W}}$ unambiguously.[11, 12]

Thus, for a vector sensor we are led to the effective measurement model for the direction measurement

$$\hat{\mathbf{W}} = A\hat{\mathbf{V}} + \Delta\hat{\mathbf{W}} \tag{5}$$

with

$$E\{\Delta\hat{\mathbf{W}}\} = 0 \tag{6a}$$

$$E\{\Delta\hat{\mathbf{W}}\,\Delta\hat{\mathbf{W}}^T\} = R_{\hat{\mathbf{W}}} \tag{6b}$$

where $R_{\hat{\mathbf{W}}}$ must be singular because of the norm constraint on $\hat{\mathbf{W}}$ and satisfy

$$R_{\hat{\mathbf{W}}}\,\hat{\mathbf{W}} = 0 \tag{7}$$

A useful approximate covariance matrix for direction measurements is the QUEST model,[2, 13] which has been used for the analysis of the QUEST and TRIAD algorithms,[2]

$$R_{\hat{\mathbf{W}}} = \sigma_{\hat{\mathbf{W}}}^2 \left(I_{3\times 3} - \hat{\mathbf{W}}\hat{\mathbf{W}}^T \right) \tag{8}$$

We expect this algorithm to be a particularly faithful representation of the effective direction-measurement error for a focal-plane sensor with a limited field of view. Henceforth, we will refer to $\hat{\mathbf{W}}$ as a direction measurement.

Since we are interested only in deterministic solutions, we will attempt to construct the direction-cosine matrix from the measurements under the assumption that we may neglect the measurement noise. We are led, therefore, to consider three measurement scenarios:

Scenario 1: Two Directions

We wish to determine the three-axis attitude given the two measurements

$$\hat{\mathbf{W}}_k = A\hat{\mathbf{V}}_k \qquad k = 1, 2 \tag{9}$$

Scenario 2: One Direction and One Angle

We wish to determine the three-axis attitude given the two measurements

$$\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1 \tag{10a}$$

$$d_2 = \hat{\mathbf{S}}_2^T A \hat{\mathbf{V}}_2 \tag{10b}$$

Scenario 3: Three Angles

We wish to determine the three-axis attitude given the three measurements

$$d_k = \hat{\mathbf{S}}_k^T A \hat{\mathbf{V}}_k\,, \qquad k = 1, 2, 3 \tag{11}$$

These represent the minimum number of measurements of each type (directions only, directions and angles, angles only) for which the attitude solution will have at most a finite degeneracy. We know already that Scenario 1 will not generally have solution because A is overdetermined, and a deterministic solution will require that one datum be discarded, as we shall see below. For the other two scenarios a solution will indeed be possible. Note that when we compute the attitude covariance matrix, the noise terms must be added to the definition of the measurement vector and taken into account explicitly.

Three-Axis Attitude from Two Directions

Scenario 1 above is simply that of the TRIAD algorithm, whose derivation we shall repeat here. because it is very short and will be of value in later discussion.

We begin by constructing two dextral (right-handed orthonormal) triads of unit vectors from the observations and the reference vectors, namely

$$\hat{\mathbf{r}}_1 = \hat{\mathbf{V}}_1\,, \qquad \hat{\mathbf{s}}_1 = \hat{\mathbf{W}}_1 \tag{12a}$$

$$\hat{\mathbf{r}}_2 = \frac{\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2}{|\hat{\mathbf{V}}_1 \times \hat{\mathbf{V}}_2|}\,, \quad \hat{\mathbf{s}}_2 = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_2|} \tag{12b}$$

$$\hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2\,, \qquad \hat{\mathbf{s}}_3 = \hat{\mathbf{s}}_1 \times \hat{\mathbf{s}}_2 \tag{12c}$$

In the absence of measurement noise, these ancillary vectors would satisfy

$$\hat{\mathbf{s}}_k = A\,\hat{\mathbf{r}}_k\,, \qquad k = 1, 2, 3 \tag{13}$$

or, equivalently,

$$M_s = A\,M_r \tag{14}$$

with

$$M_s \equiv [\hat{\mathbf{s}}_1 \quad \hat{\mathbf{s}}_2 \quad \hat{\mathbf{s}}_3]\,, \qquad M_r \equiv [\hat{\mathbf{r}}_1 \quad \hat{\mathbf{r}}_2 \quad \hat{\mathbf{r}}_3] \tag{15}$$

and the right members of Equation (15) denote 3×3 matrices labeled by their columns. The matrices M_r and M_s are both proper orthogonal because the two triads of column vectors are each dextral. Hence, we may solve Equation (14) for A to obtain

$$A = M_s M_r^{-1} \tag{16}$$

This is the TRIAD algorithm.[2, 3]

