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PARAMETER INTERFERENCE IN DISTORTION AND ALIGNMENT CALIBRATION

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PARAMETER INTERFERENCE IN DISTORTION AND ALIGNMENT CALIBRATION

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Estimation algorithms are presented for the consistent determination inflight of focal-plane distortion parameters and alignment parameters for attitude sensing instruments. The ambiguity in the specification of the parameter sets is demonstrated and a simple prescription is given for removing it for the case of polynomial representations of the focal-plane distortion. We develop estimation algorithms for both the case in which the instrument determines the distortion coefficients and the three-axis attitude (alignment to an external coordinate system) as well as for the case in which the instrument measurements are used to determine the distortion coefficients and the relative alignment from another fixed sensor.

INTRODUCTION

The practical use of scientific instruments and attitude sensors on spacecraft generally requires that these be recalibrated after launch. In addition, due to changes in the spacecraft structure arising from thermal flexure and zero-gravity effects, the alignment of these devices must also be determined. For unmanned spacecraft, the separation of alignment and distortion corrections is non-trivial. In addition, it is made more complicated by the fact that we commonly represent spacecraft alignment in terms of transformations in three dimensions, while distortions of the focal plane are most conveniently treated in two dimensions. Thus, investigations of the interference of focal-plane distortion and misalignment are compromised by a fundamental difference in treatment. A previous report[1] developed the representation of

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rotations in the focal plane. In the present work, we develop and test algorithms which apply this representation to the estimation of sensor focal-plane distortion parameters and rotational parameters.

It is not always appreciated that distortion and misalignment are not independent transformations of the focal plane. Therefore, attempts to calibrate sensors both for distortion and misalignment corrections sometimes lead to estimates of the parameters which are not meaningful individually. In ground calibration, when the sensor alignment and the calibration are commonly referred to an optical alignment cube mounted on the sensor, no ambiguity arises. In space, unfortunately, one no longer has any knowledge of the orientation of the optical alignment cube, and the unambiguous separation of alignment and distortion parameters is no longer possible.

If one is interested only in representing the transformation of sensor data from the spacecraft to an inertial coordinate system then the inherent ambiguity will not lead to any errors in the analysis of the data. However, if one wishes to trend the alignments or correlate them with other spacecraft data, such as temperature, then meaningful results are unlikely to be obtained. The principal purpose of this report is to present a methodology in which the estimated misalignment and distortion parameters will be meaningful.

We begin by reviewing the representation of misalignment and focal-plane distortion developed in Ref. 1. We then study various ways of combining these two transformations of the focal plane, demonstrate the fundamental redundancy between them, and present a simple method which removes the redundancy. The distortion of a sensor focal plane is most frequently represented by a Taylor series in the focal-plane coordinates. However, the sensor vendor may sometimes specify a functional form for the distortion calibration functions derived from the physical nature of the sensor which are not simple polynomials. In that case, one must develop a means of transforming these function to account for the redundancy. We postpone the treatment of an arbitrary parameterization of the focal-plane distortion for a later report.

Having developed the proper unambiguous representation of alignment and distortion we develop specific algorithms for estimating these parameters. There are two cases to consider. In the first case, the rotation parameters characterize the alignment of one sensor with respect to another. Thus, in this case the rotation parameters are constant in time and both the rotation and the distortion parameters are global. In the second case, the rotation parameters characterize the entire attitude. In this case, therefore, the rotation parameters change as a function of time and are frame specific, while the distortion parameters are constant and, therefore, global. Thus the treatment of data sampled at different times will not be the same in the two cases. However, the significant component of the present problem is not the formulation of batch and sequential estimators but of a measurement model, which takes account of the special character of focal-plane distortion and alignment parameters. Thus, our focus in this paper is on the development of correct parameter set and the composite measurement model.

Generally, alignment and distortion calibration activities are carried out by different groups which communicate incompletely with one another. Neither of these two groups is concerned with the problem of parameter redundancy, since each uses a non-redundant parameter set for its own purposes. We show, however, that this practice can lead to random walk effects in the estimates of both the distortion and the alignment parameters.

GEOMETRY OF ALIGNMENT AND DISTORTION

Generally, we represent a direction in space by the 3×1 matrix of its components with respect to a basis. For this study we will choose the basis so that the z-axis is perpendicular to the focal plane of the sensor and passes through the origin of the focal plane and the origin of coordinate system, which is separated from the focal plane by a distance f. Hence the x- and y-axes lie nominally in the focal plane. We say nominally because distortion and misalignment make these statements somewhat inexact. The purpose of a distortion/alignment calibration is to find corrections which theoretically make this description exact.

We write, therefore, for a general direction in three-dimensional space

$$\hat{\mathbf{W}} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} . \tag{1}$$

The caret denotes a unit vector. Let us imagine that our sensor behaves like a perfect pin-hole camera with the focal plane is at a distance f from the origin. Then a straight line through the origin in the direction of $\hat{\mathbf{W}}$ will intersect the focal plane at coordinates (X, Y) in such a way that the proportionality $X : Y : f = W_1 : W_2 : W_3$ holds. Hence,

$$\frac{X}{f} = \frac{W_1}{W_3} \quad \text{and} \quad \frac{Y}{f} = \frac{W_2}{W_3} , \tag{2}$$

or

$$X = f \frac{W_1}{W_3} \quad \text{and} \quad Y = f \frac{W_2}{W_3} . \tag{3}$$

These equations may be inverted to give

$$\hat{\mathbf{W}} = \frac{1}{\sqrt{X^2 + Y^2 + f^2}} \begin{bmatrix} X \\ Y \\ f \end{bmatrix} . \tag{4}$$

It will be advantageous, however, to define specific focal-plane coordinates given by

$$x \equiv X/f \quad \text{and} \quad y \equiv Y/f , \tag{5}$$

so that

$$x = \frac{W_1}{W_3} \quad \text{and} \quad y = \frac{W_2}{W_3} , \tag{6}$$

leading to

$$\hat{\mathbf{W}} = \frac{1}{\sqrt{x^2 + y^2 + 1}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} . \tag{7}$$

In this way we achieve a representation of the distortion which does not depend explicitly on the dimensions of the pin-hole camera. In fact, these specific focal-plane coordinates do not depend on the instrument at all and may be used to represent the data for any focal-plane sensor, no matter what its construction. Thus, we have a universal set of coordinates.

