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MAGNETOMETER CALIBRATION FOR THE FIRST ARGENTINE SPACECRAFT

Malcolm D. Shuster* and Roberto Alonso†

New algorithms have been developed for the inflight estimation of magnetometer biases and other calibration parameters for the first Argentine Spacecraft. The spacecraft will be three-axis stabilized with respect to inertial axes. The algorithms developed combine the fast convergence of an heuristic algorithm currently in use with the correct treatment of the statistics of the measurement, and does this without discarding data. The new algorithm works well even when the magnetometer bias is comparable in magnitude to the ambient magnetic field. The algorithm performance is examined using simulated data similar to that expected for the first Argentine spacecraft.

INTRODUCTION

The first Argentine spacecraft, Satelite de Aplicaciones Científicas–B (SAC-B), will be inserted into a circular orbit with an altitude of 560 km and will be inertially stabilized about all three axes in order to observe the Sun. The spacecraft orbit will have an inclination of 38 deg.

At orbit injection, the only attitude sensor which may be operating is often the vector magnetometer. Frequently, the spacecraft is spinning rapidly, and, if the spacecraft is not in an equatorial orbit or at too high an altitude, it is possible on the basis of this sensor alone (and, of course, a knowledge of the spacecraft position) to determine the spin rate and the spin-axis attitude of the spacecraft. At the same time, the accuracy of the magnetometer data may be compromised by large systematic magnetic disturbances on the spacecraft, often the result of space charging during launch or from electrical currents within the spacecraft. Thus, some means is usually needed to quickly determine this bias. Since the three-axis attitude of the spacecraft usually cannot be determined at this stage, the desired algorithm must not require a knowledge of the attitude as input.

The above situation occurs for nearly every spacecraft. For spacecraft equipped with only a vector magnetometer and a Sun sensor, three-axis attitude will be computed using the magnetometer data. In this case, the spacecraft attitude cannot be used directly to determine the magnetometer bias vector by transforming the reference magnetic field to magnetometer coordinates using the computed attitude and then comparing this transformed reference field with the magnetometer measurement. For such a mission, which occurs quite often, algorithms of the type discussed in this paper are required.

*Professor, Department of Aerospace Engineering, Mechanics and Engineering Science, University of Florida, Gainesville, Florida, 32611. email: m.shuster@ieee.org.
†Jefe, Grupo de Control de Actitud, Comisión Nacional de Actividades Espaciales (CONAE), Avenida Dorrego 4014, (1425) Buenos Aires, Argentina.

A number of algorithms have been proposed for estimating the magnetometer bias. The simplest is to solve for the bias vector by minimizing the weighted sum of the squares of residuals which are the differences in the squares of the magnitudes of the measured and modeled magnetic fields [1]. Unfortunately, this leads to a cost function which is quartic in the magnetometer bias vector. To avoid the naive minimization of a quartic function of the magnetometer bias, a number of alternative methods have been proposed. These comprise the centered algorithm of Gambhir [1, 2], Davenport's quadratic approximation [3], Acuña's model-independent method [4], and the fixed-point method of Thompson [5]. The new method, which we call **TWOSTEP**, is an improvement and considerable extension of Gambhir's algorithm. Gambhir's algorithm did not treat properly the correlations introduced by the centering process, nor did it attempt to correct for the possibly significant amount of data which the centering process discards. The new algorithm suffers from neither of these drawbacks and is very robust and efficient as well. The present paper presents the development of this new algorithm.

THE MEASUREMENT MODEL

We begin with the model

$$\mathbf{B}_k = A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k\,, \quad k = 1, \ldots, N\,, \tag{1}$$

where $\mathbf{B}_k$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time t_k; $\mathbf{H}_k$ is the corresponding value of the geomagnetic field with respect to an Earth-fixed coordinate system; A_k is the attitude of the magnetometer with respect to the Earth-fixed coordinates; $\mathbf{b}$ is the magnetometer bias; and $\boldsymbol{\varepsilon}_k$ is the measurement noise. The measurement noise, which includes both sensor errors and geomagnetic field model uncertainties, is generally assumed to be white and Gaussian. This is probably a poor approximation, since the errors in the geomagnetic field model are certainly correlated, and, in fact, generally dominate the instrument errors. However, for the sake of argument we shall assume here that the errors are white and Gaussian.

To eliminate the dependence on the attitude, we transpose terms in equation (1) and compute the square, so that at each time

$$|\mathbf{H}_k|^2 = |A_k \mathbf{H}_k|^2 = |\mathbf{B}_k - \mathbf{b} - \boldsymbol{\varepsilon}_k|^2 \tag{2a}$$

$$= |\mathbf{B}_k|^2 - 2\,\mathbf{B}_k \cdot \mathbf{b} + |\mathbf{b}|^2 - 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k + |\boldsymbol{\varepsilon}_k|^2\,. \tag{2b}$$

If we now define effective measurements and measurement noise according to

$$z_k \equiv |\mathbf{B}_k|^2 - |\mathbf{H}_k|^2\,, \qquad v_k \equiv 2(\mathbf{B}_k - \mathbf{b}) \cdot \boldsymbol{\varepsilon}_k - |\boldsymbol{\varepsilon}_k|^2\,, \tag{3ab}$$

then we can write

$$z_k = 2\,\mathbf{B}_k \cdot \mathbf{b} - |\mathbf{b}|^2 + v_k\,, \quad k = 1, \ldots, N\,. \tag{4}$$

Note that in equations (3b) and (4), $\mathbf{B}_k$ is the value about which the measurement is linearized and therefore must be interpreted as the sampled value of the measured magnetic field and not a random variable in what follows.

Even with the assumption that the original magnetometer measurement noise is white and Gaussian, the effective measurement noise is not exactly white or Gaussian. Thus, if

$$\boldsymbol{\varepsilon}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_k)\,, \tag{5}$$

and

$$E\{\boldsymbol{\varepsilon}_k \boldsymbol{\varepsilon}_\ell^{\mathrm{T}}\} = 0 \quad \text{for} \quad k \neq \ell\,, \tag{6}$$

where $E\{\cdot\}$ denotes the expectation, it follows that

$$\mu_k \equiv E\{v_k\} = -\mathrm{tr}\,(\Sigma_k)\,, \tag{7a}$$

$$\sigma_k^2 \equiv E\{v_k^2\} - \mu_k^2 = 4\,(\mathbf{B}_k - \mathbf{b})^{\mathrm{T}}\Sigma_k(\mathbf{B}_k - \mathbf{b}) + 2\sum_{i=1}^{3}(\Sigma_k)_{ii}^2\,, \tag{7b}$$

so that v_k must contain both Gaussian and χ^2 components, as is already evident from equation (3b). However, since $|\mathbf{B}_k - \mathbf{b}|$ is generally much larger than $|\boldsymbol{\epsilon}_k|$, the approximation that v_k is Gaussian is very good. In equation (7b) $\mathrm{tr}\,(\cdot)$ denotes the trace operation. In addition,

$$E\{v_k\, v_\ell\} = \mu_k \mu_\ell\,, \tag{8}$$

so that the v_k are uncorrelated but not white. If we assume that the noise $\boldsymbol{\epsilon}_k$ is small compared to the geomagnetic field, which is certainly true in low-Earth orbit, then to a large degree v_k is Gaussian and we can write approximately

$$v_k \sim \mathcal{N}(\mu_k,\, \sigma_k^2)\,. \tag{9}$$

SCORING

Given the statistical model above, the negative-log-likelihood function [6] for the magnetometer bias is given by

$$J(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{\sigma_k^2}(z_k - 2\,\mathbf{B}_k\cdot\mathbf{b} + |\mathbf{b}|^2 - \mu_k)^2 + \log\sigma_k^2 + \log 2\pi\right]\,, \tag{10}$$

which is quartic in $\mathbf{b}$. The maximum-likelihood estimate maximizes the likelihood of the estimate of the bias, which is the probability density of the measurements (evaluated at their sampled values) given as a function of the magnetometer bias. Hence, it minimizes the negative logarithm of the likelihood (equation (10)), which thus provides a cost function.

