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Focal-Plane Representation of Rotations

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Abstract

The general mathematical formalism for representing rotations in terms of focal-plane coordinates is presented. Explicit and recursive expressions are given for the focal-plane transformation expansion coefficients, and the reconstruction of the rotation matrix from the focal-plane coefficients is presented.

Introduction

While it is common to represent attitude by a 3×3 matrix transforming column vectors in three-space, attitude sensors frequently measure only two-dimensional quantities, the stereographic projection of a direction onto a plane, the *focal* plane. Instrument calibration is therefore expressed most frequently in terms of these focal-plane coordinates rather than in terms of the three-dimensional vectors. The latter would, in fact, constitute an enormous inconvenience. The most common parameterization of focal-plane calibration are the coefficients of the two-dimensional Taylor expansion of the corrected focal-plane coordinates in terms of the uncorrected focal-plane coordinates. Since these same coefficients can represent rotations, however, some ambiguity necessarily exists between sensor calibration and sensor attitude. As a first step in resolving this ambiguity, we explore in the present work the representation of rotations by focal-plane polynomial coefficients.

The focal-plane coordinates which we will examine in this work represent a very idealized instrument, essentially an idealized (non-diffractive) pin-hole camera. Real instruments do not behave exactly like pin-hole cameras. However, the purpose of sensor calibration generally is to relate the true measurements to some idealized measurement of purely geometrical significance. Thus, a deeper understanding of the idealized focal-plane coordinates which are the subject of this work is not without practical importance. Our philosophy is similar to that which we have taken previously in processing attitude measurements, in which we have considered sensors which measure a complete direction expressed as a unit three-vector [1]. Although

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no sensor works this way, such an abstraction permits us to separate sensor calibration from attitude determination. The present work falls in the middle ground between those two activities.

Focal-plane sensors are a subset of vector sensors, and the applicability of the representation presented here is somewhat more limited than the earlier line-of-sight measurement model [1]. It is limited to sensors which truly incorporate focal planes in some way, such as Sun and star sensors. It would not be a useful representation at all for the description of magnetometers, however, where one measures three independent components of a field, nor might it be enlightening for horizon scanners, where the intrinsic measurements are very different from focal-plane coordinates.

We begin by presenting the general transformation of the focal plane in two dimensions and that of a rotation in three dimensions. We then proceed to a closed-form expression for the transformation of focal-plane coordinates due to a rotation and the examination of three special cases, rotation about the two focal-plane axes and about the normal to the focal plane at the origin (the *boresight*). We next develop explicit expressions for the Taylor expansion of a rotation in terms of focal-plane coordinates. These are not the most efficient means of computing these coefficients, however. We therefore present as well an efficient recursion relation for these coefficients. Finally, we develop inverse expressions for reconstructing the 3×3 rotation matrix from the focal-plane expansion coefficients. The present work treats only the parameterization of rotations in terms of focal-plane coordinates. The estimation of distortions of the focal plane due to deformations and their separation from rotational degrees of freedom have been treated in a recent report [2]. The development of these ideas to create an attitude determination algorithm based directly in the focal plane will be presented in a later work.

Geometrical Preliminaries

Generally, we represent a direction in space by the 3×1 matrix of its components with respect to a basis. For this study we will choose the basis such that the **z**-axis is the boresight of the sensor observing the vector, and the focal plane, therefore, will be parallel to the xy-plane. We write in the usual way

$$\hat{\mathbf{W}} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \tag{1}$$

The caret denotes a unit vector. We define the focal-plane coordinates by

$$x = \frac{W_1}{W_3} \quad \text{and} \quad y = \frac{W_2}{W_3} \tag{2}$$

and, therefore,

$$\hat{\mathbf{W}} = \frac{1}{\sqrt{x^2 + y^2 + 1}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{3}$$

Equations (2) and (3) present our simple model of focal-plane coordinates, which can be related to any unit vector in a manner independent of the sensor.