PARAMETERIZATION OF DISTORTION AND ALIGNMENT

Part of the confusion surrounding alignment and distortion is that the two are represented in two different spaces. Generally, distortion is represented in terms of the focal-plane parameters, which we can write in terms of focal-plane vectors as

$$\mathbf{x}' = \mathbf{x} + F(\mathbf{x}), \tag{8}$$

or in terms of components as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix}. \tag{9}$$

Here $\mathbf{x}'$ is the observed focal-plane vector without correction for instrument distortion (i.e., the distorted vector), and $\mathbf{x}$ is the true (i.e., ideal) focal-plane vector without any distortion effects. Generally, one assumes that the two functions $F_1(x, y)$ and $F_2(x, y)$ are given by polynomial series

$$F_1(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, x^2 + a_{1,1}\, xy + a_{0,2}\, y^2 + \ldots \tag{10a}$$

$$F_2(x, y) = b_{0,0} + b_{1,0}\, x + b_{0,1}\, y + b_{2,0}\, x^2 + b_{1,1}\, xy + b_{0,2}\, y^2 + \ldots. \tag{10b}$$

Frequently, one terminates the series at second order, which leads to six terms in each component, or at third order, which leads to ten terms in each component (expansions to sixth order, or 28 terms in each component, are not unheard of). The functions $F_1(x, y)$ and $F_2(x, y)$ assume very small values over the focal plane of the sensors. Apart from the effects of misalignments, whose effect we will soon examine in detail, the largest terms will most likely be $a_{1,0}$ and $b_{0,1}$, which arise from thermal expansion of the focal plane.

Alignment, on the other hand, is a pure rotation, and we tend to represent it in the full three-dimensional space as an orthogonal transformation,[2,3] which we may write as[4]

$$S = \cos\theta\, I_{3\times3} + (1 - \cos\theta)\, \hat{\mathbf{n}}\, \hat{\mathbf{n}}^T + \sin\theta\, [[\, \hat{\mathbf{n}}\,]], \tag{11}$$

where θ is the angle of rotation and $\hat{\mathbf{n}}$, a unit vector, is the axis of rotation. As in Refs. 2 and 3, $[[\, \mathbf{v}\,]]$ denotes the 3×3 antisymmetric matrix

$$[[\, \mathbf{v}\,]] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}. \tag{12}$$

The Taylor expansions of Eqs. (10), if carried out to all orders can represent any transformation of the focal plane. Therefore, it must be true that a rotation of the focal plane about any of the three axes of the focal-plane coordinate system, must be expressible in terms of Eqs. (10) for the appropriate values of the coefficients $\{a_{0,0}, a_{1,0}, \ldots, b_{0,0}, b_{1,0}, \ldots\}$. These coefficients are, in fact, derived to all orders in Ref. 1. Therefore, it is trivially obvious that the rotations are redundant with distortions.

Clearly, one could treat the transformation of the sensor focal-plane totally in terms of distortions and never introduce a rotation matrix in our representation. Unfortunately, while the distortions of the focal plane arising from thermal distortion, zero gravity, and other environmental effects are small, the rotation of the sensor focal plane (particularly, if we measure this rotation from inertial axes) can be quite large. Therefore, the Taylor series would require an infinite number of terms. The intelligent approach, therefore, is to treat the rotation to all orders by expressing it in terms of the 3 × 3 rotation matrix (or the quaternion, or the Euler angles, etc.), and removing those degrees of freedom from the distortion equation, Eqs. (10), so that the expansion will be non-redundant. The ultimate goal of this memo is to show exactly how to accomplish this.

In order to speak about rotations and distortions together, we must be able to speak about them first in the same terms. It will be easier to represent rotations in terms of focal-plane coordinates than to express focal-plane distortions in terms of a 3 × 3 matrix.

Consider the equation

$$\mathbf{U} \equiv R \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} R_{11}x + R_{12}y + R_{13} \\ R_{21}x + R_{22}y + R_{23} \\ R_{31}x + R_{32}y + R_{33} \end{bmatrix} . \tag{13}$$

The third component of the vector $\mathbf{U}$ is obviously not unity, and therefore the first two components of $\mathbf{U}$ do not correspond to focal-plane coordinates. To make the third component unity, however, we simply divide the right member of Eq. (13) by the third component to obtain

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{1}{R_{31}x + R_{32}y + R_{33}} R \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} . \tag{14}$$

or

$$x' = \frac{R_{11}x + R_{12}y + R_{13}}{R_{31}x + R_{32}y + R_{33}} , \tag{15a}$$

$$y' = \frac{R_{21}x + R_{22}y + R_{23}}{R_{31}x + R_{32}y + R_{33}} . \tag{15b}$$

This is the focal-plane representation of a transformation in three dimensions. Equations (15) hold for any linear transformation, although in this case, R is a rotation matrix. Equations (15) are known generally as the collinearity equations and play an important role in satellite photogrammetry.[5,6]

CORRESPONDENCE OF ALIGNMENT AND DISTORTION FOR INFINITESIMAL ROTATIONS

Let us examine an infinitesimal rotation, which we write as

$$R(\boldsymbol{\theta}) \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} . \tag{16}$$

From Ref. 1, we know that the focal-plane distortion coefficients for a pure rotation are given by

$$a_{ij} = (-1)^{i+j+1} \frac{R_{31}^{i-1} R_{32}^{j-1}}{R_{33}^{i+j+1}} \left[\binom{i+j-1}{i} R_{31} (R_{12} R_{33} - R_{13} R_{32}) + \binom{i+j-1}{j} R_{32} (R_{11} R_{33} - R_{13} R_{31}) \right] , \quad (17a)$$

and

$$b_{ij} = (-1)^{i+j+1} \frac{R_{31}^{i-1} R_{32}^{j-1}}{R_{33}^{i+j+1}} \left[\binom{i+j-1}{i} R_{31} (R_{22} R_{33} - R_{23} R_{32}) + \binom{i+j-1}{j} R_{32} (R_{21} R_{33} - R_{23} R_{31}) \right] , \quad (17b)$$

where $\binom{n}{i}$ is the binomial coefficient

$$\binom{n}{i} \equiv \begin{cases} \dfrac{n!}{i!\,(n-i)!} & \text{for } 1 \leq i \leq n \,, \\ 0 & \text{otherwise} \end{cases} . \quad (18)$$