Since the domain of J has no boundaries, the maximum-likelihood estimate for $\mathbf{b}$, which we denote by $\mathbf{b}^*$, must satisfy

$$\left.\frac{\partial J}{\partial \mathbf{b}}\right|_{\mathbf{b}^*} = \mathbf{0}\,. \tag{11}$$

Note that only the first of the three terms under the summation depends on the magnetometer bias. Unless one wishes to estimate parameters of the measurement noise, there is no reason to retain the remaining two terms.[3] This quartic dependence can be avoided if complete three-axis attitude information is available, since the bias term then enters linearly into the measurement model (q.v. equation (1)) as in the work of Lerner and Shuster [7].

The most direct solution is obtained by scoring, which is just the Newton–Raphson method. Since an *a priori* estimate of the magnetometer bias is generally not available, we consider the sequence[4]

$$\mathbf{b}_0^{\mathrm{NR}} = \mathbf{0}\,, \tag{12a}$$

$$\mathbf{b}_{i+1}^{\mathrm{NR}} = \mathbf{b}_i^{\mathrm{NR}} - \left[\frac{\partial^2 J}{\partial\mathbf{b}\partial\mathbf{b}^{\mathrm{T}}}\left(\mathbf{b}_i^{\mathrm{NR}}\right)\right]^{-1}\frac{\partial J}{\partial\mathbf{b}}\left(\mathbf{b}_i^{\mathrm{NR}}\right)\,. \tag{12b}$$

[3] In fact, the standard deviations do depend on the bias vector as shown by equation (7b). However, we take the point of view that the standard deviations are functions of the true value of the bias vector. The dependence of the estimate of the bias vector on the weights is not very strong in any event.

[4] Throughout this work we shall use k as the time index and i as the iteration index.

This series is obtained by expanding $J(\mathbf{b})$ to quadratic order in $(\mathbf{b} - \mathbf{b}_i^{\mathrm{NR}})$, setting the gradient of the truncated series to zero, and solving for $\mathbf{b}_{i+1}$. If for some value of i we are sufficiently close to the maximum-likelihood estimate, then as i tends to infinity, $\mathbf{b}_i^{\mathrm{NR}}$ will tend toward a minimum or maximum of $J(\mathbf{b})$. Unfortunately, the quartic nature of $J(\mathbf{b})$ leads to multiple minima and maxima so that the convergence to the desired global minimum is by no means guaranteed.

A modification of equations (12) in frequent use is to replace the Hessian matrix (the matrix of second partial derivatives) of $J(\mathbf{b})$, by its expectation value, the Fisher information matrix F_{bb}. Under not very restrictive conditions, as the amount of data becomes infinite (or for even small samples for Gaussian measurement noise, as assumed here), the estimate error covariance matrix P_{bb} is the inverse of the Fisher information matrix. The method of replacing the Hessian matrix by the Fisher information matrix, called the Gauss-Newton method, usually results in some simplification through the discarding of complicated terms with vanishing mean, but does not solve the problem of multiple critical values.

THE CENTERED ESTIMATE

We will develop a new estimation method whose first step is similar to the RESIDG algorithm [1]. Thus, in order to avoid the minimization of a quartic cost function, let us define in a manner similar to Gambhir [1,2] the following weighted averages

$$\bar{z} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} z_k\,, \qquad \overline{\mathbf{B}} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mathbf{B}_k\,, \qquad \bar{v} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} v_k\,, \qquad \bar{\mu} \equiv \bar{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \mu_k\,, \tag{13abcd}$$

where

$$\frac{1}{\bar{\sigma}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2}\,. \tag{14}$$

Then it follows that

$$\bar{z} = 2\,\overline{\mathbf{B}} \cdot \mathbf{b} - |\mathbf{b}|^2 + \bar{v}\,. \tag{15}$$

If we define now

$$\tilde{z}_k \equiv z_k - \bar{z}\,, \qquad \widetilde{\mathbf{B}}_k \equiv \mathbf{B}_k - \overline{\mathbf{B}}\,, \qquad \tilde{v}_k \equiv v_k - \bar{v}\,, \qquad \tilde{\mu}_k \equiv \mu_k - \bar{\mu}\,, \tag{16abcd}$$

then subtracting equation (15) from equation (4) leads to

$$\tilde{z}_k = 2\,\widetilde{\mathbf{B}}_k \cdot \mathbf{b} + \tilde{v}_k\,, \quad k = 1, \ldots, N\,. \tag{17}$$

This operation is called *centering*.

The centered measurements, equation (17), are no longer quadratic in the magnetometer bias vector, so that using the centered measurements alone we can solve for $\mathbf{b}^*$ in a single iteration of the Newton–Raphson or Gauss-Newton method. However, the centered measurement noise is no longer uncorrelated. Thus, one can no longer write the negative-log-likelihood function in the form of equation (10), that is, as the sum of N squares. Nonetheless, in practice attitude ground support systems have ignored this correlation and used RESIDG to determine the bias from an approximate cost function of the form

$$J^{\mathrm{approx}}(\mathbf{b}) = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}_k - 2\,\widetilde{\mathbf{B}}_k \cdot \mathbf{b})^2\,, \tag{18}$$

and achieved reasonable results in spite of the lack of mathematical consistency and rigor, arguing that one was only discarding a single measurement out of many. In actual practice, these calculations have often assumed a constant weighting and neglected the contribution of $\tilde{\mu}_k$. Gambhir's RESIDG algorithm [2], however, is presented with variable weights, although (1) the correlations are not treated correctly; (2) it used redundant measurements; and (3) it assumed that $\tilde{\mu}_k = 0$. The redundancy of the centered mesurements if obvious from

$$\sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tilde{z}_k = 0 . \tag{19}$$

Hence, the centered measurements are not independent.

Minimizing $J^{\text{approx}}(\mathbf{b})$ over $\mathbf{b}$ leads to

$$\mathbf{b}^{*\,\text{approx}} = P_{bb}^{\text{approx}} \sum_{k=1}^{N-1} \frac{1}{\sigma_k^2} (\tilde{z}_k - \tilde{\mu}_k)\, 2\widetilde{\mathbf{B}}_k , \tag{20}$$

with the estimate error covariance matrix given approximately by

$$P_{bb}^{\text{approx}} \approx \left(F_{bb}^{\text{approx}} \right)^{-1} = \left[\sum_{k=1}^{N-1} \frac{1}{\sigma_k^2} 4\, \widetilde{\mathbf{B}}_k \widetilde{\mathbf{B}}_k^{\mathrm{T}} \right]^{-1} . \tag{21}$$

Note that $\tilde{\mu}_k$ will vanish only if the original measurement noise $\boldsymbol{\varepsilon}_k$, $k = 1, \ldots, N$, is identically distributed. Gambhir's RESIDG algorithmr converges in a single iteration because the cost function is exactly quadratic. However, equations (20) and (21) rest on incorrect statistical assumptions. The RESIDG result will not be the maximum likelihood estimate, nor will the RESIDG estimate be consistent, i.e., as the data become infinitely numerous, the RESIDG result will not converge to the true value of the magnetometer bias vector.

