International Symposium on Spacecraft Ground Control and Flight Dynamics—SCD1, São José dos Campos, Brazil, February 7–11, 1994
to appear in: *Revista Brasileira de Ciências Mecânicas*

ATTITUDE ESTIMATION FROM THE MEASUREMENT OF A DIRECTION AND AN ANGLE

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Abstract

The general problem of determining the attitude from measurements of an angle and a direction is considered. It is shown that there is no continuous ambiguity for this problem, because effectively three data are given. However, the attitude still has generally a two-fold ambiguity which can be removed only by the addition of further data.

Introduction

While numerous algorithms exist for the estimation of the attitude from the measurement of two or more directions, the best known being the TRIAD and QUEST algorithms,[1] no simple algorithm has ever been published for the estimation of the attitude from the measurement of a single vector and a single angle. This particular case is of interest, because there being only three independent equivalent scalar data, it might seem at first glance that a unique solution should exist in this case. For the case of the measurement of two directions on the other hand, a solution is, in general, not defined without additional criteria, because three parameters must be determined from four data.

In the present work a simple construction is given for determining the solution to this problem. It turns out that the solution is not unique but has a two-fold degeneracy. It is also noted that since a deterministic estimate exists, this must also be the maximum-likelihood estimate. This fact is exploited to develop a covariance analysis of the algorithm using the QUEST model[2] for the measurement errors. The algorithms for solving this attitude problem and the covariance analysis were developed in order to provide rapid analysis of attitude data from the a spacecraft equipped with a three-axis magnetometer and a Sun angle sensor.

The Problem

We seek an attitude matrix A which satisfies

$$\hat{\boldsymbol{W}}_1 = A\,\hat{\boldsymbol{V}}_1\,, \qquad \hat{\boldsymbol{S}}_2 \cdot A\,\hat{\boldsymbol{V}}_2 = d\,, \tag{1}$$

where $\hat{\boldsymbol{W}}_1$ is a measured unit vector in the body frame (the direction measurement); $\hat{\boldsymbol{S}}_2$ is a known vector in the body frame; $\hat{\boldsymbol{V}}_1$ and $\hat{\boldsymbol{V}}_2$ are known vectors in the primary reference (typically inertial) frame; and d is a measured cosine (the angle measurement). We assume that $|d| \le 1$; otherwise, a solution will not exist. For a typical spacecraft, $\hat{\boldsymbol{W}}_1$ might be the measured magnetic field vector and d might be the cosine of the Sun angle as obtained from a spinning digital solar aspect detector.

General Structure of the Solution

We begin by seeking all attitude matrices A which satisfy $\hat{\boldsymbol{W}}_1 = A\,\hat{\boldsymbol{V}}_1$. These are given by

$$A = R(\hat{\boldsymbol{W}}_1,\, \theta)\, A_o\,, \tag{2}$$

where A_o is any attitude matrix satisfying $\hat{\boldsymbol{W}}_1 = A_o\,\hat{\boldsymbol{V}}_1$; $R(\hat{\boldsymbol{W}}_1,\, \theta)$ is a the rotation matrix for a rotation about the axis $\hat{\boldsymbol{W}}_1$ through and angle θ; and θ is any angle satisfying $0 \le \theta < 2\pi$. $R(\hat{\boldsymbol{W}}_1,\, \theta)$ is given by Euler's formula

$$R(\hat{\boldsymbol{n}},\, \theta) = \cos\theta\, I_{3\times 3} + (1-\cos\theta)\,\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}^T + \sin\theta\,[[\,\hat{\boldsymbol{n}}\,]]\,, \tag{3}$$

with

$$[[\,\boldsymbol{v}\,]] \equiv \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}. \tag{4}$$

To prove the assertion of Equation (2), assume that there exist two distinct attitude matrices, A and A_o, satisfying $\hat{\boldsymbol{W}}_1 = A\,\hat{\boldsymbol{V}}_1$ and $\hat{\boldsymbol{W}}_1 = A_o\,\hat{\boldsymbol{V}}_1$, respectively. Then

$$\begin{aligned} \hat{\boldsymbol{W}}_1 &= A\,(A_o^{-1}A_o)\,\hat{\boldsymbol{V}}_1 &(5)\\ &= (AA_o^{-1})\,A_o\,\hat{\boldsymbol{V}}_1 &(6)\\ &= (AA_o^{-1})\,\hat{\boldsymbol{W}}_1\,. &(7)\end{aligned}$$