Substituting Eq. (16) leads to the corresponding expressions for infinitesimal rotations

$$a_{ij} = (-1)^i \, \theta_2^{i-1} \, \theta_1^{j-1} \left[\binom{i+j-1}{i} \theta_2 \theta_3 - \binom{i+j-1}{j} \theta_1 \right] , \quad (19a)$$

$$b_{ij} = (-1)^i \, \theta_2^{i-1} \, \theta_1^{j-1} \left[\binom{i+j-1}{i} \theta_2 + \binom{i+j-1}{j} \theta_1 \theta_3 \right] , \quad (19b)$$

where we have kept only the leading terms. Clearly, $a_{i,j}$ and $b_{i,j}$ will be at least quadratic in the angles for $i+j > 2$. Thus, infinitesimal rotations contribute to only a few distortion terms. If we evaluate Eqs. (19) for $i+j \leq 2$, we will find to first order in the infinitesimal angles that

$$x' = x - \theta_2 + \theta_3 \, y - \theta_2 \, x^2 + \theta_1 \, xy + \dots \,, \quad (20a)$$

$$y' = y + \theta_1 - \theta_3 \, x - \theta_2 \, xy + \theta_1 \, y^2 + \dots \,. \quad (20b)$$

so that eight of the first twelve distortion terms are affected by infinitesimal rotations. These same equations follow also by inserting Eq. (16) directly into Eqs. (15). Within an infinitesimal region of the center of the focal plane, corresponding to $(x, y) = (0, 0)$, we have that

$$x' = x - \theta_2 + \theta_3 \, y + \dots \,, \quad (21a)$$

$$y' = y + \theta_1 - \theta_3 \, x + \dots \,. \quad (21b)$$

In any practical application (see examples below), we will wish to drive the three angles to zero. Thus, these are the only terms we have to consider infinitesimally near the center of the focal plane.

Clearly, if we wish to remove the redundancy between the distortion parameters and rotational parameters from an examination of the behavior of the coordinates at the center of the focal plane, then we must set

$$a_{0,0} = b_{0,0} = 0\,, \tag{22}$$

since the action of these parameters is indistinguishable from that of $-\theta_2$ and θ_1. For θ_3 the argument is slightly more complicated. If we rewrite the first few terms of Eq. (9) as

$$x' = a_{0,0} + a_{1,0}\,x + [(a_{0,1} + b_{1,0})/2 + (a_{0,1} - b_{1,0})/2]\,y + \dots\,, \tag{23a}$$

$$y' = b_{0,0} + [(a_{0,1} + b_{1,0})/2 - (a_{0,1} - b_{1,0})/2]\,x + b_{0,1}\,y + \dots\,, \tag{23b}$$

Then we see that at the center of the field of view the action of an infinitesimal θ_3 is indistinguishable from that of $(a_{0,1} - b_{1,0})/2$. Therefore, to remove the redundancy, we must set

$$a_{0,1} - b_{1,0} = 0\,. \tag{24}$$

This is not the only choice we could have made. For example, we might have chosen to set

$$a_{2,0} = b_{0,2} = a_{0,1} - b_{1,0} = 0\,. \tag{25}$$

since $a_{2,0}$ and $b_{0,2}$ are also effected linearly by the infinitesimal rotations. Mathematically, this is a perfectly acceptable prescription, since it leads to a non-redundant set of variables. Physically, however, such a choice would be disastrous, since it would lead to the rotation parameters being influenced enormously by non-linear distortions of the focal plane and what we would physically consider to be the misalignment of the center of the focal plane would be dominate several of the focal-plane distortion parameters. The distortion parameters would then be very sensitive to the attitude of the sensor, and could therefore be very large. Such a prescription is mathematically consistent, but it would not accomplish the goal of suppressing a strong attitude dependence in the distortion parameters, and would be even more confusing than using the complete set of distortion parameters without explicit parameterization of a rotation. We choose, therefore, as our focal-plane parameters set the three parameters of the attitude and the distortion parameters with the constraint that

$$a_{0,0} = b_{0,0} = a_{0,1} - b_{1,0} = 0\,. \tag{26}$$

ORDER OF DISTORTION AND ALIGNMENT TRANSFORMATIONS

Let $\mathcal{E}$ denote the inertial frame and $\mathcal{I}$ denote the instrument frame. Likewise, let $\mathcal{D}$ denote the distortion transformation and $\mathcal{R}$ denote the rotation transformation expressed in focal plane coordinates. Thus, we may write the change in focal-plane coordinates due to distortion alone by

$$\mathbf{x}' = \mathcal{D}(\mathbf{x}, \Lambda)\,, \tag{27}$$

where Λ denotes the totality of distortion parameters (except, naturally for $a_{0,0}$ and $b_{0,0}$, which now vanish identically and for $b_{1,0}$, which is given identically by $a_{0,1}$). The change in focal-plane coordinates due to rotation alone is given by

$$\mathbf{x}' = \mathcal{R}(\mathbf{x}, \boldsymbol{\theta})\,, \tag{28}$$

where $\boldsymbol{\theta}$ is the (finite) rotation vector characterizing the rotation. These equations are simply a shorthand for Eqs. (10) and (15). If we consider the effect of both misalignment and distortion, then we are led to describe the combined effect either by writing

$$\mathbf{x}' = \mathcal{D}(\mathbf{x}_m, \Lambda), \tag{29a}$$

$$\mathbf{x}_m = \mathcal{R}(\mathbf{x}, \boldsymbol{\theta}). \tag{29b}$$

for the alignment transformation computed first followed by the distortion or

$$\mathbf{x}' = \mathcal{R}(\mathbf{x}_d, \boldsymbol{\theta}'), \tag{30a}$$

$$\mathbf{x}_d = \mathcal{D}(\mathbf{x}, \Lambda'), \tag{30b}$$

with the transformations taking place in the opposite order. Thus, we either first misalign and then distort or we first distort and then misalign. If $\boldsymbol{\theta}$ is large, then clearly there will be a large difference between the instrument frame and the inertial frame. If we perform the distortion first, then effectively, we are computing the distortion coefficients with respect to inertial axes, which makes little sense. Also, distortion is a phenomenon which we identify with the instrument itself, while misalignment is associated more with the spacecraft structure. But if we distort the focal plane before rotating it, then the focal-plane parameters will depend on the attitude, which we want to avoid. Thus, physically the first alternative, Eqs. (29), is more natural, and from a practical standpoint it is much simpler.

