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A Suboptimal Algorithm for Attitude Determination from Multiple Star Cameras

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Abstract

The effect of direction averaging in generating suboptimal algorithms for three-axis attitude determination is examined for attitude determination systems consisting of: (1) two star cameras; and (2) a star camera and a single-direction sensor. It is shown that for star cameras with fields of view smaller than 10 deg little accuracy is lost for the case of two star cameras or for the case of a single star camera paired with a single-direction sensor of comparable accuracy. When a star camera is paired with a sensor of much lessor accuracy, such as an infra-red horizon scanner, a three-axis magnetometer or a coarse Sun sensor, the loss in attitude accuracy about the star-camera boresight can be very significant.

INTRODUCTION

A number of algorithms have been proposed for the computation of the three-axis attitude which minimizes the cost function

$$L(A) = \frac{1}{2}\sum_{k=1}^{N} a_k \left| \hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k \right|^2 , \tag{1}$$

where A is the direction-cosine matrix[1], $\hat{\mathbf{W}}_k$, $k = 1, \ldots, k$, are directions (lines of sight, observation vectors) observed in the spacecraft body frame, $\hat{\mathbf{V}}_k$, $k = 1, \ldots, k$, are the corresponding directions known in an inertial frame (the reference vectors) and a_k, $k = 1, \ldots, k$, are a set of positive weights. A caret in this work will be used to denote a unit vector. This cost function was first proposed by G. Wahba[2] in 1965 and has been

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the starting point of many algorithms, of which the most popular has been the QUEST algorithm[3].

Many solutions to the Wahba problem begin with Davenport's q-algorithm[4]. Davenport showed that the Wahba cost function could be recast as

$$L(A) = \text{constant} - \text{tr}\left(B^T A\right) \,, \tag{2}$$

where

$$B \equiv \sum_{k=1}^{N} a_k \, \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \,. \tag{3}$$

and where tr(·) denotes the trace operation, and recast further as the quadratic form

$$L(A) = \text{constant} - \bar{q}^T K \, \bar{q} \,, \tag{4}$$

where the 4×4 matrix K is given by

$$K = \begin{bmatrix} S - s\,I & \mathbf{Z} \\ \mathbf{Z}^T & s \end{bmatrix} \,, \tag{5}$$

and

$$S = B + B^T \,, \quad s = \text{tr}\, B \,, \tag{6ab}$$

$$\mathbf{Z} = [B_{23} - B_{32},\, B_{31} - B_{13},\, B_{12} - B_{21}]^T \,. \tag{6c}$$

Here $\bar{q}$ denotes the quaternion of rotation[1].

Minimization of $L(A)$ leads to an eigenvalue equation for K, namely

$$K\bar{q}^* = \lambda_{\text{max}} \, \bar{q}^* \,, \tag{7}$$

where the asterisk denotes the optimal value and λ_{max} is the largest eigenvalue of K.

The QUEST algorithm[3] uses a very efficient method for both the determination of the maximum eigenvalue λ_{max} and the optimal quaternion. In addition, it offered a model covariance matrix based on the simple measurement model

$$\hat{\mathbf{W}}_k = A\, \hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k \,, \tag{8}$$

with the measurement error $\Delta\mathbf{W}_k$ having first and second moments

$$E\{\Delta\hat{\mathbf{W}}_k\} = \mathbf{0} \tag{9}$$

$$E\{\Delta\mathbf{W}_k \, \Delta\hat{\mathbf{W}}_k^T\} = \sigma_k^2 \, [I - (A\,\hat{\mathbf{V}}_k)\,(A\,\hat{\mathbf{V}}_k)^T] \,, \tag{10}$$

where $E\{\cdot\}$ denotes the expectation, and I is the 3×3 identity matrix. This leads to the result

$$P_{\theta\theta} = \left[\sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left(I - \hat{\mathbf{W}}_k^{\text{true}} \, \hat{\mathbf{W}}_k^{\text{true}\,T} \right) \right]^{-1} \,, \tag{11}$$

and

$$\hat{\mathbf{W}}_k^{\rm true} \equiv A\,\hat{\mathbf{V}}_k\,, \tag{12}$$

provided that the weights a_k, $k = 1, \ldots, N$, are chosen to be proportional to $1/\sigma_k^2$. Note that in actual computations we must replace $\hat{\mathbf{W}}_k^{\rm true}$ by $\hat{\mathbf{W}}_k$, because the former is not known in general. Since we will be interested in calculating quantities only to lowest nonvanishing order in $\Delta\hat{\mathbf{W}}_{\ell,k}$ this replacement will not lead to important errors in general.