Parameterization of Focal-Plane Transformations

If the 2×1 column matrix $\mathbf{x}$

$$\mathbf{x} \equiv \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

denotes the ideal undistorted focal-plane coordinates, and $\mathbf{x}'$ the distorted focal-plane coordinates, then we write, in general

$$\mathbf{x}' = \mathbf{x} + \mathbf{F}(\mathbf{x}) \tag{5}$$

or in terms of components

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} \tag{6}$$

Generally, one assumes that the two functions $F_1(x, y)$ and $F_2(x, y)$ are given by polynomial series

$$F_1(x, y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 + \dots \tag{7a}$$

$$F_2(x, y) = b_{0,0} + b_{1,0}x + b_{0,1}y + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2 + \dots . \tag{7b}$$

For transformations arising from the physical distortion of the instrument, the functions $F_1(x, y)$ and $F_2(x, y)$ generally assume very small values over the entire focal plane of the sensors. Generally, the first three terms in each series will be the largest. Coefficients arising from the misalignment of the focal plane, as measured from the *a priori* alignment, are also generally small and will affect mostly the terms $a_{0,0}$, $a_{0,1}$, $b_{0,0}$, and $b_{1,0}$. However, if the coefficients represent the attitude of the sensors, with respect to axes external to the spacecraft, then all coefficients may be quite large, as we shall see below.

Parameterization of Rotations

We tend to represent rotations in the full three-dimensional space as functions of three parameters. If we choose these three parameters to be the components of the rotation vector $\boldsymbol{\theta}$, then the rotation may be represented by a 3×3 orthogonal matrix R

$$R(\boldsymbol{\theta}) = \cos\theta I_{3\times 3} + (1 - \cos\theta)\hat{\mathbf{n}}\hat{\mathbf{n}}^{\mathrm{T}} + \sin\theta[[\hat{\mathbf{n}}]] \tag{8}$$

with

$$\theta \equiv |\boldsymbol{\theta}|, \quad \text{and} \quad \hat{\mathbf{n}} \equiv \frac{\boldsymbol{\theta}}{|\theta|} \tag{9}$$

Here, $[[v]]$ denotes the 3×3 antisymmetric matrix

$$[[v]] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} \tag{10}$$

For misalignments, the rotation vector $\boldsymbol{\theta}$ is generally quite small and can often be treated as infinitesimal. The quantity θ in equation (9) is the angle of rotation, and $\hat{\mathbf{n}}$ the axis of rotation.

Focal-Plane Representation of Rotations

Consider the action of a rotation R on a vector $[x, y, 1]^{\mathrm{T}}$.

$$\mathbf{U} \equiv R\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} R_{11}x + R_{12}y + R_{13} \\ R_{21}x + R_{22}y + R_{23} \\ R_{31}x + R_{32}y + R_{33} \end{bmatrix} \tag{11}$$

The third component of the vector $\mathbf{U}$ is generally not unity, and therefore the first two components of $\mathbf{U}$ do not correspond to focal-plane coordinates. To make the third component unity, however, we simply divide the right member of equation (11) by the third component to obtain

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{1}{R_{31}x + R_{32}y + R_{33}} R \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{12}$$

or

$$x' = \frac{R_{11}x + R_{12}y + R_{13}}{R_{31}x + R_{32}y + R_{33}} \tag{13a}$$

$$y' = \frac{R_{21}x + R_{22}y + R_{23}}{R_{31}x + R_{32}y + R_{33}} \tag{13b}$$

This is the action of a rotation in three dimensions on the focal-plane coordinates of a unit vector. Equations (13) are a special case of the *collinearity equations,* which have important applications in Photogrammetry [3, 4].

Examples of Focal-Plane Representations of a Rotation

Let us consider rotations about the three axes. These will provide additional insights into the nature of the rotations in the focal plane and give us special cases against which we can test the more general formula.