FRAMES OF REFERENCE

Consider first the case in which the rotational parameters represent the alignment. The convention which was adopted in Refs. 2 and 3 for misalignment is not entirely suited to the study of misalignment and distortion. Let $\hat{\mathbf{U}}_{i,k,\ell}$ denote the ℓ-th vector observed by sensor i at time t_k in the sensor coordinate system and let $\hat{\mathbf{V}}_{i,k,\ell}$ and $\hat{\mathbf{W}}_{i,k,\ell}$ be the corresponding column vectors in inertial coordinates and body coordinates. Then the attitude matrix causes the transformation

$$\hat{\mathbf{W}}_{i,k,\ell} = A_k \hat{\mathbf{V}}_{i,k,\ell} + \Delta\hat{\mathbf{W}}_{i,k,\ell}, \tag{31}$$

while the alignment transformation causes the transformation

$$\hat{\mathbf{W}}_{i,k,\ell} = S_i \hat{\mathbf{U}}_{i,k,\ell}. \tag{32}$$

Here $\Delta\hat{\mathbf{W}}_{i,k,\ell}$ is the measurement noise. (Note that the measurement noise appears only implicitly in Eq. (32).) Clearly, the attitude must depend on the time but not on the specific sensor, and the alignment matrix depends on the sensor but not on the time. This last statement is an idealization, because in practice alignments can change with time due, for example, to changes in the thermal loading of the spacecraft.

In Ref. 2, if S_i^o was an *a priori* value of the alignment matrix, and S_i was the correct alignment, then the misalignment matrix M_i^B (written simply as M_i in Refs. 2 and 3) was defined so that

$$S_i = M_i^B S_i^o. \tag{33}$$

From Eq. (32) it is clear that $M_i^{\mathcal{B}}$ is a transformation from an *a priori* body frame to the correct body frame. Combining Eqs. (31), (32) and (33) we have that

$$\hat{\mathbf{U}}_{i,k,\ell} = S_i^T \hat{\mathbf{W}}_{i,k,\ell} \tag{34a}$$

$$= S_i^T A_k \hat{\mathbf{V}}_{i,k,\ell} + \Delta\hat{\mathbf{U}}_{i,k,\ell} \tag{34b}$$

$$= S_i^{oT} M_i^{\mathcal{B}T} A_k \hat{\mathbf{V}}_{i,k,\ell} + \Delta\hat{\mathbf{U}}_{i,k,\ell} . \tag{34c}$$

It is the parameters of $M_i^{\mathcal{B}T}$ that we wish to estimate from the measurements $\hat{\mathbf{U}}_{i,k,\ell}$. These, obviously, are not convenient because of the transformation S_i^{oT}. We therefore, define an instrument-referenced misalignment according to

$$S_i = S_i^o M_i^{\mathcal{I}} , \tag{35}$$

so that

$$M_i^{\mathcal{B}} = S_i^o M_i^{\mathcal{I}} S_i^{oT} . \tag{36}$$

Writing in the usual way

$$M_i^{\mathcal{B}} \approx I_{3\times3} + [[\, \boldsymbol{\theta}_i^{\mathcal{B}}]] \quad \text{and} \quad M_i^{\mathcal{I}} \approx I_{3\times3} + [[\, \boldsymbol{\theta}_i^{\mathcal{I}}]] , \tag{37}$$

it follows that

$$\boldsymbol{\theta}_i^{\mathcal{B}} = S_i^o \boldsymbol{\theta}_i^{\mathcal{I}} . \tag{38}$$

Thus, we can write

$$\hat{\mathbf{U}}_{i,k,\ell} = M_i^{\mathcal{I}T} S_i^{oT} A_k \hat{\mathbf{V}}_{i,k,\ell} + \Delta\hat{\mathbf{U}}_{i,k,\ell} \tag{39a}$$

$$= M_i^{\mathcal{I}T} \hat{\mathbf{U}}_{i,k,\ell}^o + \Delta\hat{\mathbf{U}}_{i,k,\ell} , \tag{39b}$$

$$\approx \hat{\mathbf{U}}_{i,k,\ell}^o - [[\, \boldsymbol{\theta}^{\mathcal{I}}]]\, \hat{\mathbf{U}}_{i,k,\ell}^o + \Delta\hat{\mathbf{U}}_{i,k,\ell} , \tag{39c}$$

$$= \hat{\mathbf{U}}_{i,k,\ell}^o + [[\, \hat{\mathbf{U}}_{i,k,\ell}^o]]\, \boldsymbol{\theta}_i^{\mathcal{I}} + \Delta\hat{\mathbf{U}}_{i,k,\ell} , . \tag{39d}$$

where

$$\hat{\mathbf{U}}_{i,k,\ell}^o \equiv S_i^{oT} A_k \hat{\mathbf{V}}_{i,k,\ell} \tag{40}$$

is the *a priori* value of the instrument measurement in the instrument frame given the *a priori* alignment matrix and the attitude.

In practical calculations, one does not know A_k but only A_k^*, the estimated attitude, given by

$$A_k^* = e^{[[\, \boldsymbol{\xi}_k]]} A_k . \tag{41}$$

Here, $\boldsymbol{\xi}_k$ is the attitude estimation error, assumed to be Gaussian, zero-mean, and having covariance $P_{\xi,k}$. In this case Eq. (39a) becomes

$$\hat{\mathbf{U}}_{i,k,\ell} = M_i^{\mathcal{I}T} S_i^{oT} e^{-[[\, \boldsymbol{\xi}_k]]} A_k^* \hat{\mathbf{V}}_{i,k,\ell} + \Delta\hat{\mathbf{U}}_{i,k,\ell} , \tag{42}$$

which leads to

$$\hat{\mathbf{U}}_{i,k,\ell} = \hat{\mathbf{U}}_{i,k,\ell}^o + [[\, \hat{\mathbf{U}}_{i,k,\ell}^o]]\, \boldsymbol{\theta}_i^{\mathcal{I}} + \Delta\hat{\mathbf{U}}_{i,k,\ell} + [[\, \hat{\mathbf{U}}_{i,k,\ell}^o]]\, S_i^{oT} \boldsymbol{\xi}_k , \tag{43}$$

where now

$$\hat{\mathbf{U}}^{o}_{i,k,\ell} \equiv S^{o\,T}_{i} A^{*}_{k} \hat{\mathbf{V}}_{i,k,\ell}\,, \tag{44}$$

and there is an extra noise term.

In general, the complete set of absolute alignments cannot be determined with high confidence.[3] Therefore, it is common in practice to define one of the misalignment matrices $M^{\mathcal{I}}_{i}$ to be equal to the identity matrix. This effectively defines the spacecraft body coordinate system as a fixed rotation from the coordinate system of sensor i, and the sensor alignments become effectively coalignments.