The covariance matrix in equation (11) is defined in terms of error angles. If $A_{\rm true}$ is the true attitude, and A^* is the estimated attitude, then the 3×1 array of attitude error angles

$$\Delta\boldsymbol{\theta}^* \equiv [\Delta\theta_1^*, \Delta\theta_2^*, \Delta\theta_3^*]^T \tag{13}$$

are defined by

$$A^* = C(\Delta\boldsymbol{\theta}^*)\,A_{\rm true}\,, \tag{14}$$

where

$$C(\boldsymbol{\theta}) = I + \frac{\sin(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|}\,[[\,\boldsymbol{\theta}\,]] + \frac{1-\cos(|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|^2}\,[[\,\boldsymbol{\theta}\,]]^2 \tag{15}$$

is the formula for a proper orthogonal matrix parameterized by the rotation vector[1] and

$$[[\,\boldsymbol{\theta}\,]] \equiv \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \tag{16}$$

Note that for $|\Delta\boldsymbol{\theta}| << 1$ we have that

$$C(\Delta\boldsymbol{\theta}) = I + [[\,\Delta\boldsymbol{\theta}\,]] + O(|\Delta\boldsymbol{\theta}|^2)\,. \tag{17}$$

The attitude covariance matrix is defined as

$$P_{\theta\theta} \equiv E\{\Delta\boldsymbol{\theta}^*\Delta\boldsymbol{\theta}^{*T}\}\,. \tag{18}$$

Markley has developed an equally efficient algorithm FOAM[5], which works directly in terms of the direction-cosine matrix.

Another important result in the development of solutions to the Wahba problem was to show that if the measurement model of equations (8) through (10) is accepted and the measurement errors are assumed to be Gaussian as well, then the maximum-likelihood estimation[6] of the attitude leads directly to the Wahba cost function[7]. This put the Wahba problem on a firm statistical footing. The QUEST algorithm has supported numerous spacecraft missions, beginning with the Magsat mission in 1979. It has the additional advantage of providing a useful figure of merit as additional output, which allows data rejection to be automated easily.

THE PSEUDO-MEASUREMENT

In an earlier work[8], Brozenec and Bender presented a method for decreasing the computational burden for QUEST when attitude was determined from multiple star-direction data from a star camera and a second sensor mounted on the spacecraft.

The authors argued that because star cameras generally have very small fields of view (generally on the order of ±5 deg/axis), the measurements will be closely clustered. As a result, the star direction measurements will provide much less information on the attitude of the spacecraft about the star-camera boresight compared with that about the other two axes, a phenomenon generally known as *geometric dilution of precision* (GDOP). Since, a second star camera or other accurate vector sensor was assumed to be present, this second sensor would will provide a great deal of information about the attitude of the spacecraft about the first star camera's boresight and vice versa if the second sensor is also a star camera. Hence, the authors argued, it was reasonable to simply average over the directions measured in the star camera at any one time, and use the direction of this average as an effective measurement for the attitude. In this way one discards any information about the attitude about the star camera boresight. That information, as we have said, is assumed to be minuscule compared to equivalent information provided by the other sensor.

Thus, specializing now to the case where one has two star cameras, one defines

$$\widehat{\hat{\mathbf{W}}}_\ell \equiv \text{unit}\left(\sum_{k=1}^{N_\ell} \hat{\mathbf{W}}_{\ell,k}\right), \quad \widehat{\hat{\mathbf{V}}}_\ell \equiv \text{unit}\left(\sum_{k=1}^{N_\ell} \hat{\mathbf{V}}_{\ell,k}\right), \quad \ell = 1,\, 2, \tag{19}$$

where N_ℓ are the number of directions observed by star camera "ℓ," and unit$(\cdot)$ is the function which generates a unit vector in the same direction as its argument if non-vanishing. The spacecraft attitude was determined by finding the optimal attitude from the Wahba cost function