A STATISTICALLY CORRECT CENTERED ALGORITHM

The original data, z_k, $k = 1, \ldots, N$, may be replaced by the centered data, $\tilde{z}_k$, $k = 1, \ldots, N-1$, and the center value $\bar{z}$, without loss of information. The measurement equations are given by equations (15) and (17). The centered data have the advantage of depending only linearly on the magnetometer bias. However, they have the disadvantage that the centered measurement noise is correlated. Therefore, the negative-log-likelihood function for the centered data alone cannot be written as the sum of $N-1$ squares. To write a statistically correct cost function for the centered data (making the approximation that the measurement noise v_k is Gaussian) we define

$$\widetilde{\mathcal{Z}} \equiv [\, \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{N-1} \,]^{\mathrm{T}} , \qquad \widetilde{\mathcal{B}} \equiv \left[\widetilde{\mathbf{B}}_1, \widetilde{\mathbf{B}}_2, \ldots, \widetilde{\mathbf{B}}_{N-1} \right]^{\mathrm{T}} , \tag{22ab}$$

$$\widetilde{\mathcal{M}} \equiv [\, \tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_{N-1} \,]^{\mathrm{T}} , \qquad \widetilde{\mathcal{V}} \equiv [\, \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{N-1} \,]^{\mathrm{T}} , \tag{22cd}$$

and write

$$\widetilde{\mathcal{Z}} = 2\, \widetilde{\mathcal{B}} \mathbf{b} + \widetilde{\mathcal{V}} , \tag{23}$$

with

$$\widetilde{\mathcal{V}} \sim \mathcal{N}\left(\widetilde{\mathcal{M}}, \widetilde{\mathcal{R}} \right) . \tag{24}$$

Here $\widetilde{\mathcal{R}}$ is the covariance matrix of $\widetilde{\mathcal{V}}$, and $\widetilde{\mathcal{M}}$ is the mean. The stacked measurement $\widetilde{\mathcal{B}}$ is an $(N-1) \times 3$ matrix, and $\widetilde{\mathcal{R}}$ is an $(N-1) \times (N-1)$ positive-definite matrix whose elements are fully populated.

The negative-log-likelihood function for this stacked centered measurement is simply

$$\widetilde{J}(\mathbf{b}) = \frac{1}{2} \left[\left(\widetilde{\mathcal{Z}} - 2\, \widetilde{\mathcal{B}} \mathbf{b} - \widetilde{\mathcal{M}} \right)^{\mathrm{T}} \widetilde{\mathcal{R}}^{-1} \left(\widetilde{\mathcal{Z}} - 2\, \widetilde{\mathcal{B}} \mathbf{b} - \widetilde{\mathcal{M}} \right) + \log \det \widetilde{\mathcal{R}} + (N-1) \log 2\pi \right] . \tag{25}$$

Equation (18) expresses the incorrect assumption that $\widetilde{\mathcal{R}}$ is diagonal. We do not make this approximation here. Minimizing the negative-log-likelihood function of equation (25) leads directly to the correctly centered estimate

$$\widetilde{\mathbf{b}}^* = \left(4\,\widetilde{\mathcal{B}}^{\mathrm{T}}\widetilde{\mathcal{R}}^{-1}\widetilde{\mathcal{B}}\right)^{-1} 2\widetilde{\mathcal{B}}^{\mathrm{T}}\widetilde{\mathcal{R}}^{-1}\left(\widetilde{\mathcal{Z}} - \widetilde{\mathcal{M}}\right), \tag{26}$$

with estimate error covariance matrix

$$\widetilde{P}_{bb} = \left(4\,\widetilde{\mathcal{B}}^{\mathrm{T}}\widetilde{\mathcal{R}}^{-1}\widetilde{\mathcal{B}}\right)^{-1}. \tag{27}$$

For large quantities of data, the naive evaluation of equations (26) and (27) can be a formidable task. Therefore, we seek the means of inverting the matrix in equation (25) explicitly. By direct substitution,

$$\widetilde{\mathcal{R}}_{k\ell} = E\{\,(v_k - \mu_k)(v_\ell - \mu_\ell) - (v_k - \mu_k)(\overline{v} - \overline{\mu}) - (\overline{v} - \overline{\mu})(v_\ell - \mu_\ell) + (\overline{v} - \overline{\mu})^2\,\}\,. \tag{28}$$

Evaluating the individual expectations leads to

$$\widetilde{\mathcal{R}}_{k\ell} = \sigma_k^2\,\delta_{k\ell} - \overline{\sigma}^2\,, \tag{30}$$

which has the simple inverse

$$\left(\widetilde{\mathcal{R}}^{-1}\right)_{k\ell} = \frac{1}{\sigma_k^2}\,\delta_{k\ell} + \frac{\sigma_N^2}{\sigma_k^2\sigma_\ell^2}\,, \tag{31}$$

where σ_N^2 is the variance of v_N. Substituting this expression into equation (25) leads to

$$\widetilde{J}(\mathbf{b}) = \frac{1}{2}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\,(\tilde{z}_k - 2\widetilde{\mathbf{B}}_k\cdot\mathbf{b} - \tilde{\mu}_k)^2 + \text{terms independent of }\mathbf{b}\,. \tag{32}$$

The statistically correct cost function for the centered data, $\widetilde{J}(\mathbf{b})$, looks exactly like the naive expression of equation (18) except that the $\tilde{\mu}_k$ is nor correctly present, a truly remarkable result. The minimization is simple now and leads directly to

$$\widetilde{\mathbf{b}}^* = \widetilde{P}_{bb}\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\,(\tilde{z}_k - \tilde{\mu}_k)\,2\widetilde{\mathbf{B}}_k\,, \tag{33}$$

The estimate error covariance of the centered estimate is given by

$$\widetilde{P}_{bb} = \widetilde{F}_{bb}^{-1} = \left[\sum_{k=1}^{N}\frac{1}{\sigma_k^2}\,4\,\widetilde{\mathbf{B}}_k\widetilde{\mathbf{B}}_k^{\mathrm{T}}\right]^{-1}. \tag{34}$$

This correctly centered estimate is more attractive than the heuristic estimate of REDIDG. It is simple, and it treats the correlation of the centered measurement noise correctly. Although similar in form, it is very different in character from the centered estimate of Gambhir [1,2]. The only drawback to the centered algorithm lies in the exclusion of certain data, namely, the center term $\overline{z}$, the effect of which we investigate and eliminate in the next section. That the redundancy in the N centered measurements would cancel the effect of the correlations was not known to the developer of RESIDG [2].

We note in passing that the calculation of the remaining terms in equation (32) is not difficult. The result, which is developed in the appendix, is simply

$$\widetilde{J}(\mathbf{b}) = \frac{1}{2}\left\{ \sum_{k=1}^{N} \left[\frac{1}{\sigma_k^2}(\tilde{z}_k - 2\widetilde{\mathbf{B}}_k \cdot \mathbf{b} - \tilde{\mu}_k)^2 + \log \sigma_k^2 + \log 2\pi \right] - \left[\log \overline{\sigma}^2 + \log 2\pi \right] \right\} . \tag{35}$$

THE COMPLETE SOLUTION WITH CORRECTION FOR CENTERING

The rigorously centered algorithm derived above is no more complicated than the naive centered algorithm presented earlier. From the standpoint of computational burden, the more rigorous treatment of the statistics has merely included the $\tilde{\mu}_k$ in the cost function. However, equation (35), because it has been derived rigorously, affords us the possibility of computing the correction from the discarded center measurement $\overline{z}$. (Note the nomenclature: *center term* or *center measurement* for $\overline{z}$, *centered measurements* for the $\tilde{z}_k$, $k = 1, \ldots, N$.)