A rotation about the **x**-axis is given by

$$R(\hat{\mathbf{x}}, \theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{14}$$

Substituting values for the elements of R in equations (13) yields

$$x' = \frac{x}{-\sin\theta_1 y + \cos\theta_1} = \frac{1}{\cos\theta_1}\frac{x}{1 - \tan\theta_1 y} \tag{15a}$$

$$y' = \frac{\cos\theta_1 y + \sin\theta_1}{-\sin\theta_1 y + \cos\theta_1} = \frac{\tan\theta_1 + y}{1 - \tan\theta_1 y} \tag{15b}$$

A rotation about the **y**-axis is given by

$$R(\hat{\mathbf{y}}, \theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{16}$$

with corresponding focal-plane representation

$$x' = \frac{x - \tan\theta_2}{1 + \tan\theta_2 x} \tag{17a}$$

$$y' = \frac{1}{\cos\theta_2}\frac{y}{1 + \tan\theta_2 x} \tag{17b}$$

And, finally, a rotation about the **z**-axis is given alternately by

$$R(\hat{\mathbf{z}}, \theta_3) = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

and

$$x' = \cos\theta_3 x + \sin\theta_3 y \tag{19a}$$

$$y' = -\sin\theta_3 x + \cos\theta_3 y \tag{19b}$$

which is obviously a rotation in the focal plane. For infinitesimal rotation angles θ_1, θ_2, θ_3, which are the angles of rotation about the three coordinate axes, we have

$$x' = \frac{x + \theta_3 y - \theta_2}{1 + \theta_2 x - \theta_1 y} \tag{20a}$$

$$y' = \frac{x - \theta_3 x + \theta_1}{1 + \theta_2 x - \theta_1 y} \tag{20b}$$

which to linear order in the angles becomes

$$x' = x - \theta_2 + \theta_3 y - \theta_2 x^2 + \theta_1 xy + \ldots, \tag{21a}$$

$$y' = y + \theta_1 - \theta_3 x - \theta_2 xy + \theta_1 y^2 + \ldots. \tag{21b}$$

Thus, infinitesimal rotations of the sensor show up as the following sets of distortion parameters:

Rotation about the **x**-axis: $b_{0,0}$, $a_{1,1}$, $b_{0,2}$
Rotation about the **y**-axis: $a_{0,0}$, $b_{1,1}$, $a_{2,0}$
Rotation about the **z**-axis: $a_{0,1}$, $b_{1,0}$

Focal-Plane Expansion of a General Rotation

Simple expressions can be obtained for the focal-plane coefficients for an arbitrary rotation. Assuming that $R_{33} \neq 0$, let us define

$$\alpha \equiv R_{31}/R_{33} \quad \text{and} \quad \beta \equiv R_{32}/R_{33} \tag{22}$$

Then equation (13a) can be written as

$$x' = \frac{1}{R_{33}}\frac{R_{11}x + R_{12}y + R_{13}}{1 + \alpha x + \beta y} \tag{23}$$

Assuming that R_{33} is sufficiently large in magnitude that the magnitudes of α and β are less than unity, we may expand the denominator in a power series to yield

$$x' = \frac{1}{R_{33}}(R_{11}x + R_{12}y + R_{13})\sum_{k=0}^{\infty}(-1)^k(\alpha x + \beta y)^k \tag{24}$$

Applying now the binomial theorem leads to

$$x' = \frac{1}{R_{33}}(R_{11}x + R_{12}y + R_{13})\sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{k}\binom{k}{i}(\alpha x)^i(\beta y)^{k-i} \tag{25}$$

where the binomial coefficient is defined as[2]

$$\binom{k}{i} \equiv \begin{cases} \dfrac{k!}{i!(k-i)!} & 0 \le i \le k \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

We may rewrite equation (25) more symmetrically as

$$x' = \frac{1}{R_{33}}(R_{11}x + R_{12}y + R_{13})\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{i+j}{i}\alpha^i\beta^j x^i y^j \tag{27}$$