$$L(A) = \frac{1}{2}\sum_{\ell=1}^{2} a_\ell \left|\widehat{\hat{\mathbf{W}}}_\ell - A\widehat{\hat{\mathbf{V}}}_\ell\right|^2, \tag{20}$$

with

$$a_\ell = \frac{N_\ell}{N_1 + N_2} \quad \ell = 1,\, 2\,. \tag{21}$$

Thus, no matter how many stars are observed in each camera, the QUEST algorithm, or any other optimal algorithm using line-of-sight data, is applied only to the two effective observations, rather than to $(N_1 + N_2)$ individual star observations. The weighting of the two terms is based on the assumption that the two star cameras have the same accuracy and that the individual measurements of each star camera have a uniform circle of error.

The present work will examine the performance of the Brozenec-Bender approach and present the results of a detailed covariance analysis. If the $(N_1 + N_2)$ line-of-sight measurements were entered directly as inputs into the Wahba cost function, then the covariance of the resulting attitude would be simply

$$P_{\theta\theta}^{\text{QUEST}} = \left[\sum_{\ell=1}^{2}\sum_{k=1}^{N_\ell} \frac{1}{\sigma_{\ell,k}^2}\left(I - \hat{\mathbf{W}}_{\ell,k}^{\text{true}}\,\hat{\mathbf{W}}_{\ell,k}^{\text{true}\,T}\right)\right]^{-1}. \tag{22}$$

The $\sigma_{\ell,k}$, we have said, are assumed to be equal to a common value σ. The corresponding attitude covariance matrix for the Brozenec-Bender algorithm is more complex and will occupy the next section.

COVARIANCE ANALYSIS OF THE BROZENEC-BENDER PSEUDO-MEASUREMENT WITH TWO STAR CAMERAS

Because the Wahba problem yields the maximum-likelihood estimate of the attitude given measurements obeying the QUEST model, its attitude covariance matrix can be computed simply from the Hessian matrix of the Wahba cost function[7].

$$\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1} = E\left\{ \frac{\partial^2}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T} L\left(C(\boldsymbol{\theta})A_{\text{true}}\right)\right\}\Bigg|_{\boldsymbol{\theta}=\mathbf{0}}, \tag{23}$$

provided we choose

$$a_{\ell,k} = \frac{\sigma_{\text{tot}}^2}{\sigma_{\ell,k}^2}, \qquad \text{with} \qquad \frac{1}{\sigma_{\text{tot}}^2} = \sum_{\ell=1}^{2}\sum_{k=1}^{N_\ell} \frac{1}{\sigma_{\ell,k}^2}, \tag{24ab}$$

and the single summation over k in equation (1) is replaced with a double summation over ℓ and k. The evaluation of equation (23) leads directly to the result given in equation (22).

The same is not true with Brozenec-Bender averaging, because the Wahba cost function does not arise from the maximum-likelihood estimate of the attitude given the Brozenec-Bender effective measurements. Thus, the attitude errors must be computed directly in terms of the measurement errors in the Brozenec-Bender effective measurement and the covariance computed from this. This computation is the subject of this section. (Reference[3] computed the covariance matrix in this way not only for the TRIAD algorithm but also for the QUEST algorithm as well, because it was not realized at the time that the Wahba cost function followed directly from the measurement error model used to calculate the attitude covariance matrix.)

We thus define unnormalized vectors in a manner similar to that of equation (19), namely

$$\overline{\mathbf{W}}_\ell \equiv \sum_{k=1}^{N_\ell} \hat{\mathbf{W}}_{\ell,k}, \quad \overline{\mathbf{V}}_\ell \equiv \sum_{k=1}^{N_\ell} \hat{\mathbf{V}}_{\ell,k}, \quad \ell = 1, 2. \tag{25}$$

Clearly,

$$\overline{\mathbf{W}}_\ell = A\overline{\mathbf{V}}_\ell + \Delta\overline{\mathbf{W}}_\ell, \quad \ell = 1, 2, \tag{26}$$

with

$$\Delta\overline{\mathbf{W}}_\ell = \sum_{k=1}^{N_\ell} \Delta\hat{\mathbf{W}}_{\ell,k}, \quad \ell = 1, 2. \tag{27}$$