Although the attitude is over-determined by the data, the TRIAD algorithm is deterministic, as opposed to the QUEST algorithm, which finds an attitude solution optimally from the same data. To understand the nature of the TRIAD algorithm, note that it is sufficient to find A which satisfies Equation (13) for $k = 1$ and $k = 3$. By construction, the TRIAD attitude satisfies

$$\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1 \tag{17}$$

exactly. The equation for $k = 3$, however is equivalent to

$$\left(\frac{\hat{\mathbf{W}}_2 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{W}}_1}{|\hat{\mathbf{W}}_2 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_2)\hat{\mathbf{W}}_1|}\right) = A\left(\frac{\hat{\mathbf{V}}_2 - (\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)\hat{\mathbf{V}}_1}{|\hat{\mathbf{V}}_2 - (\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)\hat{\mathbf{V}}_1|}\right) \tag{18}$$

Thus, equivalently, one of the four equivalent scalar data is discarded by removing the component of $\hat{\mathbf{W}}_2$ which is in the direction of $\hat{\mathbf{W}}_1$, and similarly for $\hat{\mathbf{V}}_2$ and $\hat{\mathbf{V}}_1$, and readjusting the normalization of the two vectors to be unity. Thus, effectively, the only information contained in $\hat{\mathbf{W}}_2$ which is used by TRIAD algorithm is an angle. Nonetheless, the TRIAD algorithm is not equivalent to the construction of the attitude matrix for the second scenario, because this angle is not an arc length, as in the measurement model of Equation (1), but rather a dihedral angle, and therefore contains sign information about the rotation. It is this difference that is responsible for the fact that the result of the TRIAD algorithm is unambiguous, while the attitude computed for each of the two other scenarios will show at least a two-fold ambiguity.

Covariance Analysis of TRIAD

The attitude covariance matrix is defined as the covariance matrix of the attitude error vector, which is defined as the rotation vector of the very small rotation taking the true attitude into the estimated attitude. Thus if A_{true} denotes the true attitude and A^* denotes the estimated attitude, then

$$A^* = C(\Delta\theta^*)\, A_{\text{true}} \tag{19}$$

where

$$C(\theta) = I_{3\times3} + \frac{\sin(|\theta|)}{|\theta|}\,[[\,\theta\,]] + \frac{1-\cos(|\theta|)}{|\theta|^2}\,[[\,\theta\,]]^2 \tag{20}$$

is the formula for a proper orthogonal matrix parameterized by the rotation vector[10] and

$$[[\,\theta\,]] \equiv \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \tag{21}$$

Note that for $|\Delta\theta^*| << 1$ we have that

$$C(\Delta\theta^*) = I_{3\times3} + [[\,\Delta\theta^*\,]] + O(|\Delta\theta^*|^2) \tag{22}$$

The attitude covariance matrix is defined as

$$P_{\theta\theta} \equiv E\{\Delta\theta^*\Delta\theta^{*T}\} \tag{23}$$

The attitude error vector is related to the error in a particular attitude representation in the same way that the body-referenced angular velocity vector is related to the temporal derivatives of the same attitude representations. Defining the attitude covariance in this way eliminates the mischief created by the redundancy or the singularity associated with the attitude representations.

The covariance matrix of the TRIAD algorithm has been computed elsewhere[2] and will not be repeated here. The result, assuming the QUEST Model, is stated most simply as

$$\left(P_{\theta\theta}^{\text{TRIAD}}\right)^{-1} = \frac{1}{\sigma^2_{\hat{W}_1}}\left(I_{3\times3} - \hat{\mathbf{W}}_1\hat{\mathbf{W}}_1^T\right) + \frac{1}{\sigma^2_{\hat{W}_2}}\,\hat{\mathbf{s}}_4\hat{\mathbf{s}}_4^T \tag{24}$$

where

$$\hat{\mathbf{s}}_4 \equiv \hat{\mathbf{W}}_2 \times \hat{\mathbf{s}}_2 \tag{25}$$

Attitude from One Direction and One Angle

Consider now the set of measurements posed by Equations (10). To solve for the attitude in this case we begin by seeking all direction-cosine matrices A which satisfy $\hat{\mathbf{W}}_1 = A\,\hat{\mathbf{V}}_1$. These are given by

$$A = R(\hat{\mathbf{W}}_1, \theta)\, A_o \tag{26}$$

where A_o is any direction-cosine matrix satisfying $\hat{\mathbf{W}}_1 = A_o\hat{\mathbf{V}}_1$; $R(\hat{\mathbf{W}}_1, \theta)$ is the rotation matrix for a rotation about the axis $\hat{\mathbf{W}}_1$ through an angle θ; and θ is any angle satisfying $0 \leq \theta < 2\pi$. $R(\hat{\mathbf{W}}_1, \theta)$ is given by Euler's formula

$$R(\hat{\mathbf{n}}, \theta) = \cos\theta\, I_{3\times3} + (1-\cos\theta)\,\hat{\mathbf{n}}\hat{\mathbf{n}}^T + \sin\theta\,[[\,\hat{\mathbf{n}}\,]] \tag{27}$$

with $[[\,\hat{\mathbf{n}}\,]]$ defined in Equation (21).