Equation (39) assumes that A_k the spacecraft attitude matrix is known from some source. This may not always be the case. For example, if wish to compute instrument coalignment and distortion coefficients by first computing instrument attitude and distortion coefficients for each sensor (this assumes that each sensor can measure multiple directions) and then computing the coalignments from the individual instrument attitudes. In that situation, we define

$$A_{i,k} = S^{T}_{i} A_{k}\,, \tag{45}$$

as the instrument attitude. Then

$$A_{i,k} = (\delta A_{i,k})\, A^{o}_{i,k}\,, \tag{46}$$

where $A^{o}_{i,k}$ is the *a priori* attitude of the instrument and $\delta A_{i,k}$ is the attitude correction, which we hope is also a small correction.

The complete set of transformations which we apply to obtain the distorted and misaligned datum from the given *a priori* knowledge is as follows:

- Given $\hat{\mathbf{V}}_{i,k,\ell}$, $\ell = 1, \ldots, n_{i,k}$ and the *a priori* instrument attitude (or $\hat{\mathbf{V}}_{i,k,\ell}$, $\ell = 1, \ldots, n_{i,k}$, the spacecraft attitude and the *a priori* instrument alignment matrix) we compute $\hat{\mathbf{U}}^{o}_{i,k,\ell}$, $\ell = 1, \ldots, n_{i,k}$, and we write

  $$\hat{\mathbf{U}}_{i,k,\ell} = (\delta C)\, \hat{\mathbf{U}}^{o}_{i,k,\ell}\,, \quad \ell = 1, \ldots, n_{i,k}\,,$$

  where δC denotes $\delta A_{i,k}$ (or $M^{\mathcal{I}\,T}_{i}$).

- Using Eq. (6) we compute $\mathbf{x}$, the focal-plane representation of $\hat{\mathbf{U}}^{o}_{i,k,\ell}$.

- Using Eq. (20) we compute $\mathbf{x}_m$, the focal-plane coordinates corrected for misalignment.

- Using Eqs. 10 and the reduced set of coefficients (Eq. (21)), we compute the distorted focal-plane coordinates $\mathbf{x}'$.

THE MEASUREMENT MODEL

We are now prepared to estimate alignment and distortion parameters, that is, the parameters of δC and the reduced set of distortion parameters $\{ a_{1,0}, a_{0,1}, a_{2,0}, \ldots, b_{0,1}, b_{2,0}, \ldots \}$.

The two cases presented above, in which one is estimating either the parameters of $M_i^{\mathcal{I}}$ or of $\delta A_{i,k}$ are fundamentally different. If knowledge of the attitude is available from some source other than the instrument being considered, then we can regard δC (i.e., $M_i^{\mathcal{I}}$) as being the same for every frame of data (i.e., every value of k). All of the parameters being estimated in this case are global. However, in the case where attitude is being determined from the same data, then δC (i.e., $A_{i,k}$ in this case) will be different in every frame. We must, therefore, estimate a mixture of global and frame-specific parameters. We will examine these two cases both within the framework of batch estimation and within the framework of the Kalman filter.

We compute the measurement model (i.e., the measurement sensitivity matrix) in three steps. In each case above we are given an *a priori* rotation, which we denote here by $R^o_{i,k}$, and a set of *a priori* measurement vectors $\hat{\mathbf{U}}^o_{i,k,\ell}$, $\ell = 1, \ldots, n_{i,k}$. From this $R^o_{i,k}$ and the *a priori* directions we compute $\mathbf{x}(i, k, \ell)$, the *a priori* focal plane coordinates, according to Eq. (15). We then apply the (infinitesimal) rotation correction $\delta R_{i,k}$, which in focal-plane coordinates is accomplished by

$$\mathbf{x}_m(i, k, \ell) \equiv \begin{bmatrix} x_m(i, k, \ell) \\ y_m(i, k, \ell) \end{bmatrix} \tag{47a}$$

$$= \begin{bmatrix} x(i, k, \ell) \\ y(i, k, \ell) \end{bmatrix} + \begin{bmatrix} -\theta_2 + \theta_3\, y - \theta_2\, x^2 + \theta_1\, xy \\ \theta_1 - \theta_3\, x - \theta_2\, xy + \theta_1\, y^2 \end{bmatrix}_{i,k,\ell} + O(|\boldsymbol{\theta}|^2)\,. \tag{47b}$$

Finally, we apply the distortion correction

$$\mathbf{x}' \equiv \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x_m \\ y_m \end{bmatrix} + \begin{bmatrix} F_1(x_m, y_m) \\ F_2(x_m, y_m) \end{bmatrix}, \tag{48}$$

and

$$F_1(x_m, y_m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}\, x_m^i y_m^j\,, \tag{49a}$$

$$F_2(x_m, y_m) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{i,j}\, x_m^i y_m^j\,, \tag{49b}$$

with the restriction that

$$a_{0,0} = 0\,, \quad b_{0,0} = 0\,, \quad \text{and} \quad b_{1,0} = a_{0,1}\,. \tag{50}$$

For simplicity we have not written the subscripts i,k,ℓ for each pair of focal-plane coordinates.

The measurement is given by

$$\mathbf{z}_{i,k,\ell} = \mathbf{x}'_{i,k,\ell} + \mathbf{v}_{i,k,\ell} \tag{51}$$

where $\mathbf{v}_{i,k,\ell}$ is the measurement noise, which we assume generally to be white and Gaussian.