Thus, given the QUEST model for the individual line-of-sight measurements, we have that $\Delta\mathbf{W}_\ell$ has mean zero and covariance matrix

$$R_{\overline{\mathbf{W}}_\ell} = \sum_{k=1}^{N_\ell} \sigma_{\ell,k}^2 \left(I - \hat{\mathbf{W}}_{\ell,k}^{\text{true}}\hat{\mathbf{W}}_{\ell,k}^{\text{true},T}\right), \quad \ell = 1, 2, \tag{28}$$

From

$$\widehat{\overline{\mathbf{W}}}_\ell = \overline{\mathbf{W}}_\ell / |\overline{\mathbf{W}}_\ell|, \quad \ell = 1, 2, \tag{29}$$

it follows that to lowest order in $\Delta\overline{\mathbf{W}}_\ell$, $\ell = 1, 2$,

$$\Delta\widehat{\overline{\mathbf{W}}}_\ell = \frac{1}{|\overline{\mathbf{W}}_\ell|}\left(I - \widehat{\overline{\mathbf{W}}}_\ell\widehat{\overline{\mathbf{W}}}_\ell^T\right)\Delta\overline{\mathbf{W}}_\ell\,, \quad \ell = 1, 2\,. \tag{30}$$

Thus, the Brozenec-Bender effective measurement satisfies

$$\widehat{\overline{\mathbf{W}}}_\ell = A\widehat{\overline{\mathbf{V}}}_\ell + \Delta\widehat{\overline{\mathbf{W}}}_\ell\,, \quad \ell = 1, 2\,, \tag{31}$$

with

$$E\{\,\Delta\widehat{\overline{\mathbf{W}}}_\ell\,\} = \mathbf{0}\,, \quad \ell = 1, 2\,, \tag{32a}$$

$$E\{\,\Delta\widehat{\overline{\mathbf{W}}}_\ell\Delta\widehat{\overline{\mathbf{W}}}_\ell^T\,\} = R_{\widehat{\overline{\mathbf{W}}}_\ell}\,, \quad \ell = 1, 2\,, \tag{32b}$$

and

$$R_{\widehat{\overline{\mathbf{W}}}_\ell} = \frac{1}{|\overline{\mathbf{W}}_\ell|^2}\left(I - \widehat{\overline{\mathbf{W}}}_\ell\widehat{\overline{\mathbf{W}}}_\ell^T\right) R_{\overline{\mathbf{W}}_\ell}\left(I - \widehat{\overline{\mathbf{W}}}_\ell\widehat{\overline{\mathbf{W}}}_\ell^T\right), \quad \ell = 1, 2\,, \tag{33}$$

COVARIANCE ANALYSIS OF THE BROZENEC-BENDER ATTITUDE-DETERMINATION ALGORITHM FOR TWO STAR CAMERAS

Now that we have a complete model for the Brozenec-Bender measurement, we may compute the spacecraft attitude. The mechanization of the QUEST algorithm is straightforward and has been described in detail elsewhere[3] and need not concern us here. What does concern us is the attitude error. To compute the attitude error, we are interested only in computing $C(\Delta\boldsymbol{\theta}^*) = A^*A_{\text{true}}^{-1}$, after which we will extract $\Delta\boldsymbol{\theta}^*$ using equation (17). We can compute $C(\Delta\boldsymbol{\theta}^*)$ most easily be replacing $\widehat{\overline{\mathbf{V}}}_\ell$ with $\widehat{\overline{\mathbf{W}}}_\ell^{\text{true}}$ in equation (20), leading to

$$L\,(C(\Delta\boldsymbol{\theta})) = \frac{1}{2}\sum_{\ell=1}^{2} a_\ell\left|\widehat{\overline{\mathbf{W}}}_\ell - C(\Delta\boldsymbol{\theta})\widehat{\overline{\mathbf{W}}}_\ell^{\text{true}}\right|^2, \tag{34}$$

Substituting equation (17) and minimizing over $\Delta\boldsymbol{\theta}$ leads straightforwardly to

$$\Delta\boldsymbol{\theta}^* = \left[\sum_{\ell=1}^{2} a_\ell\left(I - \widehat{\overline{\mathbf{W}}}_\ell\widehat{\overline{\mathbf{W}}}_\ell^T\right)\right]^{-1}\sum_{\ell=1}^{2} a_\ell\,[[\,\widehat{\overline{\mathbf{W}}}_\ell\,]]\,\Delta\widehat{\overline{\mathbf{W}}}_\ell\,, \tag{35}$$