To prove the assertion of Equation (26), assume that there exist two distinct direction-cosine matrices, A and A_o, satisfying $\hat{\mathbf{W}}_1 = A\,\hat{\mathbf{V}}_1$ and $\hat{\mathbf{W}}_1 = A_o\,\hat{\mathbf{V}}_1$, respectively. Then

$$\hat{\mathbf{W}}_1 = A\,(A_o^{-1}A_o)\,\hat{\mathbf{V}}_1 \tag{28a}$$

$$= (AA_o^{-1})\,A_o\,\hat{\mathbf{V}}_1 \tag{28b}$$

$$= (AA_o^{-1})\,\hat{\mathbf{W}}_1 \tag{28c}$$

Thus, $\hat{\mathbf{W}}_1$ must be the axis of rotation of the rotation matrix AA_o^{-1}. Since AA_o^{-1} must be different from the identity matrix, the axis of rotation is well-defined and unique. Hence,

$$AA_o^{-1} = R(\hat{\mathbf{W}}_1, \theta) \tag{29}$$

for some angle θ. Equation (26) now follows from Equations (28). Every direction-cosine matrix given by Equation (26) satisfies $\hat{\mathbf{W}}_1 = A\hat{\mathbf{V}}_1$. Therefore, there is a continuum of solutions satisfying this equation.

Equation (26) is equivalent to

$$A = A_o\, R(\hat{\mathbf{V}}_1, \theta) \tag{30}$$

with identical A_o and θ. This follows from Ref. 10

$$\begin{aligned} A_o R(\hat{\mathbf{V}}_1, \theta) &= A_o R(\hat{\mathbf{V}}_1, \theta) A_o^T A_o \\ &= R(A_o\hat{\mathbf{V}}_1, \theta) A_o \\ &= R(\hat{\mathbf{W}}_1, \theta) A_o \end{aligned} \tag{31}$$

Having found a candidate matrix A_o which characterizes the set of direction-cosine matrix which satisfies Equation (10a), we now determine the values of θ_o for which Equation (10b) is also satisfied.

We must now find a single A_o which satisfies $\hat{\mathbf{W}}_1 = A_o \hat{\mathbf{V}}_1$. Let us look for an A_o of the form

$$A_o = R(\hat{\mathbf{n}}_o, \theta_o) \tag{32}$$

For the special case that $\hat{\mathbf{W}}_1 = \hat{\mathbf{V}}_1$, the choice of $\hat{\mathbf{n}}_o$ is arbitrary provided we choose $\theta_o = 0$. Likewise, for the special case that $\hat{\mathbf{W}}_1 = -\hat{\mathbf{V}}_1$, we may choose $\hat{\mathbf{n}}_o$ to be any direction perpendicular to $\hat{\mathbf{V}}_1$ and $\theta_o = \pi$. In all other cases, we may chose

$$\hat{\mathbf{n}}_o \equiv \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|} \tag{33}$$

Thus, in every case, we can chose $\hat{\mathbf{n}}_o$ to satisfy

$$\hat{\mathbf{n}}_o \cdot \hat{\mathbf{V}}_1 = \hat{\mathbf{n}}_o \cdot \hat{\mathbf{W}}_1 = 0 \tag{34}$$

a fact which will be useful later. Assuming Equation (33)

$$\begin{aligned}[[\hat{\mathbf{n}}_o]]\hat{\mathbf{V}}_1 &= -\hat{\mathbf{n}}_o \times \hat{\mathbf{V}}_1 \\ &= -\frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|} \times \hat{\mathbf{V}}_1 \\ &= -\frac{\hat{\mathbf{V}}_1 \times (\hat{\mathbf{V}}_1 \times \hat{\mathbf{W}}_1)}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|} \\ &= \frac{\hat{\mathbf{W}}_1 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)\hat{\mathbf{V}}_1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}\end{aligned} \tag{35}$$

Thus,

$$\begin{aligned}R(\hat{\mathbf{n}}_o, \theta_o)\hat{\mathbf{V}}_1 &= \cos\theta_o \hat{\mathbf{V}}_1 \\ &\quad + \frac{\sin\theta_o}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}(\hat{\mathbf{W}}_1 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)\hat{\mathbf{V}}_1) \\ &= \left[\cos\theta_o - \frac{(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}\sin\theta_o\right]\hat{\mathbf{V}}_1 \\ &\quad + \frac{\sin\theta_o}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}\hat{\mathbf{W}}_1\end{aligned} \tag{36}$$

For $\hat{\mathbf{W}}_1 \neq \pm\hat{\mathbf{V}}_1$, $\hat{\mathbf{W}}_1$ and $\hat{\mathbf{V}}_1$ are linearly independent, and a unique solution exists for θ_o, namely,

$$\sin\theta_o = |\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|, \qquad \cos\theta_o = (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1) \tag{37}$$

which yields

$$\theta_o = \arctan_2(|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|, (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)) \tag{38}$$

where $\arctan_2(y, x)$ is the function which computes the arc tangent of y/x and in the correct quadrant. This is just the familiar FORTRAN function ATAN2.