Instead of the measurement set $\{ \tilde{z}_k,\ k = 1, \ldots, N-1;\ \overline{z} \}$, we may now consider the measurements to be effectively $\{ \widetilde{\mathbf{b}}^*, \overline{z} \}$, since for a linear Gaussian estimation problem, the maximum-likelihood estimate is a sufficient statistic [6], as we shall demonstrate explicitly below. Therefore, to determine the exact maximum likelihood estimate $\mathbf{b}^*$, we must develop the statistics of these two effective measurements more completely.

To see that $\widetilde{\mathbf{b}}^*$ is a sufficient statistic for $\mathbf{b}$, substitute equation (17) into equation (33). This leads to

$$\widetilde{\mathbf{b}}^* = \widetilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (2\widetilde{\mathbf{B}}_k \cdot \mathbf{b} + \tilde{v}_k - \tilde{\mu}_k)\, 2\widetilde{\mathbf{B}}_k , \tag{36}$$

which we may rewrite as

$$\widetilde{\mathbf{b}}^* = \mathbf{b} + \widetilde{P}_{bb} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} 2\widetilde{\mathbf{B}}_k (\tilde{v}_k - \tilde{\mu}_k) \tag{37a}$$

$$\equiv \mathbf{b} + \widetilde{\mathbf{v}}_b . \tag{37b}$$

The last term is just the (zero-mean) estimate error. Obviously,

$$\widetilde{\mathbf{v}}_b \sim \mathcal{N}\left(\mathbf{0},\ \widetilde{P}_{bb}\right) . \tag{38}$$

It follows that we can write

$$\widetilde{J}(\mathbf{b}) = \frac{1}{2}\left(\mathbf{b} - \widetilde{\mathbf{b}}^*\right)^{\mathrm{T}} \widetilde{P}_{bb}^{-1} \left(\mathbf{b} - \widetilde{\mathbf{b}}^*\right) + \text{terms independent of } \mathbf{b} , \tag{39}$$

which can be verified by expanding equation (32) and completing the square in $\mathbf{b}$. But this is just the data-dependent term of the negative-log likelihood function of $\mathbf{b}$ given equations (37b) and (38). It is equation (39) which makes $\widetilde{\mathbf{b}}^*$ a sufficient statistic for $\mathbf{b}$. Equation (39) is very useful, because it allows us to investigate the effect of corrections to the centered formula using only our knowledge of $\widetilde{\mathbf{b}}^*$ and $\widetilde{P}_{bb}$. We do not have to refer again to the N centered measurements $\tilde{z}_k$, $k = 1, \ldots, N$.

We must now combine $\widetilde{\mathbf{b}}^*$ and $\overline{z}$ to obtain a complete representation of our data for the computation of $\mathbf{b}$. Recall equation (15),

$$\overline{z} = 2\overline{\mathbf{B}} \cdot \mathbf{b} - |\mathbf{b}|^2 + \overline{v} , \tag{15}$$

with

$$\bar{v} \sim \mathcal{N}(\bar{\mu}, \bar{\sigma}^2)\,. \tag{40}$$

Note that $\bar{z}$, which, unfortunately, is a nonlinear function of $\mathbf{b}$, is nonetheless an extremely accurate measurement, more accurate than the other measurements by typically a factor of $1/\sqrt{N}$, because $\bar{\sigma}$ is smaller typically than the other variances by this factor. Thus, simply centering the data can entail a significant loss of accuracy if the sensitivity of $\bar{z}$ to $\mathbf{b}$ is not small.

What is the correlation between $\tilde{\mathbf{v}}_b$ and $\bar{v}$? Calculating this explicitly, gives

$$E\{\,\tilde{\mathbf{v}}_b(\bar{v} - \bar{\mu})\,\} = \tilde{P}\sum_{k=1}^{N} \frac{1}{\sigma_k^2}\,\tilde{\mathbf{B}}_k E\{\,(\tilde{v}_k - \tilde{\mu}_k)(\bar{v} - \bar{\mu})\,\} \tag{41a}$$

$$= \tilde{P}\sum_{k=1}^{N} \frac{1}{\sigma_k^2}\,\tilde{\mathbf{B}}_k\bar{\sigma}^2 \tag{41b}$$

$$= \mathbf{0}\,, \tag{41c}$$

since from equation (16b)

$$\sum_{k=1}^{N} \frac{1}{\sigma_k^2}\,\tilde{\mathbf{B}}_k = \mathbf{0}\,. \tag{42}$$

Thus, $\tilde{\mathbf{v}}_b$ and $\bar{v}$ are uncorrelated. Since the measurement errors were assumed to be Gaussian, it follows that $\tilde{\mathbf{v}}_b$ and $\bar{v}$ are independent. The joint probability density function of $\tilde{\mathbf{v}}_b$ and $\bar{v}$ is therefore the product of the two individual probability density functions. Thus, the two corresponding negative-log-likelihood functions add,

$$J(\mathbf{b}) = \tilde{J}(\mathbf{b}) + \bar{J}(\mathbf{b})\,, \tag{43}$$

with $\tilde{J}(\mathbf{b})$ given by equation (39) and

$$\bar{J}(\mathbf{b}) = \frac{1}{2}\left[\frac{1}{\bar{\sigma}^2}(\bar{z} - 2\,\bar{\mathbf{B}}\cdot\mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})^2 + \log\bar{\sigma}^2 + \log 2\pi\right]\,. \tag{44}$$

The weight associated with the center term $\bar{J}(\mathbf{b})$ is equal to the sum of all the weights of $\tilde{J}(\mathbf{b})$. Thus, when $\bar{\mathbf{B}} - \mathbf{b}^{\text{true}}$ is not small, the loss of accuracy from discarding the center time can be substantial, as we shall see explicitly in some of the numerical examples. We can determine the relative importance of these terms to the estimate accuracy by computing the Fisher information matrix F_{bb} to obtain

$$F_{bb} = E\left\{\frac{\partial^2 J}{\partial\mathbf{b}\partial\mathbf{b}^{\mathrm{T}}}\right\} = E\left\{\frac{\partial^2\tilde{J}}{\partial\mathbf{b}\partial\mathbf{b}^{\mathrm{T}}}\right\} + E\left\{\frac{\partial^2\bar{J}}{\partial\mathbf{b}\partial\mathbf{b}^{\mathrm{T}}}\right\} = \tilde{F}_{bb} + \bar{F}_{bb} \tag{45a}$$

$$= \tilde{P}_{bb}^{-1} + \frac{4}{\bar{\sigma}^2}(\bar{\mathbf{B}} - \mathbf{b})(\bar{\mathbf{B}} - \mathbf{b})^{\mathrm{T}} \tag{45b}$$

$$= P_{bb}^{-1}\,. \tag{45c}$$

The estimate error covariance matrix will be the inverse of this quantity. If the distribution of the magnetometer measurements is "isotropic," that is, if $\bar{\mathbf{B}} - \mathbf{b}^{\text{true}}$ vanishes, then $\bar{J}(\mathbf{b})$ will be insensitive to $\mathbf{b}$. It is in this case that the centering approximation obviously leads to the best results. If, however, one attempts to determine the magnetometer bias from a short data span, say, from an inertially stabilized or Earth-pointing spacecraft, then $\bar{\mathbf{B}} - \mathbf{b}^{\text{true}}$ will be equal to the similar expression for a typical value of the magnetic field, and the formerly discarded center term which will provide half or more of the accuracy, especially for the component along $\bar{\mathbf{B}} - \mathbf{b}^{\text{true}}$.