Whence the attitude covariance matrix for the Brozenec-Bender algorithm is given by

$$P_{\theta\theta}^{\text{BB}} = \left[\sum_{\ell=1}^{2} a_\ell\left(I - \widehat{\overline{\mathbf{W}}}_\ell\widehat{\overline{\mathbf{W}}}_\ell^T\right)\right]^{-1} \sum_{\ell=1}^{2} a_\ell^2\,[[\,\widehat{\overline{\mathbf{W}}}_\ell\,]]\,R_{\widehat{\overline{\mathbf{W}}}_\ell}\,[[\,\widehat{\overline{\mathbf{W}}}_\ell\,]]^T\left[\sum_{\ell=1}^{2} a_\ell\left(I - \widehat{\overline{\mathbf{W}}}_\ell\widehat{\overline{\mathbf{W}}}_\ell^T\right)\right]^{-1}, \tag{36}$$

which should be compared with the result for the QUEST algorithm in equation (21).

MODEL COVARIANCE ANALYSIS

It follows from the Cramér-Rao Theorem[6] that

$$P_{\theta\theta}^{\text{QUEST}} \leq P_{\theta\theta}^{\text{BB}} . \tag{37}$$

The important question is how large is the difference between the two attitude covariance matrices. To answer this question, we examine the two covariances in a simple model, in which the star cameras are assumed to have a circular field of view of angular radius ρ and the stars are distributed uniformly over the field of view of each sensor. We will assume that one star camera has its boresight along the spacecraft x-axis and the other about the spacecraft y-axis. In the frame of each of the star cameras, the boresight will be taken to be the z-axis. We assume, as in Equation (21) that the two star cameras are characterized by the same variance σ^2, which is the same for all observations in the fields of view of the two star cameras.

In the limit that N_1 and N_2 are large we may replace the summation over the observations by an integral, thus if $f(\hat{\mathbf{W}})$ is any function of the observations we may write

$$\sum_{k=1}^{N_\ell} f(\hat{\mathbf{W}}_{\ell,k}) \to \frac{N_\ell}{\Omega} \int_0^{2\pi} \int_0^{\rho} f\left(\hat{\mathbf{W}}(\vartheta, \varphi)\right) \sin\vartheta \, d\vartheta \, d\varphi , \quad \ell = 1, 2 , \tag{38}$$

with Ω the solid angle subtended by the star camera field of view,

$$\Omega = 2\pi(1 - \cos\rho) , \tag{39}$$

and

$$\hat{\mathbf{W}}(\vartheta, \varphi) = \begin{bmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{bmatrix} . \tag{40}$$

With these substitutions, and assuming the distribution of observed vectors to be uniform in the star camera field of view, the inverse covariance matrix for each star camera using the QUEST algorithm for computing the attitude is

$$\left(P_{\theta\theta}^{\text{QUEST}}\right)_\ell^{-1} = \frac{N_\ell}{\sigma^2} \operatorname{diag}(a, a, b) , \quad \ell = 1, 2 , \tag{41}$$

where

$$\operatorname{diag}(a, b, c) \equiv \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{42}$$

and

$$a = (4 + \cos\rho + \cos^2\rho)/6 , \tag{43a}$$

$$b = (2 - \cos\rho - \cos^2\rho)/3 . \tag{43b}$$

Note that as $\rho \to 0$ we have that $a \to 1$ and $b \to 0$. Noting that the sensor boresights are along the x- and y-axes, we have for the QUEST algorithm

$$\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1} = R(\hat{\mathbf{x}}, \pi/2)\left(P_{\theta\theta}^{\text{QUEST}}\right)_1^{-1} R^T(\hat{\mathbf{x}}, \pi/2) + R(\hat{\mathbf{y}}, \pi/2)\left(P_{\theta\theta}^{\text{QUEST}}\right)_2^{-1} R^T(\hat{\mathbf{y}}, \pi/2) \tag{44a}$$

$$= \frac{N_1}{\sigma^2}\,\text{diag}(b,\, a,\, a) + \frac{N_2}{\sigma^2}\,\text{diag}(a,\, b,\, a) \tag{44b}$$