We denote the complete set of parameters (rotation plus distortion) by

$$\boldsymbol{\lambda} \equiv \begin{bmatrix} \boldsymbol{\theta} \\ \Lambda \end{bmatrix} \tag{52a}$$

$$= [\theta_1, \theta_2, \theta_3, a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, \ldots, b_{0,1}, b_{2,0}, b_{1,1}, \ldots]^T\,. \tag{52b}$$

Since we want to estimate the parameters, we wish to write

$$\mathbf{z}_{i,k,\ell} = \mathbf{z}^{o}_{i,k,\ell} + H_{i,k,\ell}\boldsymbol{\lambda} + \mathbf{v}_{i,k,\ell}\,, \tag{53}$$

with

$$\mathbf{z}^{o}_{i,k,\ell} = \mathbf{x}_{i,k,\ell}\,, \tag{54}$$

which was determined from the $\hat{\mathbf{U}}^{o}_{i,k,\ell}$, $\ell = 1, \dots, n_{i,k}$, and $R^{o}_{i,k}$, and we may partition the measurement sensitivity matrix as

$$H_{i,k,\ell} = \left[H_{\theta\, i,k,\ell} \;\; H_{\Lambda\, i,k,\ell} \right] . \tag{55}$$

Clearly,

$$H_{\theta\, i,k,\ell} = \frac{\partial \mathbf{z}_{i,k,\ell}}{\partial \boldsymbol{\theta}} = \sum_{j=1}^{2} \frac{\partial \mathbf{z}_{i,k,\ell}}{\partial x_m(i, j, \ell)} \frac{\partial x_m(i, j, \ell)}{\partial \boldsymbol{\theta}}\,, \tag{56}$$

and

$$H_{\Lambda\, i,k,\ell} = \frac{\partial \mathbf{z}_{i,k,\ell}}{\partial \Lambda}\,. \tag{57}$$

The individual partial derivatives are given by

$$\frac{\partial \mathbf{z}}{\partial x_m} = \begin{bmatrix} 1 + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} i\, a_{i,j}\, x_m^{i-1} y_m^{j} \\ \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} i\, b_{i,j}\, x_m^{i-1} y_m^{j} \end{bmatrix} \tag{58a}$$

$$\frac{\partial \mathbf{z}}{\partial y_m} = \begin{bmatrix} \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} j\, a_{i,j}\, x_m^{i} y_m^{j-1} \\ 1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} j\, b_{i,j}\, x_m^{i} y_m^{j-1} \end{bmatrix} \tag{58b}$$

$$\frac{\partial x_m}{\partial \boldsymbol{\theta}} = \left[x_m y_m \quad (-1 - x_m^2) \quad y_m \right] \tag{59a}$$

$$\frac{\partial y_m}{\partial \boldsymbol{\theta}} = \left[(1 + y_m^2) \quad -x_m y_m \quad -x_m \right] \tag{59b}$$

$$\frac{\partial \mathbf{z}}{\partial a_{i,j}} = \begin{bmatrix} x_m^i y_m^j \\ 0 \end{bmatrix} \quad \text{for} \quad (i, j) \neq (0, 1) \text{ or } (0, 0)\,, \tag{60a}$$

$$\frac{\partial \mathbf{z}}{\partial b_{i,j}} = \begin{bmatrix} 0 \\ x_m^i y_m^j \end{bmatrix} \quad \text{for} \quad (i, j) \neq (1, 0) \text{ or } (0, 0)\,, \tag{60b}$$

$$\frac{\partial \mathbf{z}}{\partial a_{0,1}} = \begin{bmatrix} y_m \\ x_m \end{bmatrix}. \tag{60c}$$

In all of the above formulas we emphasize once more that $a_{0,0} = b_{0,0} = 0$ and $b_{1,0} = a_{1,0}$. In the formulas above we have adopted the convention that the derivative of a scalar with respect to a column vector is a row vector.

NUMERICAL EXAMPLES

To illustrate the need to estimate alignment and distortion parameters in this way we have considered a typical example. In normal operations (Method 1), the flight operations team will estimate the attitude and the alignments, assuming a given set of distortion parameters, while distortion parameters will be estimated by the instrument team, assuming a given attitude and alignment. Generally, each team will use different data sets and carry out their operations at different times. Since each team's parameter sets are well defined, there will be no problem of redundancy in each individual case. Communication between each team is usually limited to transmitting the estimated parameter sets. This is the normal mode of operations for all spacecraft known to the authors. (In fact, this is the normal mode even if the distortion coefficients and the misalignments are being estimated by the same team!)

Clearly, there will be a redundancy of parameters between the two teams but not for each individual teams activities. In the figures which follow we have considered this method, which we label "Method 1," and two other methods. Method 2 is identical to Method 1 except that the focal-plane distortion coefficients have been constrained to have

$$a_{0,0} = b_{0,0} = a_{0,1} - b_{1,0} = 0 .$$

In Method 3 the alignments and constrained distortion coefficients are estimated simultaneously. In each case we assumed that the true value of all of the distortion coefficients was 0 and the attitude matrix of the sensor was the identity matrix, which means that the true value of the misalignments was zero. For sixteen frames of data, 50 directions were measured in each frame. It was assumed that the error in each direction measurement was 1 deg/axis. The sensor field of view was 20 deg full width in each direction.

Since the true value of all of the parameters is zero, the estimates are equivalent to the estimate errors. Ten experiments were performed for each method and the sampled covariance computed. Figure 1 shows the results for the estimate of θ_1. The solid line shows the result of estimating the misalignment and the reduced (non-redundant) set of distortion parameters simultaneously (Method 3). The dotted and dashed line gives the case of alternate estimation of the misalignments and the reduced set of distortion coefficients (Method 2). The dashed line shows Method 1, in which alternately one estimates the misalignments and the redundant set of parameters (Method 1). Methods 2 and 3 are barely distinguishable, while Method 1 shows a large random walk. The same phenomenon is apparent in the estimation of θ_2 and θ_3, shown in Figures 2 and 3. Figure 4 shows the performance for $a_{1,0}$, which is not expected to be as sensitive to the redundancy with the misalignment parameters. In fact, the estimate errors for this quantity shows the same behavior for all three methods.

Figure 5 shows the behavior of $b_{0,0}$ (dotted and dashed line), θ_1 (dashed line) and $b_{0,0} + \theta_1$ (solid line) when these are estimated using Method 1. The obvious random walk in the sample variances or the individual variables contrasts dramatically with the more or less constant behavior of the sum of the two estimates, as was to be expected.