Because $\tilde{\mathbf{b}}^*$ provides a consistent estimator of the magnetometer bias vector, its errors are characterized by the Fisher information matrix, which can then be used to assess the need to compute the correction due to the discarded center term. If a diagonal element of the Fisher information $\bar{F}_{bb}$ of the center term alone computed at $\tilde{\mathbf{b}}^*$ is large compared to the corresponding element of $\tilde{F}_{bb}$ then we must compute the center correction. If it is smaller in all three cases, the center term may be discarded without worry. We are thus led to a two-step algorithm, which we call **TWOSTEP**, which is as follows:

- We compute the centered estimate $\tilde{\mathbf{b}}^*$ of the magnetometer bias and the covariance matrix $\tilde{P}_{bb}$ using the centered data and equations (33) and (34).

- At the centered estimate $\tilde{\mathbf{b}}^*$ we compute $\tilde{F}_{bb}$ and $\bar{F}_{bb}$. If the diagonal elements of $\bar{F}_{bb}$ are sufficiently small compared with the corresponding elements of $\tilde{F}_{bb}$,

$$[\bar{F}_{bb}]_{mm} < c\,[\tilde{F}_{bb}]_{mm}\,, \qquad m = 1,\, 2,\, 3\,, \tag{46}$$

  then we will terminate the computation of the magnetometer bias at the computation of $\tilde{\mathbf{b}}^*$ and accept this value as the estimate with the estimate error covariance matrix given by the inverse of $\tilde{F}_{bb}$. Otherwise,

- Using the centered estimate $\tilde{\mathbf{b}}^*$ as an initial estimate, the correction due to the center term is computed using the Gauss–Newton method

$$\mathbf{b}_{i+1} = \mathbf{b}_i - F_{bb}^{-1}(\mathbf{b}_i)\,\mathbf{g}(\mathbf{b}_i)\,, \tag{47}$$

  where the Fisher information matrix $F_{bb}(\mathbf{b})$ is given by equation (45), and the gradient vector is given by the sum of the gradients of equations (39) and (44)

$$\begin{aligned}\mathbf{g}(\mathbf{b}) &= \tilde{\mathbf{g}}(\mathbf{b}) + \bar{\mathbf{g}}(\mathbf{b}) \\ &= \tilde{P}_{bb}^{-1}\,(\mathbf{b} - \tilde{\mathbf{b}}^*) - \frac{1}{\bar{\sigma}^2}\,(\bar{z} - 2\,\bar{\mathbf{B}} \cdot \mathbf{b} + |\mathbf{b}|^2 - \bar{\mu})\,2\,(\bar{\mathbf{B}} - \mathbf{b})\,.\end{aligned} \tag{48}$$

- The iteration is continued until

$$\eta_i \equiv (\mathbf{b}_i - \mathbf{b}_{i-1})^{\mathrm{T}} F_{bb}(\mathbf{b}_{i-1})\,(\mathbf{b}_i - \mathbf{b}_{i-1}) \tag{49}$$

  is less than some predetermined small quantity.

Since the centered estimate was consistent, we expect that

$$\delta \equiv (\mathbf{b}^* - \tilde{\mathbf{b}}^*)^{\mathrm{T}}\, \tilde{P}_{bb}^{-1}(\mathbf{b}^* - \tilde{\mathbf{b}}^*) \tag{50}$$

will not be large. If $\mathbf{b}^*$ were the exact value of $\mathbf{b}$, then we should expect that this quantity would be $\chi^2(3)$, which has mean 3 and variance 6. The mean and variance of δ should be typically smaller than this. A large value of δ might indicate convergence to a non-global minimum of $J(\mathbf{b})$.

How large should c be in the test for computing the center correction, equation (46)? If we choose c to be 0.5, then the center correction will be computed only if it improves the accuracy by at least 20 per cent. If we choose c to be 0.1, then the center correction will be computed only if it improves the accuracy by at least 5 per cent. A reasonable value for c seems to be somewhere between these two numbers, depending on the taste of the user.

Table 1. Performance of TWOSTEP for SAC-B.

	step	bias estimate (mG)
$\mathbf{b}^{\text{true}} = [\,10., 20., 30.\,]$ mG.		
	centering approximation	[9.92, 20.00, 29.68] ±[0.14, 0.33, 0.98]
	with center correction	[9.94, 19.94, 29.92] ±[0.11, 0.17, 0.11]
$\mathbf{b}^{\text{true}} = [\,100., 200., 300.\,]$ mG.		
	centering approximation	[99.92, 200.01, 299.68] ±[0.14, 0.33, 0.98]
	with center correction	[99.94, 199.94, 299.92] ±[0.11, 0.17, 0.11]

NUMERICAL EXAMPLES FOR THE MAGNETOMETER BIAS ESTIMATOR

The new algorithm developed in this work has been examined for the SAC-B orbit parameters and inertial three-axis stabilization. The geomagnetic field in our studies has been simulated using the International Geomagnetic Reference Field (IGRF (1985)) [8], which has been extrapolated to 1994. More recent field models are available, but IGRF (1985) is adequate for our simulation needs.

For purposes of simulation we have assumed an effective white Gaussian magnetometer measurement error with isotropic error distribution and a standard deviation per axis of 2.0 mG, corresponding to an angular error of approximately 0.5 deg at the equator. We have assumed also, following the example of SAC-B, that the x-axis of the magnetometer is parallel to the spacecraft spin axis, which always points toward the Sun. The Sun direction makes an angle of approximately 40 degrees with the orbit plane. The magnetometer data were sampled every eight seconds, which is the sampling frequency for SAC-B. All entries in the tables for the estimated magnetometer bias and the associated standard deviations are in mG.

We have generally displayed all iterations up to convergence to two decimal places. The results are seen to be quite good in all cases. In only a few cases (in Table 2 below) were more than one iteration of the center correction required to this accuracy, and this only for grossly mismodeled errors. In most cases, the centering approximation alone was sufficient to this level of accuracy. Nearly 200 different cases were simulated in testing the algorithm. The above cases were typical except that we have modified the field model slightly so that the third component of the bias would be less observable from the centered data alone. This was done to illustrate more acutely the possible importance of the center correction and the performance of the special algorithm developed for cases of poor observability.

Note in Table 1 the similarity of the fractional parts of the estimates for the small and large values of the bias, the result of using the same seed in each case.

ROBUSTNESS OF TWOSTEP

Thus far, both the estimator and the data have used identical statistical assumptions, in particular, it has been assumed that the fundamental magnetometer measurement noise is white and Gaussian. In general, it is neither of these, although estimators nearly always assume such a measurement noise model. This is the case for **TWOSTEP**. To test the sensitivity to these sweeping and not totally correct modeling assumptions, we have examined two cases. In the first case, we have replaced the white Gaussian noise sequence $\boldsymbol{\varepsilon}_k$ by a colored noise sequence described by a first-order Markov process driven by white noise. The "time constant" of the Markov process has been chosen to correspond to an orbital arc length of 18 deg, consistent with the correlation length associated with the neglected orders of the harmonic expansion of the magnetic field model. The power spectral density of the white-noise driving term has been chosen so that the covariance matrix of the stationary first-order Markov process will match that of the Gaussian white-noise model used in Table 1. The results are shown in Table 2. The iteration index "1" is the centering approximation, further indices refer to iterations of the center correction. The quality of the estimates has deteriorated somewhat because the estimator now contains model errors. As a result, the actual errors are outside the error bounds computed by **TWOSTEP** based on its now incorrect assumptions on the nature of the measurement noise. However, for all practical purposes the results are still quite good.