For the Brozenec-Bender algorithm, we obtain straightforwardly in the individual star-camera frames (boresight $= \hat{\mathbf{z}}$)

$$\overline{\mathbf{W}} = N_\ell \left(\frac{1 + \cos\rho}{2}\right)\hat{\mathbf{z}}, \qquad \text{and} \qquad \widehat{\overline{\mathbf{W}}} = \hat{\mathbf{z}}, \tag{45}$$

and

$$R_{\overline{\mathbf{W}}_\ell} = N_\ell\, \sigma^2\, \text{diag}(a,\, a,\, b), \tag{46a}$$

$$R_{\widehat{\overline{\mathbf{W}}}_\ell} = \frac{\sigma^2}{N_\ell}\left(\frac{2}{1 + \cos\rho}\right)^2 \text{diag}(a,\, a,\, 0), \tag{46b}$$

From this it follows that in the spacecraft body frame

$$\sum_{\ell=1}^{2} a_\ell^2\, [[\widehat{\overline{\mathbf{W}}}_\ell]]\, R_{\widehat{\overline{\mathbf{W}}}_\ell}\, [[\widehat{\overline{\mathbf{W}}}_\ell]]^T = \frac{\sigma^2}{(N_1 + N_2)^2}\left(\frac{2}{1 + \cos\rho}\right)^2 \left(N_1\, \text{diag}(0,\, a,\, a) + N_2\, \text{diag}(a,\, 0,\, a)\right) \tag{47}$$

Likewise,

$$\left[\sum_{\ell=1}^{2} a_\ell \left(I - \widehat{\overline{\mathbf{W}}}_\ell \widehat{\overline{\mathbf{W}}}_\ell^T\right)\right] = \frac{1}{N_1 + N_2}\left(N_1\, \text{diag}(0,\, 1,\, 1) + N_2\, \text{diag}(1,\, 0,\, 1)\right) \tag{48}$$

whence the inverse attitude covariance for the Brozenec-Bender algorithm is easily shown to be

$$\left(P_{\theta\theta}^{\text{BB}}\right)^{-1} = \frac{1}{\sigma^2}\left(\frac{2}{1 + \cos\rho}\right)^2 \frac{1}{a}\,\text{diag}(N_2,\, N_1,\, N_1 + N_2) \tag{49}$$

which should be compared with equation (44b) above.

If we consider the special case $N_1 = N_2 = N$, we obtain the simple results

$$\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1} = \frac{N}{\sigma^2}\,\text{diag}(a + b,\, a + b,\, 2a), \tag{50a}$$

$$\left(P_{\theta\theta}^{\text{BB}}\right)^{-1} = \frac{N}{\sigma^2}\left(\frac{1 + \cos\rho}{2}\right)^2 \frac{1}{a}\,\text{diag}(1,\, 1,\, 2), \tag{50b}$$

For $\rho << 1$ these reduce to

$$P_{\theta\theta}^{\text{QUEST}} = \frac{\sigma^2}{N} \,\text{diag}[1 - \rho^2/4,\; 1 - \rho^2/4,\; (1 + \rho^2/4)/2]\,, \tag{51a}$$

$$P_{\theta\theta}^{\text{BB}} = \frac{\sigma^2}{N} (1 + 3\rho^2/4)\,\text{diag}[1,\; 1,\; 1/2]\,. \tag{51b}$$

Thus, the fractional loss in accuracy is on the order of ρ^2, but for typical star-camera fields of view, ρ is only 5. deg, leading to a loss of accuracy of only one percent. This is certainly negligible. For $\rho = \pi/2$, corresponding to a star camera whose field of view encompasses half the sky, we have

$$P_{\theta\theta}^{\text{QUEST}} = \frac{3}{4}\frac{\sigma^2}{N} \,\text{diag}[1,\; 1,\; 1]\,, \tag{52a}$$

$$P_{\theta\theta}^{\text{BB}} = \frac{8}{3}\frac{\sigma^2}{N} \,\text{diag}[1,\; 1,\; 1/2]\,, \tag{52b}$$

so that the QUEST algorithm is better (in variance) by a factor of from 1.77 to 3.55 about any axis. This, however, is a very unusual case and clearly outside the expected range of application of the Brozenec-Bender algorithm. For $\rho = \pi$, the full sky case, the covariance of the Brozenec-Bender algorithm is infinite, because the $\overline{\mathbf{W}}_\varrho$ vanish in our example. The efficacy of the Brozenec-Bender algorithm when the field of view of the star camera is small has been demonstrated for two star cameras.