Carrying out the multiplications and redefining the indices as needed leads to

$$x' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}x^i y^j \tag{28}$$

with

$$a_{i,j} = \frac{(-1)^{i+j}}{R_{33}}\left\{\binom{i+j}{i}\alpha^i\beta^j R_{13} - \binom{i+j-1}{i-1}\alpha^{i-1}\beta^j R_{11} - \binom{i+j-1}{i}\alpha^i\beta^{j-1}R_{12}\right\} \tag{29}$$

Substituting for α and β leads finally to

$$a_{i,j} = \frac{(-1)^{i+j}}{R_{33}}\left\{\binom{i+j}{i}\frac{R_{13}R_{31}^iR_{32}^j}{R_{33}^{i+j}} - \binom{i+j-1}{i-1}\frac{R_{11}R_{31}^{i-1}R_{32}^j}{R_{33}^{i+j-1}} - \binom{i+j-1}{i}\frac{R_{12}R_{31}^iR_{32}^{j-1}}{R_{33}^{i+j-1}}\right\} \tag{30}$$

though equation (29) is easier to manipulate, and for greater clarity we shall retain α and β in our formulas in the sequel. The equivalent expression for y' can be obtained by making the substitutions

$$x' \to y', \qquad a_{i,j} \to b_{i,j} \quad \text{and} \quad R_{1\ell} \to R_{2\ell}, \qquad \ell = 1,2,3 \tag{31}$$

to yield

$$y' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{i,j}x^i y^j \tag{32}$$

with

$$b_{i,j} = \frac{(-1)^{i+j}}{R_{33}}\left\{\binom{i+j}{i}\frac{R_{23}R_{31}^iR_{32}^j}{R_{33}^{i+j}} - \binom{i+j-1}{i-1}\frac{R_{21}R_{31}^{i-1}R_{32}^j}{R_{33}^{i+j-1}} - \binom{i+j-1}{i}\frac{R_{22}R_{31}^iR_{32}^{j-1}}{R_{33}^{i+j-1}}\right\} \tag{33}$$

[2]Normally, the binomial coefficients are not defined outside the interval $0 \le i \le k$. The definition here will eliminate the need for indicating case restrictions in many of the formulas which follow.

An Alternate Formula for the Coefficients[3]

An alternate formula can be derived for the coefficients by noting that for $i > 0$, $j > 0$

$$\binom{i+j-1}{i-1} = \frac{i}{i+j}\binom{i+j}{i} \quad \text{and} \quad \binom{i+j-1}{i} = \frac{j}{i+j}\binom{i+j}{i} \tag{34}$$

Substituting these expressions into equation (29) yields

$$a_{ij} = \frac{(-1)^{i+j+1}}{R_{33}^2}\left[\binom{i+j-1}{i}\alpha^i\beta^{j-1}(R_{12}R_{33} - R_{13}R_{32}) + \binom{i+j-1}{j}\alpha^{i-1}\beta^j(R_{11}R_{33} - R_{13}R_{31})\right] \tag{35a}$$

and likewise

$$b_{ij} = \frac{(-1)^{i+j+1}}{R_{33}^2}\left[\binom{i+j-1}{i}\alpha^i\beta^{j-1}(R_{22}R_{33} - R_{23}R_{32}) + \binom{i+j-1}{j}\alpha^{i-1}\beta^j(R_{21}R_{33} - R_{23}R_{31})\right] \tag{35b}$$

Recalling that R is a proper orthogonal matrix, we have

$$R_{12}R_{33} - R_{13}R_{32} = -R_{21} \tag{36a}$$

$$R_{11}R_{33} - R_{13}R_{31} = R_{22} \tag{36b}$$

$$R_{22}R_{33} - R_{23}R_{32} = R_{11} \tag{36c}$$

$$R_{21}R_{33} - R_{23}R_{31} = -R_{12} \tag{36d}$$

Equations (35) become

$$a_{ij} = \frac{(-1)^{i+j+1}}{R_{33}^2}\left[-\binom{i+j-1}{i}\alpha^i\beta^{j-1}R_{21} + \binom{i+j-1}{j}\alpha^{i-1}\beta^jR_{22}\right] \tag{37a}$$

$$b_{ij} = \frac{(-1)^{i+j+1}}{R_{33}^2}\left[\binom{i+j-1}{i}\alpha^i\beta^{j-1}R_{11} - \binom{i+j-1}{j}\alpha^{i-1}\beta^jR_{12}\right] \tag{37b}$$