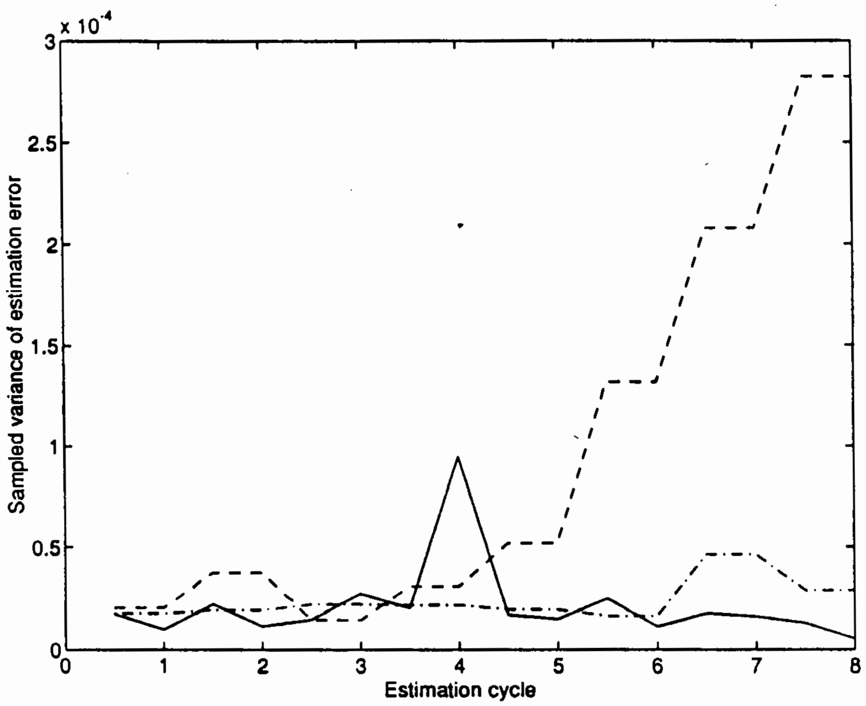

Figure 1. Estimation of θ_1

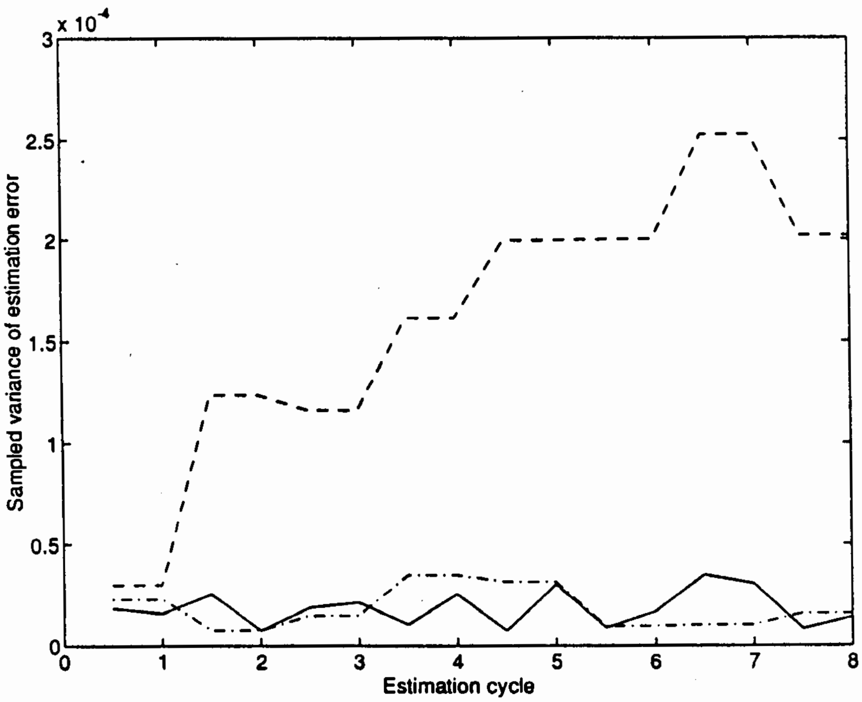

Figure 2. Estimation of θ_2

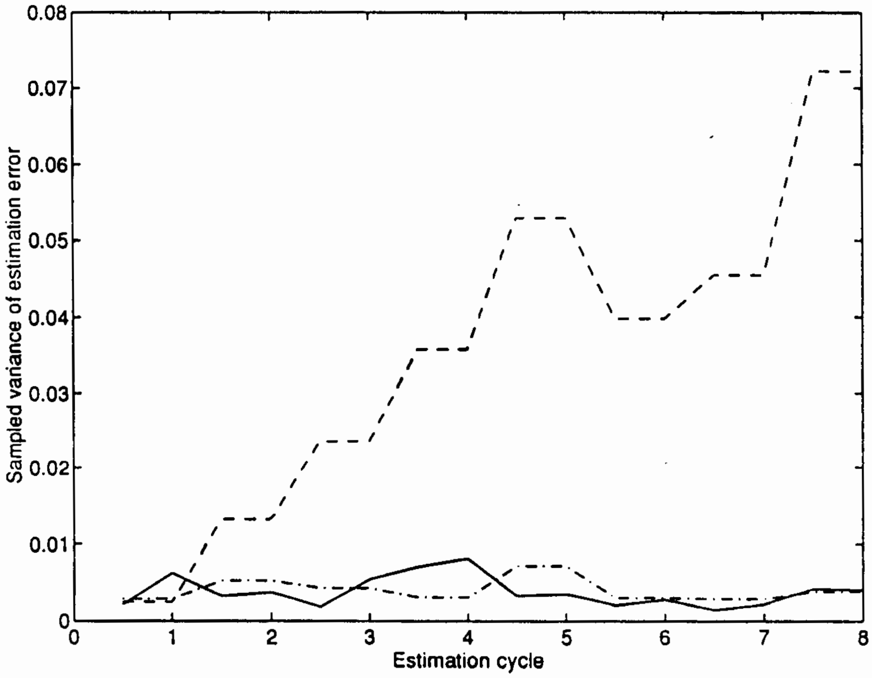

Figure 3. Estimation of θ_3

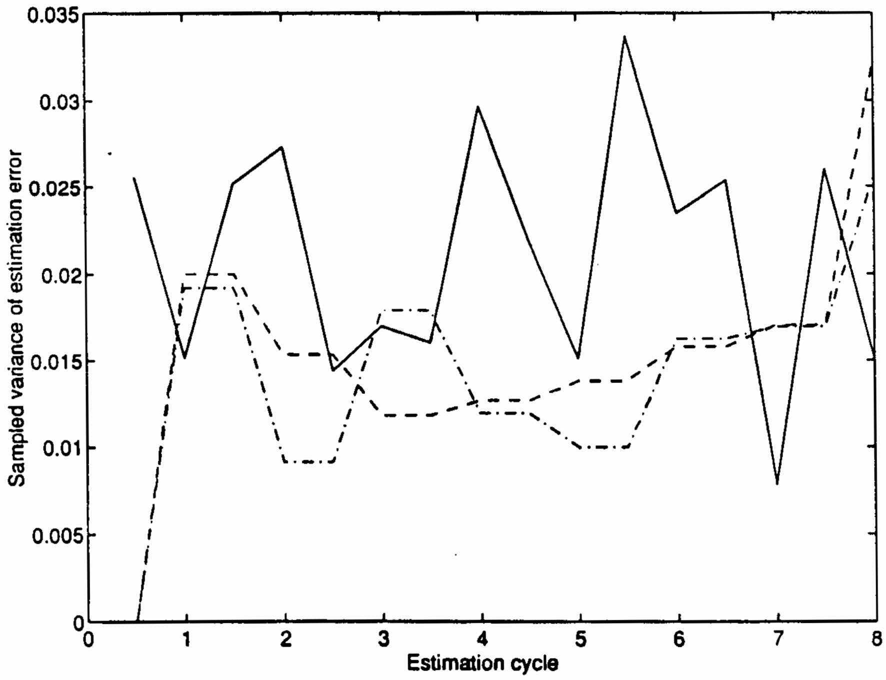

Figure 4. Estimation of $a_{1,0}$

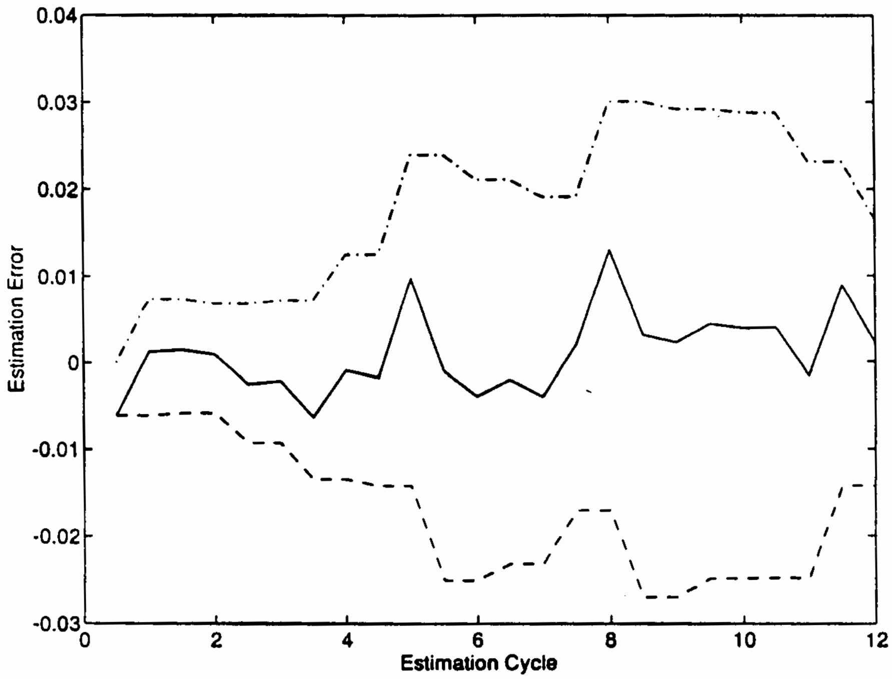

Figure 5. Estimation of θ_1 and $b_{0,0}$

To understand the random walk phenomenon, consider the following simple model

$$\mathbf{z}_k = \boldsymbol{\theta} + \mathbf{a} + \mathbf{v}_k \,. \tag{61}$$

and $\mathbf{v}_k$ is a white Gaussian sequence with covariance R. Here, $\boldsymbol{\theta}$ represents the misalignments and $\mathbf{a}$ represents the corresponding redundant distortion coefficients. The above model is somewhat simplification but contains the important component that the two parameter sets are completely redundant. This measurement model is not far-fetched, however. In practice, it can always be constructed as the maximum-likelihood estimate of $(\boldsymbol{\theta} + \mathbf{a})$.

If the variable $\boldsymbol{\theta}$ is estimated at the odd times assuming the previous estimate (or *a priori* value) for $\mathbf{a}$, then we obtain for the estimate of $\boldsymbol{\theta}_{2k+1}$

$$\boldsymbol{\theta}^*_{2k+1} = \mathbf{z}_{2k+1} - \mathbf{a}^*_{2k} \,. \tag{62}$$

Likewise at the next interval, estimating $\mathbf{a}$ assuming the previous estimate for $\boldsymbol{\theta}$ yields

$$\mathbf{a}^*_{2k+2} = \mathbf{z}_{2k+2} - \boldsymbol{\theta}^*_{2k+1} \,. \tag{63}$$

Combining these two results yields

$$\mathbf{a}^*_{2k+2} = \mathbf{a}^*_{2k} + \mathbf{z}_{2k+2} - \mathbf{z}_{2k+1} = \mathbf{a}^*_{2k} + \mathbf{v}_{2k+2} - \mathbf{v}_{2k+1} \,. \tag{64}$$

Likewise for the estimates of $\boldsymbol{\theta}$ one obtains