In a further numerical experiment, we have attempted to model the measurement noise as realistically as possible. To this end we have considered the properties of magnetometers constructed at NASA Goddard Space Flight Center [4]. These are characterized by a white noise and ripple effects of about $\sigma_o = 0.6$ mG per axis. In addition, the usable range of the magnetometer, from −600 mG to +600 mG is usually represented digitally by 12 bits, corresponding to a resolution of 0.29 mG $\equiv \Delta$. Thus, we may regard the telemetered field to be given (in counts) by

$$\mathbf{B}_k^{\mathrm{T/M}} = \mathrm{Int}[(A_k\,\mathbf{H}_k + \mathbf{b} + \mathbf{w}_k)/\Delta]\,, \tag{51}$$

where $\mathrm{Int}(\cdot)$ is the function which computes the greatest integer for each component of its argument, and $\mathbf{w}_k$ is Gaussian white noise whose covariance is given by $(0.6\,\mathrm{mG})^2\; I_{3\times3}$. The measurements would then be reconstructed from telemetry according to the prescription

$$\mathbf{B}_k = \Delta\,\left[\mathbf{B}_k^{\mathrm{T/M}} + [0.5,\,0.5,\,0.5]^{\mathrm{T}}\right]\,. \tag{52}$$

The model geomagnetic field model errors we have used the harmonic expansion coefficients of IGRF(85) up to order 10 to compute the raw measurements, but have used the coefficients only up to order 8 in the estimator. The **TWOSTEP** estimator has considered an estimator based solely on the known random and quantization errors, that is, it assumes

$$\Sigma_k = \left(\sigma_o^2 + \frac{\Delta^2}{12}\right)\,I_{3\times3}\,. \tag{53}$$

The results of the magnetometer bias determination given this mismatch between measurement noise and estimator are shown in Table 3. The results again clearly show errors that are significantly larger than the statistical limits computed from the estimators error model but are quite acceptable also in this case. Note that proportionately the agreement is greater for the larger biases in both Table 2 and Table 3, because the modeling errors are proportionately smaller. We see in these examples of mismodeling some of the few cases where more than one iteration of the center correction has been needed. The result of that further iteration can hardly be called significant, however.

ESTIMATION OF THE MAGNETOMETER BIAS, SCALE FACTORS, AND NON-ORTHOGONALITY CORRECTIONS

The algorithm presented above is easily extended to include the estimation of scale-factor and non-orthogonality corrections. Like the scale factor corrections, non- orthogonality corrections have their

Table 2. Performance of TWOSTEP for Colored Noise.

iteration	bias estimate (mG)
magnetometer bias = [10., 20., 30.] mG	
1	[10.80, 19.76, 32.91] ± [0.17, 0.26, 0.77]
2	[10.56, 20.16, 31.68] ± [0.09, 0.09, 0.11]
3	[10.56, 20.16, 31.69] ± [0.09, 0.09, 0.11]
magnetometer bias = [100., 200., 300.] mG	
1	[100.16, 198.71, 302.99] ± [0.17, 0.26, 0.76]
2	[99.67, 199.53, 300.47] ± [0.09, 0.09, 0.11]
3	[99.67, 199.52, 300.49] ± [0.09, 0.09, 0.11]

Table 3. Performance of TWOSTEP for a "Realistic" Measurement Noise Simulation.

iteration	bias estimate (mG)
magnetometer bias = [10., 20., 30.] mG	
1	[9.85, 20.26, 30.53] ± [0.05, 0.08, 0.23]
2	[9.73, 20.45, 30.52] ± [0.03, 0.03, 0.03]
magnetometer bias = [100., 200., 300.] mG	
1	[98.82, 200.36, 300.02] ± [0.05, 0.08, 0.23]
2	[99.82, 200.36, 300.02] ± [0.03, 0.03, 0.03]

origin solely in the magnetometer, and occur because the individual magnetometer axes are not orthonormal due typically to thermal gradients within the magnetometer or to mechanical stresses from the spacecraft.

We assume now that the magnetometer measurements can be modeled as

$$\mathbf{B}_k = (I + D)^{-1}(\mathcal{O}^T A_k \mathbf{H}_k + \mathbf{b} + \boldsymbol{\varepsilon}_k), \tag{54}$$

where $\mathcal{O}$ is a proper orthogonal matrix (which is not observable from measurements of magnitudes alone), D is a fully-populated symmetric matrix and, therefore, depends on six parameters, which we may take to be the upper triangular elements of D.

To estimate D and $\mathbf{b}$ define the quantities

$$E \equiv 2D + D^2, \mathbf{c} = (I + D)\,\mathbf{b}. \tag{55ab}$$

The matrix E is symmetric but not diagonal. Thus, in terms of the quantities

$$z_k = -\mathbf{B}_k^{\mathrm{T}} E \mathbf{B}_k + 2\mathbf{B}_k^{\mathrm{T}}\mathbf{c} - |\mathbf{b}(\mathbf{c}, \mathbf{E})|^2 + v_k. \tag{56}$$

We may write

$$\mathbf{B}_k^{\mathrm{T}} E \mathbf{B}_k = K_k \mathbf{E}\,, \tag{57}$$

with

$$K_k \equiv [\, B_{1,k}^2,\ B_{2,k}^2,\ B_{3,k}^2,\ 2B_{1,k}B_{2,k},\ 2B_{1,k}B_{3,k},\ 2B_{2,k}B_{3,k}\,]\,, \tag{58a}$$

$$\mathbf{E} \equiv [\, E_{11},\ E_{22},\ E_{33},\ E_{12},\ E_{13},\ E_{23}\,]^{\mathrm{T}}\,. \tag{58b}$$

Thus,

$$z_k = -K_k\mathbf{E} + 2\,\mathbf{B}_k^{\mathrm{T}}\mathbf{c} + |\mathbf{b}(\mathbf{c},\,\mathbf{E})|^2 + v_k \tag{59a}$$

$$= L_k\,\boldsymbol{\theta}' + |\mathbf{b}(\boldsymbol{\theta}')|^2 + v_k\,, \tag{59b}$$

with

$$L_k \equiv [\,2\mathbf{B}_k^{\mathrm{T}}\ \mid\ -K_k\,]\,, \quad \boldsymbol{\theta}' \equiv \begin{bmatrix} \mathbf{c} \\ \mathbf{E} \end{bmatrix}\,. \tag{60}$$

$\boldsymbol{\theta}'$ is 9×1 and L_k 1×9. Defining

$$\overline{L} \equiv \overline{\sigma}^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} L_k\,, \qquad \widetilde{L}_k \equiv L_k - \overline{L}\,, \tag{61}$$

the centered and center measurements become

$$\tilde{z}_k = \widetilde{L}_k \cdot \boldsymbol{\theta}' + \tilde{v}_k\,, \quad k = 1,\,\ldots\,,\,N\ \overline{z} = \overline{L}\boldsymbol{\theta}' + |\mathbf{b}(\boldsymbol{\theta}')|^2 + \overline{v}\,. \tag{62ab}$$

Solving equations (46) for $\mathbf{b}(\boldsymbol{\theta}')$ leads to

$$|\mathbf{b}(\boldsymbol{\theta}')|^2 = \mathbf{c}^{\mathrm{T}}(I + D)^{-2}\mathbf{c} = \mathbf{c}^{\mathrm{T}}(I + E)^{-1}\mathbf{c} \tag{63}$$

The partial derivatives of $|\mathbf{b}(\boldsymbol{\theta}')|^2$ are given by

$$\frac{\partial}{\partial c_m} |\mathbf{b}(\boldsymbol{\theta}')|^2 = 2\,\left((I + E)^{-1}\mathbf{c}\right)_m\,, \tag{64a}$$

$$\frac{\partial}{\partial E_{m,n}} |\mathbf{b}(\boldsymbol{\theta}')|^2 = -(2 - \delta_{mn})\,\left((I + E)^{-1}\mathbf{c}\right)_m \left((I + E)^{-1}\mathbf{c}\right)_n\,, \tag{64b}$$

where $\left((I + E)^{-1}\mathbf{c}\right)_m$ denotes the mth element of $\left((I + E)^{-1}\mathbf{c}\right)$. Note that the intermediate parameters $\mathbf{c}$ and $\mathbf{E}$ have been introduced in order that the only nonlinear dependence will be found in $|\mathbf{b}(\boldsymbol{\theta}')|^2$.