The computation of the focal-plane expansion coefficients up to and including third-order terms is straightforward and given by

$$\begin{aligned} x' = {} & \left[\frac{R_{13}}{R_{33}}\right] + \left[\frac{R_{22}}{R_{33}^2}\right]x + \left[-\frac{R_{21}}{R_{33}^2}\right]y + \left[-\frac{R_{22}R_{31}}{R_{33}^3}\right]x^2 \\ & + \left[\frac{1}{R_{33}^3}(-R_{22}R_{32} + R_{21}R_{31})\right]xy + \left[\frac{R_{21}R_{32}}{R_{33}^3}\right]y^2 + \left[\frac{R_{22}R_{31}^2}{R_{33}^4}\right]x^3 \\ & + \left[\frac{R_{31}}{R_{33}^4}(2R_{22}R_{32} - R_{21}R_{31})\right]x^2y + \left[\frac{R_{32}}{R_{33}^4}(R_{22}R_{32} - 2R_{21}R_{31})\right]xy^2 \\ & + \left[-\frac{R_{21}R_{32}^2}{R_{33}^4}\right]y^3 + \ldots \end{aligned} \tag{38a}$$

[3]The formulas of this section hold only for $i > 0, j > 0$.

$$y' = \left[\frac{R_{23}}{R_{33}}\right] + \left[-\frac{R_{12}}{R_{33}^2}\right]x + \left[\frac{R_{11}}{R_{33}^2}\right]y + \left[\frac{R_{12}R_{31}}{R_{33}^3}\right]x^2$$
$$+ \left[\frac{1}{R_{33}^3}(R_{12}R_{32} - R_{11}R_{31})\right]xy + \left[-\frac{R_{11}R_{32}}{R_{33}^3}\right]y^2 + \left[\frac{-R_{21}R_{31}^2}{R_{33}^4}\right]x^3$$
$$+ \left[\frac{R_{31}}{R_{33}^4}(-2R_{21}R_{32} + R_{11}R_{31})\right]x^2y + \left[\frac{R_{32}}{R_{33}^4}(-R_{21}R_{32} + 2R_{11}R_{31})\right]xy^2$$
$$+ \left[\frac{R_{11}R_{32}^2}{R_{33}^4}\right]y^3 + \ldots \tag{38b}$$

A Recursion Relation for the Coefficients

The computation of these coefficients can be made considerably less burdensome by the use of a recursion relation. Let us recall equation (23)

$$x' = \frac{1}{R_{33}} \frac{R_{11}x + R_{12}y + R_{13}}{1 + \alpha x + \beta y} \tag{23}$$

Noting that

$$\frac{1}{1 + \alpha x + \beta y} = 1 - \frac{\alpha x + \beta y}{1 + \alpha x + \beta y} \tag{39}$$

and substituting this expression into equation (23) leads to

$$x' = \frac{1}{R_{33}}(R_{11}x + R_{12}y + R_{13}) - (\alpha x + \beta y)x' \tag{40}$$

and substituting equation (28) into equation (40) leads to

$$x' = \frac{1}{R_{33}}(R_{11}x + R_{12}y + R_{13}) - \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \alpha a_{i-1,j}x^iy^j - \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \beta a_{i,j-1}x^iy^j \tag{41}$$

Comparing this result with equations (37) now yields

$$a_{i,j} = \frac{1}{R_{33}}(R_{13}\delta_{i0}\delta_{j0} + R_{11}\delta_{i1}\delta_{j0} + R_{12}\delta_{i0}\delta_{j1}) - (1 - \delta_{i0})\alpha a_{i-1,j} - (1 - \delta_{j0})\beta a_{i,j-1} \tag{42a}$$

and similarly for $b_{i,j}$

$$b_{i,j} = \frac{1}{R_{33}}(R_{23}\delta_{i0}\delta_{j0} + R_{21}\delta_{i1}\delta_{j0} + R_{22}\delta_{i0}\delta_{j1}) - (1 - \delta_{i0})\alpha b_{i-1,j} - (1 - \delta_{j0})\beta b_{i,j-1} \tag{42b}$$

In equation (42), δ_{ij} is the Kronecker symbol, and it is to be understood that a_{ij} and b_{ij} vanish whenever one of the indices is negative. It is easily verified that this formula generates immediately the coefficients of equations (37).