Thus, $\hat{\boldsymbol{W}}_1$ must be the axis of rotation of the rotation matrix AA_o^{-1}. Since AA_o^{-1} must be different from the identity matrix, the axis of rotation is well-defined and unique. Hence,

$$AA_o^{-1} = R(\hat{\boldsymbol{W}}_1, \theta) \tag{8}$$

for some angle θ. Equation (2) now follows from Equations (6) and (8). Every attitude matrix given by Equation (2) satisfies $\hat{\boldsymbol{W}}_1 = A\hat{\boldsymbol{V}}_1$. Therefore, there is a continuum of solutions, satisfying this equation.

Equation (2) is equivalent to

$$A = A_o\,R(\hat{\boldsymbol{V}}_1, \theta)\,, \tag{9}$$

with identical A_o and θ. This follows from[3]

$$A_o\,R(\hat{\boldsymbol{V}}_1, \theta)A_o^T = R(A_o\hat{\boldsymbol{V}}_1, \theta)\,. \tag{10}$$

A Single Solution for $\mathbf{A_o}$

We must now find a single A_o which satisfies $\hat{\boldsymbol{W}}_1 = A_o\hat{\boldsymbol{V}}_1$. Let us look for an A_o of the form

$$A_o = R(\hat{\boldsymbol{n}}_o, \theta_o)\,, \tag{11}$$

where

$$\hat{\boldsymbol{n}}_o \equiv \frac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}\,, \tag{12}$$

which is defined provided that $\hat{\boldsymbol{W}}_1 \neq \pm\hat{\boldsymbol{V}}_1$. Then trivially

$$\hat{\boldsymbol{n}}_o \cdot \hat{\boldsymbol{V}}_1 = 0\,, \tag{13}$$

and

$$\begin{aligned} [[\hat{\boldsymbol{n}}_o]]\,\hat{\boldsymbol{V}}_1 &= -\hat{\boldsymbol{n}}_o \times \hat{\boldsymbol{V}}_1 \\ &= \frac{\hat{\boldsymbol{W}}_1 - (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)\hat{\boldsymbol{V}}_1}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}\,. \end{aligned} \tag{14}$$

Thus,

$$\begin{aligned} &R(\hat{\boldsymbol{n}}_o, \theta_o)\,\hat{\boldsymbol{V}}_1 \\ &= \cos\theta_o\,\hat{\boldsymbol{V}}_1 \\ &\quad + \frac{\sin\theta_o}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}\left(\hat{\boldsymbol{W}}_1 - (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)\hat{\boldsymbol{V}}_1\right) \\ &= \left[\cos\theta_o - \frac{(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}\sin\theta_o\right]\hat{\boldsymbol{V}}_1 \\ &\quad + \frac{\sin\theta_o}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}\,\hat{\boldsymbol{W}}_1\,. \end{aligned} \tag{15}$$

Since $\hat{\boldsymbol{W}}_1$ and $\hat{\boldsymbol{V}}_1$ are linearly independent, a unique solution exists for θ_o, namely,

$$\sin\theta_o = |\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|\,, \quad \cos\theta_o = (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)\,, \tag{16}$$

which yields

$$\theta_o = \mathrm{ATAN2}\left(|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|, (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)\right)\,, \tag{17}$$

where ATAN2 is the familiar FORTRAN function, which for the purpose of calculating θ_o we will adjust so that the values always lie in the interval $-\pi < \theta_o \leq \pi$.

Note that once $\hat{\boldsymbol{n}}_o$ is fixed there can be only one solution for θ_o. We could equally well have chosen

$$\hat{\boldsymbol{n}}_o' \equiv -\frac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}\,, \tag{18}$$

in which case we would have been led to

$$\theta_o' = \mathrm{ATAN2}\left(-|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|, (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)\right)\,. \tag{19}$$

The two solutions are equivalent.