The calculation of the centered estimate leads to

$$\widetilde{J} = \frac{1}{2} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left(\tilde{z}_k - \widetilde{L}_k\,\boldsymbol{\theta}' - \tilde{\mu}_k\right)^2 + \text{terms independent of } \boldsymbol{\theta}'\,, \tag{65}$$

whence,

$$\tilde{\boldsymbol{\theta}}'^{*} = \widetilde{P}_{\theta'\theta'} \sum_{k=1}^{N} \frac{1}{\sigma_k^2} (\tilde{z}_k - \tilde{\mu}_k)\,\widetilde{L}_k^{\mathrm{T}}\,, \qquad \widetilde{P}_{\theta'\theta'}^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \widetilde{L}_k \widetilde{L}_k^{\mathrm{T}}\,. \tag{66ab}$$

and the center cost function is

$$\overline{J}(\boldsymbol{\theta}') = \frac{1}{2\,\overline{\sigma}^2} (\overline{z} - \overline{L}\boldsymbol{\theta}' + |\mathbf{b}(\boldsymbol{\theta}')|^2 - \overline{\mu})^2\,. \tag{67}$$

The center contribution to the Fisher information matrix (for us the inverse covariance) is simply

$$\overline{P}^{-1}_{\theta'\theta'} = \frac{1}{\overline{\sigma}^2}\left(\overline{L} - \frac{\partial|\mathbf{b}|^2}{\partial\boldsymbol{\theta}'^{\mathrm{T}}}\right)^{\mathrm{T}}\left(\overline{L} - \frac{\partial|\mathbf{b}|^2}{\partial\boldsymbol{\theta}'^{\mathrm{T}}}\right) . \tag{68}$$

Following the calculation of $\mathbf{c}^*$ and $\mathbf{E}^*$, we must compute D^* and $\mathbf{b}^*$. To compute D^* we write

$$E^* = USU^{\mathrm{T}} , \tag{69}$$

where U is orthogonal and S diagonal,

$$S = \mathrm{diag}(s_1,\, s_2,\, s_3) . \tag{70}$$

We define W to be the diagonal matrix $\mathrm{diag}(w_1,\, w_2,\, w_3)$ satisfying

$$S = 2W + W^2 , \tag{71}$$

In general, the elements of S are much less than unity so that a solution will exist. The diagonal elements of W have the solution

$$w_j = -1 + \sqrt{1 + s_j} , \quad j = 1,\, 2,\, 3 . \tag{72}$$

The maximum likelihood estimate of the scale-factor-nonorthogonality matrix D is then given by

$$D^* = UWU^{\mathrm{T}} , \tag{73}$$

with U the orthogonal matrix of equation (50). The maximum likelihood estimate of the magnetometer bias vector is then given finally by

$$\mathbf{b}^* = (I + D^*)^{-1}\mathbf{c}^* . \tag{74}$$

To transform the covariance matrix of $\boldsymbol{\theta}'$ to the covariance matrix of $\boldsymbol{\theta}$ we perform the transformation

$$P_{\theta\theta} = \left(\frac{\partial(\mathbf{b},\, \mathbf{D})}{\partial(\mathbf{c},\, \mathbf{E})}\right) P_{\theta'\theta'} \left(\frac{\partial(\mathbf{b},\, \mathbf{D})}{\partial(\mathbf{c},\, \mathbf{E})}\right)^{\mathrm{T}} \tag{75}$$

where we have defined

$$\mathbf{D} \equiv [\, D_{11},\, D_{22},\, D_{33},\, D_{12},\, D_{13},\, D_{23}\,]^{\mathrm{T}} . \tag{76a}$$

$$\boldsymbol{\theta} = \begin{bmatrix} \mathbf{b} \\ \mathbf{D} \end{bmatrix} \tag{76b}$$

Then

$$\left(\frac{\partial(\mathbf{b},\, \mathbf{D})}{\partial(\mathbf{c},\, \mathbf{E})}\right) = \left(\frac{\partial(\mathbf{c},\, \mathbf{E})}{\partial(\mathbf{b},\, \mathbf{D})}\right)^{-1} = \begin{bmatrix} (I + D) & M_{cD}(\mathbf{b}) \\ O_{6\times 3} & 2\, I_{6\times 6} + M_{ED}(\mathbf{D}) \end{bmatrix}^{-1} \tag{77}$$

with

$$M_{cD}(\mathbf{b}) = \begin{bmatrix} b_1 & 0 & 0 & b_2 & b_3 & 0 \\ 0 & b_2 & 0 & b_1 & 0 & b_3 \\ 0 & 0 & b_3 & 0 & b_1 & b_2 \end{bmatrix} , \tag{78a}$$

and

$$M_{ED}(\mathbf{D}) = \begin{bmatrix} 2D_1 & 0 & 0 & 2D_4 & 2D_5 & 0 \\ 0 & 2D_2 & 0 & 2D_4 & 0 & 2D_6 \\ 0 & 0 & 2D_3 & 0 & 2D_5 & 2D_6 \\ D_4 & D_4 & 0 & D_1 + D_2 & D_6 & D_5 \\ D_5 & 0 & D_5 & D_6 & D_1 + D_3 & D_5 \\ 0 & D_6 & D_6 & D_5 & D_4 & D_2 + D_3 \end{bmatrix} . \tag{78b}$$

The extension of the calibration is possible only for the linear corrections of the magnetometer bias vector.

NUMERICAL EXAMPLES FOR THE ESTIMATION OF OTHER PARAMETERS

The algorithm for the estimation of magnetometer biass, scale factors, and nonorthogonality corrections was examined for the SAC-B spacecraft using the same simulation parameters as in the previous numerical section. The **TWOSTEP** algorithm converged to the correct solution with customary rapidity. We have simulated these results for the case of a fully populated matrix D. These results are shown in Table 4. The agreement is quite good and the errors in the estimates are consistent with the computed estimate error covariance matrix.

In Tables 5 and 6 we have examined the behavior of the algorithm when the measurement noise has been mismodeled. Table 5 used the colored noise model in simulating the measurements. Table 6 used the "realistic" noise model. In all three cases two orbits of data were used. The confidence intervals were calculated on the basis of the Gaussian statistics by the estimator. Clearly, despite the fact that the estimator continues to assume Gaussian white noise, the agreement is quite good. As expected, the actual errors are typically much larger than would be expected from the computed confidence intervals, which asssume a different noise model.

Table 4. Estimation of Magnetometer Biases, Scale factors, and Non-Orthogonality Corrections for the SAC-B Spacecraft using TWOSTEP. $\boldsymbol{\theta}$ is the parameter, $\tilde{\boldsymbol{\theta}}^*$ is the centered estimate, and $\boldsymbol{\theta}^*$ is the centered estimate with center correction.

	$\boldsymbol{\theta}$	$\tilde{\boldsymbol{\theta}}^*$	$\boldsymbol{\theta}^*$
b_1	200. mG	196.97 ± 2.7	191.00 ± 2.5
b_2	100. mG	87.93 ± 1.1	98.95 ± 1.0
b_3	−200. mG	-166.71 ± 4.0	-204.40 ± 3.0
D_{11}	.05	$.032 \pm .018$	$.022 \pm .018$
D_{22}	.10	$.110 \pm .014$	$.093 \pm .010$
D_{33}	.05	$.219 \pm .210$	$-.073 \pm .040$
D_{12}	.05	$.037 \pm .014$	$.056 \pm .005$
D_{13}	.05	$.070 \pm .050$	$.063 \pm .023$
D_{23}	.05	$-.018 \pm .056$	$.063 \pm .007$

Table 5. Estimation of Magnetometer Biases, Scale factors, and Non-Orthogonality Corrections for the SAC-B Spacecraft and Colored Measurement Noise using TWOSTEP. $\boldsymbol{\theta}$ is the parameter, $\widetilde{\boldsymbol{\theta}}^*$ is the centered estimate, and $\boldsymbol{\theta}^*$ is the centered estimate with center correction.