The quaternion corresponding to A_o is has a very simple form. To calculate this quaternion we note first that

$$\begin{aligned}\cos(\theta_o/2) &= \sqrt{\frac{1+\cos\theta_o}{2}} \\ &= \sqrt{\frac{1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}{2}}\end{aligned} \tag{39}$$

and

$$\cos(\theta_o/2) \geq 0 \quad \text{for} \quad |\theta_o| \leq \pi \tag{40}$$

Likewise,

$$\sin(\theta_o) = 2\sin(\theta_o/2)\cos(\theta_o/2) \tag{41}$$

so that

$$\begin{aligned}\sin(\theta_o/2) &= \frac{\sin(\theta_o)}{2\cos(\theta_o/2)} \\ &= \frac{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}{2\sqrt{\frac{1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}{2}}} \\ &= \frac{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}{\sqrt{2(1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)}}\end{aligned} \tag{42}$$

Hence,

$$\begin{aligned}\sin(\theta_o/2)\hat{\mathbf{n}}_o &= \frac{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|}{\sqrt{2(1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)}}\frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1|} \\ &= \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{\sqrt{2(1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)}},\end{aligned} \tag{43}$$

and the corresponding quaternion is given by

$$\begin{aligned}\bar{q}_o &= \begin{bmatrix}\dfrac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{\sqrt{2(1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1)}} \\ \sqrt{\dfrac{1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}{2}}\end{bmatrix} \\ &= \sqrt{\frac{1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}{2}}\begin{bmatrix}\left(\dfrac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{1+\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1}\right) \\ 1\end{bmatrix}\end{aligned} \tag{44}$$

which can now be computed without the need to compute θ_o. The Rodrigues vector (also called the Gibbs vector) $\boldsymbol{\rho}_o$ is given obviously by Ref. 10

$$\boldsymbol{\rho}_o = \frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1} \tag{45}$$

and the matrix A_o is given equivalently by

$$\begin{aligned} A_o &= (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1) I_{3\times3} + \frac{(\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1)(\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1)^T}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1} \\ &\quad + [[\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1]] && (46a) \\ &= I_{3\times3} + [[\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1]] \\ &\quad + \frac{1}{1 + \hat{\mathbf{W}}_1 \cdot \hat{\mathbf{V}}_1} [[\hat{\mathbf{W}}_1 \times \hat{\mathbf{V}}_1]]^2 && (46b) \end{aligned}$$

Given θ_o we must now compute θ. Define

$$\hat{\mathbf{W}}_3^* \equiv A_o \hat{\mathbf{V}}_2 \tag{47}$$

Then θ is a solution of

$$\hat{\mathbf{S}}_2 \cdot R(\hat{\mathbf{W}}_1, \theta)\, \hat{\mathbf{W}}_3^* = d_2 \tag{48}$$

Substituting Euler's formula leads to

$$\begin{aligned} &\hat{\mathbf{S}}_2 \cdot \Big[\hat{\mathbf{W}}_3^* - \sin\theta\, (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*) \\ &\quad + (1 - \cos\theta)\, (\hat{\mathbf{W}}_1 \times (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*)\Big] = d_2 \end{aligned} \tag{49}$$

which can be rearranged to yield

$$\begin{aligned} &\left[\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*))\right] \cos\theta \\ &+ \left[\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*)\right] \sin\theta \\ &= (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^*) - d_2 \end{aligned} \tag{50}$$

There are clearly two solutions for θ, in general. To see this define

$$\begin{aligned} B &\equiv \Big\{\left[\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*)\right]^2 \\ &\quad + \left[\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*))\right]^2\Big\}^{1/2} && (51a) \\ &= |\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1||\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*|, && (51b) \end{aligned}$$

$$\begin{aligned} \beta &\equiv \arctan_2\Big(\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*), \\ &\qquad \hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times (\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*))\Big) \end{aligned} \tag{51c}$$

Then Equation (49) can be rewritten as

$$B \cos(\theta - \beta) = (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^*) - d_2 \tag{52}$$

From Equation (52) we see that a necessary condition that a solution exist is that

$$|(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^*) - d_2| \le |\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1||\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*| \tag{53}$$

If this condition is satisfied, then θ has the solutions

$$\theta = \beta + \cos^{-1}\left[\frac{(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_1)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^*) - d_2}{|\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1||\hat{\mathbf{W}}_1 \times \hat{\mathbf{W}}_3^*|}\right] \tag{54}$$

and the inverse cosine is indeed two-valued. Given A_o and θ we can now construct the direction-cosine matrix solutions according to Equations (26) and (46). This is the algorithm that was developed in support of the Oscar-30 mission.