The quaternion corresponding to A_o is has a very simple form. To calculate the quaternion we note first that

$$\cos(\theta_o/2) = \sqrt{\frac{1+\cos\theta_o}{2}} = \sqrt{\frac{1+\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1}{2}}\,, \tag{20}$$

and

$$\cos(\theta_o/2) \geq 0 \qquad \text{for} \qquad |\theta_o| \leq \pi\,. \tag{21}$$

Likewise,

$$\sin(\theta_o) = 2\,\sin(\theta_o/2)\,\cos(\theta_o/2)\,, \tag{22}$$

so that

$$\begin{aligned} \sin(\theta_o/2) &= \frac{\sin(\theta_o)}{2\,\cos(\theta_o/2)} \\ &= \frac{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|}{\sqrt{2\,(1+\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)}}\,. \end{aligned} \tag{23}$$

Hence,

$$\sin(\theta_o/2)\,\hat{\boldsymbol{n}}_o = \frac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1}{\sqrt{2\,(1+\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1)}}\,, \tag{24}$$

and the corresponding quaternion is given by

$$\begin{aligned} \bar{q}_o &= \sqrt{\frac{1+\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1}{2}} \\ &\quad \times \begin{bmatrix} \left(\dfrac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1}{1+\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1}\right) \\ 1 \end{bmatrix}\,, \end{aligned} \tag{25}$$

which can now be computed without the need to compute θ_o. The Rodrigues vector $\boldsymbol{\rho}_o$ (also called the Gibbs vector) is given obviously by[3]

$$\boldsymbol{\rho}_o = \frac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1}{1+\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1} \tag{26}$$

and the matrix A_o is given equivalently by

$$A_o = (\hat{W}_1 \cdot \hat{V}_1) I_{3\times3} + \frac{(\hat{W}_1 \times \hat{V}_1)(\hat{W}_1 \times \hat{V}_1)^T}{1+\hat{W}_1 \cdot \hat{V}_1} + [[\hat{W}_1 \times \hat{V}_1]] . \quad (27)$$

Complete Solution for A

Given θ_o we must now compute θ. Define

$$\hat{W}_3 \equiv A_o \hat{V}_2 . \quad (28)$$

Then θ is a solution of

$$\hat{S}_2 \cdot R(\hat{W}_1, \theta)\, \hat{W}_3 = d . \quad (29)$$

Substituting Euler's formula leads to

$$\hat{S}_2 \cdot \Big[\hat{W}_3 - \sin\theta\,(\hat{W}_1 \times \hat{W}_3) + (1-\cos\theta)\,(\hat{W}_1 \times (\hat{W}_1 \times \hat{W}_3)\Big] = d , \quad (30)$$

which can be rearranged to yield

$$\Big[\hat{S}_2 \cdot (\hat{W}_1 \times (\hat{W}_1 \times \hat{W}_3))\Big] \cos\theta + \Big[\hat{S}_2 \cdot (\hat{W}_1 \times \hat{W}_3)\Big] \sin\theta = (\hat{S}_2 \cdot \hat{W}_1)(\hat{W}_1 \cdot \hat{W}_3) - d . \quad (31)$$

There are clearly two solutions for θ, in general. To see this define

$$B \equiv |\hat{S}_2 \times \hat{W}_1|\,|\hat{W}_1 \times \hat{W}_3| \quad (32)$$

$$\beta \equiv \mathrm{ATAN2}\left(\Big[\hat{S}_2 \cdot (\hat{W}_1 \times \hat{W}_3)\Big], \Big[\hat{S}_2 \cdot (\hat{W}_1 \times (\hat{W}_1 \times \hat{W}_3))\Big]\right) . \quad (33)$$

Then Equation (31) can be rewritten as

$$B\cos(\theta-\beta) = (\hat{S}_2 \cdot \hat{W}_1)(\hat{W}_1 \cdot \hat{W}_3) - d . \quad (34)$$