$$\boldsymbol{\theta}^*_{2k+1} = \boldsymbol{\theta}^*_{2k-1} + \mathbf{z}_{2k+1} - \mathbf{z}_{2k} = \boldsymbol{\theta}^*_{2k-1} + \mathbf{v}_{2k+1} - \mathbf{v}_{2k} \,. \tag{65}$$

Let us suppose that the true and *a priori* values of $\boldsymbol{\theta}$ and $\mathbf{a}$ are zero. Then both $\boldsymbol{\theta}^*_{2k+1}$ and $\mathbf{a}^*_{2k}$ will execute a random walk with covariance $2R$ (since each interval spans two time intervals). If calibrations are performed every day over the course of five years, then, at the end of five years, an estimate error in $\boldsymbol{\theta}$ and $\mathbf{a}$ with covariance on the order of 1600 R will be introduced. The individual instrument and flight operations teams, on the other hand, will believe incorrectly that their estimate error covariance matrices are on the order of R, amounting to an error in confidence in the standard deviation of a factor of 40. If the single frame estimation error in this case is on the order of 5 arc seconds, then at the end of five years in the above example, the random walk standard deviation will be on the order of 3.5 arc minutes, which is generally unacceptable. The solution, of course, is to estimate only $\mathbf{a}$, or better, only $\boldsymbol{\theta}$ but not both.

CONCLUSIONS

A consistent methodology has been developed for estimating the alignment and distortion parameters of a focal plane sensor. This methodology discards poorly identifiable parameters from the distortion model which become redundant in the limit that the number of focal plane distortion parameters becomes infinite. Specific algorithms have been given for the carrying out computations efficiently.

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