To compute the focal-plane expansion coefficients recursively we therefore follow the following steps:

- Calculate

$$a_{0,0},\ a_{1,0},\ a_{0,1}b_{0,0},\ b_{1,0},\ b_{0,1}$$

from the first three terms of equations (38a) and (38b), respectively.

- Given $\{a_{i,j}, b_{i,j} | i + j < k\}$, compute the coefficients for $i + j = k$ according to

$$a_{k,0} = -\alpha a_{k-1,0}, \qquad b_{k,0} = -\alpha b_{k-1,0} \tag{43a}$$

$$a_{i,k-i} = -\alpha a_{i-1,k-i} - \beta a_{i,k-i-1}, \qquad i = 1, 2, \ldots, k-1 \tag{43b}$$

$$b_{i,k-i} = -\alpha b_{i-1,k-i} - \beta b_{i,k-i-1}, \qquad i = 1, 2, \ldots, k-1 \tag{43c}$$

$$a_{0,k} = -\beta a_{0,k-1}, \qquad b_{0,k} = -\beta b_{0,k-1} \tag{43d}$$

Construction of *R* From the Focal-Plane Coefficients

The first three terms of each of equations (38) provide us with the elements of the first and second rows of the rotation matrix R, to within factors of the unknown R_{33} or its square. R_{33} and the remaining two elements may be determined from

$$R_{33} = R_{11}R_{22} - R_{12}R_{21} \tag{44a}$$

$$R_{31} = R_{12}R_{23} - R_{13}R_{22} \tag{44b}$$

$$R_{32} = R_{13}R_{21} - R_{11}R_{23} \tag{44c}$$

Thus, the rotation matrix is given in terms of the focal-plane coefficients by

$$R = \begin{bmatrix} \gamma^2 b_{01} & -\gamma^2 b_{10} & \gamma a_{00} \\ -\gamma^2 a_{01} & \gamma^2 a_{10} & \gamma b_{00} \\ -\gamma^3(a_{00}a_{01} + b_{00}b_{01}) & -\gamma^3(a_{00}a_{10} + b_{00}b_{10}) & \gamma \end{bmatrix} \tag{45}$$

where

$$\gamma = \frac{1}{\sqrt[3]{a_{10}b_{01} - a_{01}b_{10}}} \tag{46}$$

Note that all of the equations containing the focal-plane coefficients become singular when $R_{33} = 0$. Hence, this case must always be excluded. Such a case is obviously not of practical concern, since the convergence of the series is not guaranteed unless

$$|R_{33}| > 1/\sqrt{2} \tag{47}$$

and the series will certainly diverge if

$$|R_{33}| < 1/\sqrt{3} \tag{48}$$

At intermediate values the convergence will be determined by the specific values of R_{31} and R_{32}. In practical applications, we will wish to apply the formulas derived here not to an arbitrary rotation but to a misalignment, in which case R_{33} is always very close to unity.

Summary

We have developed a complete set of formulae for computing the two-dimensional Taylor series for the transformation of a focal plane caused by a rotation and the inverse transformation. Such formulae could be used, for example, to estimate the attitude directly from the focal-plane measurements, though it is generally easier to work in three dimensions [1] with unit vectors rather than contend with a multitude of terms in a Taylor series. The estimation of distortion coefficients in the (unavoidable) presence of misalignment, therefore, requires special care. This has been examined in a previous work [2] and is the subject of further investigations [5].

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