From Equation (34) we see that a necessary condition that a solution exist is that

$$|(\hat{S}_2 \cdot \hat{W}_1)(\hat{W}_1 \cdot \hat{W}_3) - d| \le |\hat{S}_2 \times \hat{W}_1|\,|\hat{W}_1 \times \hat{W}_3| . \quad (35)$$

If this condition is satisfied, then θ has the solutions

$$\theta = \beta + \cos^{-1}\left[\frac{(\hat{S}_2 \cdot \hat{W}_1)(\hat{W}_1 \cdot \hat{W}_3) - d}{|\hat{S}_2 \times \hat{W}_1|\,|\hat{W}_1 \times \hat{W}_3|}\right] , \quad (36)$$

and the inverse cosine is indeed two-valued. Given A_o and θ we can now construct the attitude matrix solutions according to Equations (2) and (27).

Covariance Analysis

The attitude matrices constructed by the above algorithm solve Equations (1) exactly. Therefore, if attitude solutions exist, they each certainly minimize the cost function

$$J(A) = \frac{1}{\sigma_1^2}\,|\hat{W}_1 - A\,\hat{V}_1|^2 + \frac{1}{\sigma_d^2}\,|\hat{S}_2 \cdot A\hat{V}_2 - d|^2 , \quad (37)$$

where σ_1 and σ_d are standard deviations characterizing the weights. If a deterministic attitude solution constructed according to the above methodology does not exist (say, because Equation (35) is not satisfied due to the effect of measurement noise) then one can at least find an attitude solution (generally two) which minimizes the cost function of Equation (37). The discussion of this section will still be valid in the latter case.

We recognize the first term as the negative-log-likelihood corresponding to the error model[2]

$$\hat{W}_1 = A^{\mathrm{true}}\,\hat{V}_1 + \Delta\hat{W}_1 , \quad (38)$$

where $\Delta\hat{W}_1$ is the equivalent measurement noise, which is assumed to be Gaussian and to satisfy

$$E\{\Delta\hat{W}_1\} = 0 , \quad (39)$$

and

$$E\{\Delta\hat{W}_1 \Delta\hat{W}_1^T\} = \sigma_1^2\,(I_{3\times3} - \hat{W}_1^{\mathrm{true}}\,\hat{W}_1^{\mathrm{true}\,T}) , \quad (40)$$

where $E\{\cdot\}$ denotes the expectation. Likewise, the second term of Equation (37) is the negative-log-likelihood corresponding to the error model

$$d = \hat{S}_2 \cdot A\hat{V}_2 + \Delta d , \quad (41)$$

where Δd is a zero-mean Gaussian random noise with variance σ_d^2. Thus, the attitude matrices computed by the above algorithm are also (non-unique) maximum-likelihood estimates of the attitude. We may, therefore, compute the attitude covariance matrix as the inverse of the Fisher information matrix by interpreting Equation (37) as a negative-log-likelihood function.[2,4] The calculation of the Fisher information is tedious but straightforward. The result for the attitude covariance matrix, which we define as the covariance matrix of the infinitesimal rotation vector, $\boldsymbol{\xi}$, connecting the true attitude to the estimated attitude, is

$$P_{\xi\xi} = \left[\frac{1}{\sigma_1^2}\left(I_{3\times3} - \hat{W}_1\hat{W}_1^T\right) + \frac{1}{\sigma_d^2}\,(\hat{W}_2 \times \hat{S}_2)(\hat{W}_2 \times \hat{S}_2)^T\right]^{-1} , \quad (42)$$

where $\hat{W}_2$ is defined as

$$\hat{W}_2 \equiv A\,\hat{V}_2 . \quad (43)$$

Note that generally

$$\hat{\boldsymbol{W}}_2 \neq \hat{\boldsymbol{S}}_2 \,, \tag{44}$$

even in the absence of measurement noise. For this reason we have used the notation $\hat{\boldsymbol{S}}_2$ rather than $\hat{\boldsymbol{W}}_2$. Note also that $P_{\xi\xi}$ will not exist unless

$$\hat{\boldsymbol{W}}_1 \cdot (\hat{\boldsymbol{W}}_2 \times \hat{\boldsymbol{S}}_2) \neq 0 \,, \tag{45}$$

or, equivalently, unless

$$\hat{\boldsymbol{S}}_2 \cdot (\hat{\boldsymbol{W}}_1 \times (A\,\hat{\boldsymbol{V}}_2)) = (A\,\hat{\boldsymbol{V}}_2) \cdot (\hat{\boldsymbol{S}}_2 \times \hat{\boldsymbol{W}}_1) \neq 0 \,. \tag{46}$$

Even though the attitude matrix may be defined in this case the geometry represents an extremum situation in which the sensitivity of the attitude to the measurements vanishes along one direction in parameter space.