Covariance Analysis

The two direction-cosine matrices constructed by the above algorithm solve Equations (10) exactly. Therefore, if attitude solutions exist, they each certainly minimize the cost function

$$J(A) = \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2} |\hat{\mathbf{W}}_1 - A\hat{\mathbf{V}}_1|^2 + \frac{1}{\sigma_{d_2}^2} |\hat{\mathbf{S}}_2 \cdot A\hat{\mathbf{V}}_2 - d_2^2| \tag{55}$$

where $\sigma_{\hat{\mathbf{W}}_1}$ and σ_{d_2} are standard deviations defined earlier.

The calculation of the Fisher information is tedious but straightforward. The result for the attitude covariance matrix is

$$\begin{aligned} P_{\theta\theta}^{-1} &= \frac{1}{\sigma_{\hat{\mathbf{W}}_1}^2} \left(I_{3\times3} - \hat{\mathbf{W}}_1 \hat{\mathbf{W}}_1^T\right) \\ &\quad + \frac{1}{\sigma_{d_2}^2} (\hat{\mathbf{W}}_3^* \times \hat{\mathbf{S}}_2)(\hat{\mathbf{W}}_3^* \times \hat{\mathbf{S}}_2)^T . \end{aligned} \tag{56}$$

Note that generally

$$\hat{\mathbf{W}}_3^* \neq \hat{\mathbf{S}}_2 \tag{57}$$

even in the absence of measurement noise. For this reason we have used the notation $\hat{\mathbf{S}}_2$ rather than $\hat{\mathbf{W}}_3^*$. Note also that $P_{\theta\theta}$ will not exist unless

$$\hat{\mathbf{W}}_1 \cdot (\hat{\mathbf{W}}_3^* \times \hat{\mathbf{S}}_2) \neq 0 \tag{58}$$

or, equivalently, unless

$$\hat{\mathbf{S}}_2 \cdot (\hat{\mathbf{W}}_1 \times (A\,\hat{\mathbf{V}}_2)) = (A\,\hat{\mathbf{V}}_2) \cdot (\hat{\mathbf{S}}_2 \times \hat{\mathbf{W}}_1) \neq 0 \quad (59)$$

Even though the direction-cosine matrix may be defined in this case the geometry represents an extremum situation in which the sensitivity of the attitude to the measurements vanishes along one direction in parameter space.

A TRIAD-like Algorithm

Instead of first calculating the direction-cosine matrix from the data and then determining a vector $\hat{\mathbf{W}}_3^*$ which satisfies Equation (47), we might try instead to calculate this $\hat{\mathbf{W}}_3^*$ directly, without first determining the attitude, and, once this vector has been determined, calculate A using the TRIAD algorithm.[2]

To compute $\hat{\mathbf{W}}_3^*$ we write

$$\hat{\mathbf{W}}_3^* = a\,\hat{\mathbf{W}}_1 + b\,\hat{\mathbf{S}}_2 + c\,\frac{\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|} \quad (60)$$

which is possible provided that $\hat{\mathbf{W}}_1 \neq \pm\hat{\mathbf{S}}_2$. It then follows that

$$\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{W}}_3^* = a + b\,(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) = \hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2 \quad (61a)$$

$$\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_3^* = a\,(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b = d_2\,, \quad (61b)$$

$$\hat{\mathbf{W}}_3^* \cdot \hat{\mathbf{W}}_3^* = a^2 + 2ab\,(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b^2 + c^2 = 1 \quad (61c)$$

The solution for a and b is immediate and is given by

$$a = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|^2}\left((\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2) - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2)\,d_2\right), \quad (62a)$$

$$b = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|^2}\left(d_2 - (\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)\right) \quad (62b)$$

The solution for c is now given by

$$c = \pm\sqrt{1 - (a^2 + 2ab\,(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b^2)}\,. \quad (63)$$

This last calculation can be simplified by noting that

$$a^2 + 2ab\,(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + b^2 = \frac{1}{|\hat{\mathbf{W}}_1 \times \hat{\mathbf{S}}_2|^2}\left[d_2^2 - 2d_2\,(\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)(\hat{\mathbf{W}}_1 \cdot \hat{\mathbf{S}}_2) + (\hat{\mathbf{V}}_1 \cdot \hat{\mathbf{V}}_2)^2\right] \quad (64)$$

The lack of a unique solution is now obvious from Equation (59). Although the TRIAD algorithm[2] can now be used to calculate the attitude from the four vectors $\hat{\mathbf{V}}_1$, $\hat{\mathbf{V}}_2$, $\hat{\mathbf{W}}_1$, and $\hat{\mathbf{W}}_3^*$, the measured unit vectors are no longer uncorrelated and the attitude covariance matrix is still that computed earlier (Equations (40) or (49)).