	$\boldsymbol{\theta}$	$\widetilde{\boldsymbol{\theta}}^*$	$\boldsymbol{\theta}^*$
b_1	30. mG	30.42 ± .23	30.61 ± .17
b_2	60. mG	60.17 ± .16	60.22 ± .14
b_3	90. mG	90.14 ± .18	90.21 ± .14
D_{11}	.05	.0505 ± .0016	.0514 ± .0010
D_{22}	.10	.0990 ± .0016	.0999 ± .0010
D_{33}	.05	.0494 ± .0015	.0503 ± .0010
D_{12}	.05	.0508 ± .0007	.0509 ± .0007
D_{13}	.05	.0509 ± .0007	.0509 ± .0007
D_{23}	.05	.0497 ± .0008	.0498 ± .0008

Table 6. Estimation of Magnetometer Biases, Scale factors, and Non-Orthogonality Corrections for the SAC-B Spacecraft and "Realistic" Measurement Noise using TWOSTEP. $\boldsymbol{\theta}$ is the parameter, $\widetilde{\boldsymbol{\theta}}^*$ is the centered estimate, and $\boldsymbol{\theta}^*$ is the centered estimate with center correction.

	$\boldsymbol{\theta}$	$\widetilde{\boldsymbol{\theta}}^*$	$\boldsymbol{\theta}^*$
b_1	30. mG	30.58 ± .08	30.73 ± .06
b_2	60. mG	50.76 ± .06	60.81 ± .05
b_3	90. mG	90.86 ± .06	90.92 ± .05
D_{11}	.05	.052 ± .0006	.053 ± .0003
D_{22}	.10	.102 ± .0006	.103 ± .0004
D_{33}	.05	.053 ± .0005	.053 ± .0003
D_{12}	.05	.050 ± .0002	.050 ± .0002
D_{13}	.05	.050 ± .0002	.050 ± .0002
D_{23}	.05	.050 ± ..0003	.050 ± .0003

DISCUSSION

A new algorithm, TWOSTEP, has been developed, which is efficient and robust, and which leads to a consistent estimate of the magnetometer bias at both steps of the algorithm. Its ability to converge in all cases (nearly 200 have been simulated by the authors) is due to the fact that, if the magnetometer bias is observable at all, the centering approximation will yield a consistent and unambiguous result. Thus, the center correction, in most cases, makes little improvement in the estimate.

Important components in the development of the algorithm was the correct treatment of the correlations introduced by the centering process and the avoidance of double counting of the measurements. We have shown that the correct treatment leads to a the centered negative-log-likelihood function which is the sum of squares.

An obvious characteristic of the centered estimate, the first step in **TWOSTEP** is that it is often good enough.[6] The Fisher information associated with $\tilde{\mathbf{b}}$ genuinely characterizes the quality of the centered estimate. A comparison of this and the Fisher information associated with the center term can be used to decide whether it is worthwhile to carry out the center correction.

Note that the variances σ_k^2 given by equation (7b) are functions of $\mathbf{b}$. We have taken them to be functions of the true value of $\mathbf{b}$ and not of the corresponding model variable which appears in the cost function. Had we taken $\mathbf{b}$ to be a parameter of σ_k^2 also, then we would have differentiated also the factors $1/\sigma_k^2$ and the terms $\log \sigma_k^2$ appearing in equation (10). This latter approach would, in principal, have been more correct, but might have led to convergence problems because of the nonlinearity. However, a consistent estimate of $\mathbf{b}$ can be obtained for any set of the values for the σ_k^2, so that the added complication of making σ_k^2 a function of $\mathbf{b}$ in the cost function is not justified. Nonetheless, for consistency, once $\tilde{\mathbf{b}}^*$ has been determined from our initial set of σ_k^2, which were computed using $\mathbf{b} = \mathbf{0}$, we have recomputed the σ_k^2 using $\tilde{\mathbf{b}}^*$ as the "true" value and repeated the centering step to obtain an "improved" but hardly very different value for $\tilde{\mathbf{b}}^*$. Thus, our two-step method typically incorporates at least two iterations in the first step alone, and combines both scoring and fixed-point techniques. In a more realistic calculation, of course, one should give up the approximation that the effective magnetometer errors are isotropic and white. However, experience has shown us that the estimates are not very sensitive to the choice of the σ_k^2, at least not for the SAC-B orbit, which never comes within 50 degrees of the poles. Thus, the choice of the σ_k^2 does not seem to be important to the estimation problem. The difficulties that have been encountered up to now in estimating the magnetometer bias vector without knowledge of the attitude did not arise from an unrealistic modeling of the error levels but rather from the improper treatment of the non-quadratic nature of the cost function. Our goal in developing the **TWOSTEP** algorithm was not to make insignificant gains in computation times but to develop an algorithm which was more reliable than its predecessors.

More interesting would be the computation of the parameters of Σ_k, which are of fundamental importance. However, experience has shown that the most significant errors are those associated with the magnetic field model, which, to be meaningful, should be represented in a topocentric coordinate system associated with the geomagnetic dipole field. Such a representation of Σ_k is impossible without a knowledge of the spacecraft attitude. One may question also the wisdom of modeling the geomagnetic field errors as a white Gaussian process. Therefore, the estimation of error-level parameters, except at the crudest level, is not appropriate to the present study. For a detailed discussion of the errors in geomagnetic field models the reader is referred to Langel [8] and the two special issues devoted to the Magsat mission [9, 10].

TWOSTEP provides insights into the nature of ill-conditioned cases. It is very clear from our discussion that observability of the magnetometer bias is tantamount to observability from the centered data alone. Thus, in order to measure the three components of the bias one requires at least four magnetometer measurements. Otherwise, the quadratic dependence of the measurement on the bias will lead to a two-fold ambiguity. In some cases the ambiguity can be eliminated, in others, however, the solution may remain indeterminate. This is a problem not of the method but of the data. Other methods will fail to produce a result with even greater frequency, and provide less understanding of the reasons for failure.

TWOSTEP has been shown to work well even when the assumption of white Gaussian statistics is incorrect. The main reason for this is that the separation of the cost function into $\tilde{J}(\mathbf{b})$ and $\bar{J}(\mathbf{b})$ doesn't really depend on the statistical assumptions. Thus, the **TWOSTEP** algorithm will lead to an exact minimization of the cost function even if the statistical assumptions are not justified. There is still a price to be paid for incorrectly modeled statistics, however, which is that the computed confidence intervals will not be correct, as we have already seen in the numerical examples.

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[6] Perhaps we should call it **ONESTEP**, not very danceable until one recalls the "hop!"

TWOSTEP is certainly more sophisticated statistically and more capable than its predecessor algorithms for attitude-independent magnetometer calibration, more efficient computationally, and more reliable. Perhaps, most importantly, the new algorithm makes manifest the physical quantities which determine the behavior of the bias estimator. We hasten to point out that the algorithm can only be as good as the validity of its statistical model. If the effective measurement noise is incorrectly modeled, then the new algorithm will certainly show systematic errors (if the μ_k have incorrect values) or at least larger errors. This has been seen in some of the cases examined above where the measurement noise has been intentionally mismodeled. Although the errors levels were much larger than the naive statistical predictions in this case, as expected, the accuracy level was certainly usable.

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