Remarks

Note that the fact that we have equivalently three independent measurements (two for the direction and one for the cosine) does not guarantee a unique solution, only that the solutions be elements of a discrete set. Uniqueness would be obtained only if the equations for the three independent attitude parameters were linear, which is almost never the case.

We are not restricted to choosing

$$\hat{\boldsymbol{n}}_o \equiv \pm \frac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{V}}_1|} \,. \tag{47}$$

In fact, any vector $\hat{\boldsymbol{n}}_o$ satisfying

$$\hat{\boldsymbol{n}}_o \cdot \hat{\boldsymbol{W}}_1 = \hat{\boldsymbol{n}}_o \cdot \hat{\boldsymbol{V}}_1 \,, \tag{48}$$

will do. We have selected one of the simpler cases. An alternate choice is examined below.

In fact, the construction of $\hat{\boldsymbol{n}}_o$ fails if $\hat{\boldsymbol{W}}_1 = \pm\hat{\boldsymbol{V}}_1$. If $\hat{\boldsymbol{W}}_1 = \hat{\boldsymbol{V}}_1$, we may choose $A_o = I_{3\times3}$. If, on the other hand, $\hat{\boldsymbol{W}}_1 = -\hat{\boldsymbol{V}}_1$, the we may choose $\hat{\boldsymbol{n}}_o$ to be any vector perpendicular to $\hat{\boldsymbol{W}}_1$ and $\theta_o = \pi$.

Note that we have avoided using the relation,

$$\sin(\theta_o/2) = \sqrt{\frac{1-\cos\theta_o}{2}} \,, \tag{49}$$

in developing an analytic expression for the quaternion. This would have led to an unnecessary sign ambiguity which would have been complicated to resolve.

In developing the expression for the attitude covariance matrix we assumed a particular model for the measurement errors of the direction. We could, in fact have used an arbitrary measurement model for $\hat{\boldsymbol{W}}_1$, namely,

$$E\{\Delta\hat{\boldsymbol{W}}_1\} = \boldsymbol{0} \,, \tag{50}$$

$$E\{\Delta\hat{\boldsymbol{W}}_1 \Delta\hat{\boldsymbol{W}}_1^T\} = P_{\hat{\boldsymbol{W}}_1} \,, \tag{51}$$

where $P_{\hat{\boldsymbol{W}}_1}$ is an arbitrary covariance matrix for $\hat{\boldsymbol{W}}_1$, which must, because of the unit-norm constraint, satisfy

$$P_{\hat{\boldsymbol{W}}_1} \hat{\boldsymbol{W}}_1 = \boldsymbol{0} \,, \tag{52}$$

so that $P_{\hat{\boldsymbol{W}}_1}$ is singular (rank deficient). Equation (37) then generalizes to

$$\begin{aligned} J(A) &= (\hat{\boldsymbol{W}}_1 - A\,\hat{\boldsymbol{V}}_1)^T P^{\#}_{\hat{\boldsymbol{W}}_1} (\hat{\boldsymbol{W}}_1 - A\,\hat{\boldsymbol{V}}_1) \\ &\quad + \frac{1}{\sigma_d^2} |\hat{\boldsymbol{S}}_2 \cdot A\hat{\boldsymbol{V}}_2 - d|^2 \,, \end{aligned} \tag{53}$$

where # denotes the pseudo-inverse. The attitude covariance matrix generalizes to

$$\begin{aligned} P_{\xi\xi} &= \Big[[[\hat{\boldsymbol{W}}_1]]^T P^{\#}_{\hat{\boldsymbol{W}}_1} [[\hat{\boldsymbol{W}}_1]] \\ &\quad + \frac{1}{\sigma_d^2} (\hat{\boldsymbol{W}}_2 \times \hat{\boldsymbol{S}}_2)(\hat{\boldsymbol{W}}_2 \times \hat{\boldsymbol{S}}_2)^T \Big]^{-1} \,. \end{aligned} \tag{54}$$