While the present algorithm is clearly more efficient than that developed above, it also suffers from some numerical problems. Because of round-off error it is not guaranteed that $\hat{\mathbf{W}}_3^*$ is a unit vector. Worse still, large measurement errors may cause the argument of the square root in Equation (63) to be negative.

Three-Axis Attitude from Three Angles

We now seek a direction cosine matrix which satisfies Equations (11). In general, there will be an eight-fold degeneracy for this problem. To see this let us define

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \hat{\mathbf{u}}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad (65)$$

and consider the special case

$$\hat{\mathbf{u}}_k^T A \hat{\mathbf{u}}_k = d_k\,, \qquad k = 1, 2, 3 \quad (66)$$

Substituting Equation (27) into this equation leads to

$$\cos\theta + (1 - \cos\theta)\,n_k^2 = d_k\,, \qquad k = 1, 2, 3 \quad (67)$$

where n_k denotes the n-th component of $\hat{\mathbf{n}}$. This leads immediately to the solution for θ

$$\theta = \arccos\left(\frac{1}{2}\left[\sum_{k=1}^{3} d_k - 1\right]\right) \quad (68)$$

The arc cosine is two-valued, but only the principal value need be taken, since θ can be restricted without loss of generality to the interval $0 \leq \theta \leq \pi$. Solving for the components of $\hat{\mathbf{n}}$ now leads to

$$n_k = \pm\sqrt{\frac{d_k - \cos\theta}{1 - \cos\theta}} \quad (69)$$

revealing the sign ambiguity in each component of $\hat{\mathbf{n}}$. The attitude computed from three angles, therefore, will generally display an eight-fold ambiguity.

To construct the attitude we must distinguish two cases: (1) that two of the $\hat{\mathbf{V}}_k$, $k = 1, 2, 3$, are identical, and (2) that the three are distinct. The case that all three $\hat{\mathbf{V}}$ are identical may be excluded as that case is equivalent to knowing only the direction cosines of a single unit vector, which leads to a continuous degeneracy in the attitude solution.

Case 1: Two Reference Vectors Identical

If, say, $\hat{\mathbf{V}}_1 = \hat{\mathbf{V}}_2$, then we can construct $\hat{\mathbf{W}}_3^*$, defined by Equation (47) from

$$\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{W}}_3^* = d_1 \quad \text{and} \quad \hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_3^* = d_2 \tag{70}$$

We therefore know two direction cosines of $\hat{\mathbf{W}}_3^*$ and we can write in a manner similar to an earlier calculation above

$$\hat{\mathbf{W}}_3^* = a\hat{\mathbf{S}}_1 + b\hat{\mathbf{S}}_2 + c\,\frac{\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|} \tag{71}$$

and

$$\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{W}}_3^* = a + b\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) = d_1 \tag{72a}$$

$$\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{W}}_3^* = a\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) + b = d_2 \tag{72b}$$

$$\hat{\mathbf{W}}_3^* \cdot \hat{\mathbf{W}}_3^* = a^2 + 2ab\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) + b^2 + c^2 = 1 \tag{72c}$$

which has the solutions

$$a = \frac{1}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|^2}\left(d_1 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)\,d_2\right) \tag{73a}$$

$$b = \frac{1}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|^2}\left(d_2 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)\,d_1\right). \tag{73b}$$

$$c = \pm\sqrt{1 - \left(a^2 + 2ab\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) + b^2\right)} \tag{73c}$$

and

$$a^2 + 2ab\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) + b^2 = \frac{1}{|\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2|^2}\left[d_1^2 + 2\,(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)\,d_1 d_2 + d_2^2\right] \tag{74}$$

There are two possible solutions for $\hat{\mathbf{W}}_3^*$. Apart from this two-fold degeneracy, the problem now reduces to Problem 1. Therefore, in this case, there are four possible solutions, from which the true attitude solution can be determined only on the basis of additional information.

The above case includes also the situation when $\hat{\mathbf{V}}_1 = -\hat{\mathbf{V}}_2$, since one may simulataneously change the signs of $\hat{\mathbf{V}}_2$ and $\hat{\mathbf{S}}_2$ without changing the attitude.

Case 2: Reference Vectors Distinct

To construct a solution in this case, we note first that the attitude matrix may be written in terms of generalized Euler angles[14] as

$$A = R(\hat{\mathbf{S}}_3, \psi)\,R(\hat{\mathbf{m}}_o, \vartheta)\,R(\hat{\mathbf{V}}_3, \varphi) \tag{75}$$

provided that

$$\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{m}}_o = \hat{\mathbf{V}}_3 \cdot \hat{\mathbf{m}}_o = 0 \tag{76}$$

Assuming that we can order our inputs so that $\hat{\mathbf{S}}_3 \neq \pm\hat{\mathbf{V}}_3$, we can chose

$$\hat{\mathbf{m}}_o = \frac{\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3}{|\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3|} \tag{77}$$