BROZENEC-BENDER AVERAGING WITH ONE STAR CAMERA AND ONE SINGLE-VECTOR SENSOR

Let us consider now the alternate case where the first sensor is a CCD star camera with a circular field of view of radius ρ and single-direction standard deviation σ_1 and with generally $N_1 = N$ stars in the field of view. We will assume as a typical value $\rho = 5$ deg and $\sigma_1 = 10$ arc seconds or approximately 50 microradians, and $N = 10$. Sensor 2 is a single-direction sensor with standard deviation $sigma_2$ and, clearly, $N_2 = 1$. If Sensor 2 is a precise Sun sensor then we can expect σ_2 to have values close to 10 arc seconds or 50 microradians . Otherwise, if Sensor 2 is a coarse sensor, its accuracy will be taken as 0.3 deg or approximately 5 milliradians. For definiteness, we will assume that Sensor 1, the CCD star camera, has its boresight aligned with the spacecraft body x-axis, while Sensor 2 measures a single vector along the spacecraft body y-axis.

The computation of the spacecraft covariance matrix follows procedures similar to those of the previously considered case. For the application of the QUEST algorithm to all of the data without averaging we have for the inverse covariance matrix

$$\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1} = \frac{N_1}{\sigma_1^2} \,\text{diag}(b,\; a,\; a) + \frac{1}{\sigma_2^2} \,\text{diag}(1,\; 0,\; 1) \tag{53}$$

Clearly showing the two contributions to the inverse covariance matrix. Note that the inverse-covariance (information) will be smallest about the boresight, hence, about the x-axis for the star camera (Sensor 1) and about the y-axis for the single-direction sensor (Sensor 2). With Brozenec-Bender averaging, however, we obtain a slightly less transparent expression.

We note first that relative weights of the two sensors, according to the earlier discussion will be

$$a_1 = \frac{N/\sigma_1^2}{N/\sigma_1^2 + 1/\sigma_2^2}\,, \qquad a_2 = \frac{1/\sigma_2^2}{N/\sigma_1^2 + 1/\sigma_2^2}\,. \tag{54}$$

The pseudo-measurement covariance matrix for the star camera is again following Equations (28), (33) and (46)

$$R_{\widehat{\overline{\mathbf{W}}}_1} = \frac{\sigma_1^2}{N_\varrho}\left(\frac{2}{1+\cos\rho}\right)^2 \operatorname{diag}(a,\, a,\, 0)\,, \tag{55}$$

with a as in Equation (43a), which we write as

$$R_{\widehat{\overline{\mathbf{W}}}_1} = \sigma_{\text{eff}}^2 \operatorname{diag}(0,\, 1,\, 1) \tag{56}$$

with

$$\sigma_{\text{eff}}^2 \equiv \frac{\sigma_1^2}{N}\left(\frac{2}{1+\cos\rho}\right)^2 = \beta\left(\frac{\sigma_1^2}{N}\right) \tag{57}$$

and trivially for the single-direction sensor

$$R_{\widehat{\overline{\mathbf{W}}}_2} = \sigma_2^2 \operatorname{diag}(1,\, 0,\, 1)\,. \tag{58}$$

Note that for the specified star camera $\beta \approx 1$. Equation (47) for the present case becomes equivalently

$$\sum_{\varrho=1}^{2} a_\varrho^2\, [[\,\widehat{\overline{\mathbf{W}}}_\varrho\,]]\, R_{\widehat{\overline{\mathbf{W}}}_\varrho}\, [[\,\widehat{\overline{\mathbf{W}}}_\varrho\,]]^T = a_1^2\,\sigma_{\text{eff}}^2 \operatorname{diag}(0,\, 1,\, 1) + a_2^2\,\sigma_2^2 \operatorname{diag}(1,\, 0,\, 1) \tag{59}$$

Evaluating Equation (36) in this case leads after some manipulation to

$$\left(P_{\theta\theta}^{\text{BB}}\right)^{-1} = \operatorname{diag}\left[\frac{1}{\sigma_2^2},\, \frac{1}{\sigma_{\text{eff}}^2},\, \frac{1}{a_1^2\,\sigma_{\text{eff}}^2 + a_2^2\,\sigma_2^2}\right] \tag{60}$$