Otherwise, any unit vector satisfying Eqs. 76 will do. In analogy with Scenario 2, we have now that

$$\hat{\mathbf{S}}_3^T R(\hat{\mathbf{m}}_o, \vartheta)\hat{\mathbf{V}}_3 = d_3 \tag{78}$$

whence

$$\vartheta = \gamma \pm \cos^{-1}\left[\frac{(\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{m}}_o)(\hat{\mathbf{m}}_o \cdot \hat{\mathbf{V}}_3) - d_3}{C}\right] \tag{79}$$

where

$$\gamma = \arctan_2\Big(\hat{\mathbf{S}}_3 \cdot (\hat{\mathbf{m}}_o \times \hat{\mathbf{V}}_3),\ \hat{\mathbf{S}}_3 \cdot (\hat{\mathbf{m}}_o \times (\hat{\mathbf{m}}_o \times \hat{\mathbf{V}}_3))\Big) \tag{80a}$$

$$C = \Big\{[\hat{\mathbf{S}}_3 \cdot (\hat{\mathbf{m}}_o \times \hat{\mathbf{V}}_3)]^2 + [\hat{\mathbf{S}}_3 \cdot (\hat{\mathbf{m}}_o \times (\hat{\mathbf{m}}_o \times \hat{\mathbf{V}}_3))]^2\Big\}^{1/2} \tag{80b}$$

$$= |\hat{\mathbf{S}}_3 \times \hat{\mathbf{m}}_o||\hat{\mathbf{m}}_o \times \hat{\mathbf{V}}_3| \tag{80c}$$

and we have made the sign ambiguity in the arc cosine explicit in Equation (79), it being understood that the arc cosine itself supplies the principal value. Clearly, a solution will exist if and only if

$$|(\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{m}}_o)(\hat{\mathbf{m}}_o \cdot \hat{\mathbf{V}}_3) - d_3| \leq |\hat{\mathbf{S}}_3 \times \hat{\mathbf{m}}_o||\hat{\mathbf{m}}_o \times \hat{\mathbf{V}}_3| \quad (81)$$

If one is able to choose $\hat{\mathbf{m}}_o$ in accordance with Equation (77), one has more simply

$$\gamma = \arctan_2\left(-|\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3|, -(\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{V}}_3)\right) \quad (82)$$

leading to

$$\vartheta = \arctan_2\left(|\hat{\mathbf{S}}_3 \times \hat{\mathbf{V}}_3|, (\hat{\mathbf{S}}_3 \cdot \hat{\mathbf{V}}_3)\right) \pm \cos^{-1}(d_3) \quad (83)$$

Let us denote the two solutions by $\vartheta(\pm)$ and define

$$A_o(\pm) \equiv R(\hat{\mathbf{m}}_o, \vartheta(\pm)) \quad (84)$$

The remaining two equations of Equation (11) may now be written

$$\hat{\mathbf{S}}_1^T R(\hat{\mathbf{S}}_3, \psi)\, A_o(\pm)\, R(\hat{\mathbf{V}}_3, \varphi)\hat{\mathbf{V}}_1 = d_1 \quad (85a)$$

$$\hat{\mathbf{S}}_2^T R(\hat{\mathbf{S}}_3, \psi)\, A_o(\pm)\, R(\hat{\mathbf{V}}_3, \varphi)\hat{\mathbf{V}}_2 = d_2 \quad (85b)$$

If we define now

$$\hat{\mathbf{U}}_k(\pm) \equiv A_o(\pm)\hat{\mathbf{V}}_k, \quad k = 1, 2, 3 \quad (86)$$

Then Equations (85) become

$$\hat{\mathbf{S}}_1^T R(\hat{\mathbf{S}}_3, \psi)\, R(\hat{\mathbf{U}}_3(\pm), \varphi)\hat{\mathbf{U}}_1(\pm) = d_1 \quad (87a)$$

$$\hat{\mathbf{S}}_2^T R(\hat{\mathbf{S}}_3, \psi)\, R(\hat{\mathbf{U}}_3(\pm), \varphi)\hat{\mathbf{U}}_2(\pm) = d_2 \quad (87b)$$

eliminating one rotation. We are now left with two non-linear equations to solve in two unknowns.