It will be useful to define

$$c = \frac{N\sigma_2^2}{\sigma_1^2}\,. \tag{61}$$

Then

$$a_1 = \frac{c}{1+c}\,, \quad \text{and} \quad a_2 = \frac{1}{1+c} \tag{62}$$

Then

$$\sigma_{\text{eff}}^2 = \beta c\, \sigma_2^2\,. \tag{63}$$

For $\sigma_2 = 0.3$ deg we have then $c \approx 100{,}000$, while for $\sigma_2 = 3$ arc seconds we have instead $c = 1$.

Comparing the diagonal elements of the inverse covariance matrix we obtain

$$\frac{\left[\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1}\right]_{22}}{\left[\left(P_{\theta\theta}^{\text{BB}}\right)^{-1}\right]_{22}} = \beta a \tag{64a}$$

$$\frac{\left[\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1}\right]_{33}}{\left[\left(P_{\theta\theta}^{\text{BB}}\right)^{-1}\right]_{33}} = \beta a \left(\frac{c}{1+c}\right)^2 + \left(\frac{c}{1+c}\right)^2 \left(\frac{\beta + a}{c} + \frac{1}{c^2}\right) \tag{64b}$$

Note that for $\rho = 5$ deg we have $a = 0.988$, $\beta = 1.0002$ and $\beta a = 1.000005$ so that the attitude accuracy about the y-axis is not effected by Brozenec-Bender averaging, independent of the nature of Sensor 2. For $c = 100{,}000$ (Sensor 2 is, say, an infra-red horizon scanner) the right member of Equation (64b) differs from unity again by terms of order 10^{-5}. For $c = 1$ (Sensor 2 is a precise Sun sensor), the right member of Equation (64b) becomes $(\beta a + \beta + a + 1)/4$, which is also close to unity. Thus, the attitude determination accuracy about any axis perpendicular to the star-camera boresight is not sensitive to the nature of Sensor 2 or to Brozenec-Bender averaging.

The situation changes for the component about the star-camera boresight. In that case we find

$$\frac{\left[\left(P_{\theta\theta}^{\text{QUEST}}\right)^{-1}\right]_{11}}{\left[\left(P_{\theta\theta}^{\text{BB}}\right)^{-1}\right]_{11}} = 1 + bc \tag{64c}$$

where b was defined in Equation (43b). For $\rho = 5$ deg we have $b = 4 \times 10^{-3}$, so that the ratio of the inverse covariances is 400 when $c = 100{,}000$ and 1.004 when $c = 1$. Brozenec-Bender averaging leads to little loss in attitude determination accuracy about the boresight when the single-vector sensor is of the same accuracy roughly as the star camera but a considerable degradation of the attitude accuracy when the single-vector sensor is not very accurate.

DISCUSSION

The reasons for this great disparity in accuracy can be understood more simply than the above derivation. The geometric dilution of precision (GDOP) factor of a sensor with a narrow field of view is approximately $1/\sin(\alpha)$, where α is the half-cone angle of the sensor. For a typical star camera with a field of view 8 deg × 8 degrees, $\alpha \approx \text{FOV}/\sqrt{(12)} \approx 4$ deg, and the GDOP factor will be about 25. Thus, if the attitude accuracy of the star camera is 3 arcsec per star and the star camera measures typically 9 stars, the attitude accuracy about the average star direction will be

$$\frac{\text{GDOP} \times \sigma}{\sqrt{N}} = 25 \text{ arcsec} \tag{65}$$

This is considerably better than the accuracy of one of the three coarse sensors listed above, which is typically only about 0.5 deg. Thus, in this case, the Brozenec-Bender algorithm results in a considerable worsening of the attitude accuracy about the average star direction by two orders of magnitude. The Brozenec-Bender algorithm should not be used in this case. However, if the second sensor is a precise Sun sensor of accuracy 5.0 arc sec, the Brozenec-Bender algorithm can be used with assurance.

CONCLUSION

The Brozenec-Bender algorithm, while not adequate when a star-camera is paired with a coarse attitude sensor, nonetheless performs extremely well when the attitude sensors consist of two star cameras or a star camera and a second sensor of comparable accuracy.

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