Let us define the 3×1 matrices

$$\Phi \equiv \begin{bmatrix} 1 \\ \sin\varphi \\ \cos\varphi \end{bmatrix}, \quad \text{and} \quad \Psi \equiv \begin{bmatrix} 1 \\ \sin\psi \\ \cos\psi \end{bmatrix} \quad (88)$$

Rewriting Equation (27) as

$$R(\hat{\mathbf{n}}, \theta) = \hat{\mathbf{n}}\hat{\mathbf{n}}^T + \sin\theta\,[[\hat{\mathbf{n}}]] + \cos\theta\,(I_{3\times 3} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T) \equiv F_1(\hat{\mathbf{n}}) + \sin\theta\, F_2(\hat{\mathbf{n}}) + \cos\theta\, F_3(\hat{\mathbf{n}}) \quad (89)$$

we define 3×3 matrices $M(k, \pm)$ with elements

$$M_{ij}(k, \pm) = \hat{\mathbf{S}}_k^T F_i(\hat{\mathbf{S}}_3)\, F_j(\hat{\mathbf{U}}_3(\pm))\, \hat{\mathbf{U}}_k(\pm) - d_k\delta_{i0}\delta_{j0}, \quad k = 1, 2 \quad (90)$$

and δ_{ij} is the usual Kronecker symbol, which is unity when the two indices are equal and zero otherwise. With this new notation, Equation (87) takes the form

$$\Psi^T M(k, \pm)\, \Phi = 0, \quad k = 1, 2. \quad (91)$$

If we define now the 1×3 matrix $\Lambda(k, \pm)$ according to

$$\Lambda(k, \pm) \equiv \Psi^T M(k, \pm), \quad k = 1, 2 \quad (92)$$

then Equation (88) becomes

$$\Lambda(k, \pm)\, \Phi = 0, \quad k = 1, 2 \quad (93)$$

or, equivalently,

$$\Lambda_1(k, \pm) + \Lambda_2(k, \pm)\sin\varphi + \Lambda_3(k, \pm)\cos\varphi = 0, \quad k = 1, 2 \quad (94)$$

Straightforward application of Cramér's rule to the two equations of Equations (94) leads to

$$\sin\varphi = \frac{\Lambda_1(1, \pm)\Lambda_3(2, \pm) - \Lambda_3(1, \pm)\Lambda_1(2, \pm)}{\Lambda_2(1, \pm)\Lambda_3(2, \pm) - \Lambda_3(1, \pm)\Lambda_2(2, \pm)} \quad (95a)$$

$$\cos\varphi = \frac{\Lambda_1(1, \pm)\Lambda_2(2, \pm) - \Lambda_2(1, \pm)\Lambda_1(2, \pm)}{\Lambda_3(1, \pm)\Lambda_2(2, \pm) - \Lambda_2(1, \pm)\Lambda_3(2, \pm)} \quad (95b)$$

and, because $\sin^2\varphi + \cos^2\varphi = 1$, we must have

$$\left[\frac{\Lambda_1(1, \pm)\Lambda_3(2, \pm) - \Lambda_3(1, \pm)\Lambda_1(2, \pm)}{\Lambda_2(1, \pm)\Lambda_3(2, \pm) - \Lambda_3(1, \pm)\Lambda_2(2, \pm)}\right]^2 + \left[\frac{\Lambda_1(1, \pm)\Lambda_2(2, \pm) - \Lambda_2(1, \pm)\Lambda_1(2, \pm)}{\Lambda_3(1, \pm)\Lambda_2(2, \pm) - \Lambda_2(1, \pm)\Lambda_3(2, \pm)}\right]^2 = 1 \quad (96)$$

which must be solved for ψ. Rationalizing the denominators of Equation (96) leads to an equation which is quartic in each of $\sin\psi$ and $\cos\psi$. By collecting terms appropriately and squaring, the equation may be recast in the form

$$f(\cos\psi, \pm) = 0 \quad (97)$$

where the $f(x, \pm)$ are polynomials of order 16. Since we anticipate no more than a four-fold degeneracy for each sign, this means that 12 of the roots of each of these two polynomials must be spurious, leading to impossible values for $\cos\psi$ or via Equations (95) for $\cos\varphi$ or $\sin\varphi$. Hence, it is likely advantageous to perform numerical searches on Equation (96) directly as a function of the angle. Expecting, on intuitive grounds, that the degeneracy is indeed no worse than four-fold for each value of the sign in Equation (79) or Equation (83), a numerical search should indeed be feasible.

Covariance Analysis for the Case of Three Angles

The covariance matrix may be calculated straightforwardly using the results of Scenario 2.

$$P_{\theta\theta}^{-1} = \sum_{k=1}^{3} \frac{1}{\sigma_{d_k}^2} (\hat{\mathbf{S}}_k \times A\hat{\mathbf{V}}_k)(\hat{\mathbf{S}}_k \times A\hat{\mathbf{V}}_k)^T \qquad (98)$$

whose calculation requires, of course, that we know the correct value of A.

Discussion

We have examined the three minimum data cases for constructing spacecraft attitude deterministically when the measurement consists of directions or angles. The simplest case, when the data consists of two directions, is that of the well-known TRIAD algorithm. The case when the data consist of one direction and one angle is only slightly more complicated and shows a two-fold degeneracy in the attitude solutions. The case when the data consist of three angles is much more complicated. The attitude solution in this case is more elaborate and displays, in general, an eight-fold degeneracy. While the case of one direction and one angle has been implemented in actual mission support, it is unlikely that this will ever be the case when only three angles are measured.

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