If the measured direction is that of the geomagnetic field, then in general the entire three-vector is known and need not have unit norm. In that case the we replace $\hat{\boldsymbol{W}}_1$ and $\hat{\boldsymbol{V}}_1$ by the unnormalized $\boldsymbol{W}_1$ and $\boldsymbol{V}_1$ and the covariance matrix is now full rank (Equation (52) no longer applies). Equations (53) and (54) remain unaltered except for the replacement

$$P^{\#}_{\hat{\boldsymbol{W}}_1} \to P^{-1}_{\boldsymbol{W}_1} \,. \tag{55}$$

In many practical circumstances, however, the simple model of Equation (37) has proven to be adequate.

Note that the covariance matrix characterizes the probability of the estimate compared to other solutions within its immediate neighborhood. The algorithm, however, has two solutions, the correct one of which cannot be identified except by bringing additional information to bear on the attitude problem. Thus, even though the attitude variances may be small the estimated solution, if it happens to be the false solution, may be very far from the truth.

An Alternate Choice for the Initial Rotation

Instead of Equation (12) we could have chosen

$$\hat{\boldsymbol{n}}'_o = \frac{\hat{\boldsymbol{W}}_1 + \hat{\boldsymbol{V}}_1}{|\hat{\boldsymbol{W}}_1 + \hat{\boldsymbol{V}}_1|} \,. \tag{56}$$

Then it is easy to show that

$$A'_o = R(\hat{\boldsymbol{n}}'_o, \pi) \tag{57}$$

$$\quad = -I_{3\times3} + \frac{(\hat{\boldsymbol{W}}_1 + \hat{\boldsymbol{V}}_1)(\hat{\boldsymbol{W}}_1 + \hat{\boldsymbol{V}}_1)^T}{1 + \hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{V}}_1} \,. \tag{58}$$

The quaternion in this case is simply

$$\bar{q}'_o = \begin{bmatrix} \hat{\boldsymbol{n}}'_o \\ 0 \end{bmatrix} \,. \tag{59}$$

The special case $\hat{\boldsymbol{W}}_1 = \hat{\boldsymbol{V}}_1$ no longer requires special attention for this choice of A'_o. The treatment of the special case $\hat{\boldsymbol{W}}_1 = -\hat{\boldsymbol{V}}_1$ is as previously. The computation of β' (corresponding to the earlier β) and θ' (corresponding to the earlier θ) proceeds as before.

A More Direct Solution for the Attitude Matrix

Instead of first calculating the attitude matrix from the data and then determining a vector $\hat{\boldsymbol{W}}_2$ which satisfies

$$\hat{\boldsymbol{W}}_2 = A\,\hat{\boldsymbol{V}}_2\,. \tag{60}$$

in order to carry out the covariance analysis, we might try instead to calculate this $\hat{\boldsymbol{W}}_2$ directly and, once this vector has been determined, calculate A using the triad algorithm.[1]

To compute $\hat{\boldsymbol{W}}_2$ we write

$$\hat{\boldsymbol{W}}_2 = a\,\hat{\boldsymbol{W}}_1 + b\,\hat{\boldsymbol{S}}_2 + c\,\frac{\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{S}}_2}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{S}}_2|}\,, \tag{61}$$

which is possible provided that $\hat{\boldsymbol{W}}_1 \neq \pm\hat{\boldsymbol{S}}_2$. It then follows that

$$\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{W}}_2 = a + b\,(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2) = \hat{\boldsymbol{V}}_1 \cdot \hat{\boldsymbol{V}}_2\,, \tag{62}$$

$$\hat{\boldsymbol{S}}_2 \cdot \hat{\boldsymbol{W}}_2 = a\,(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2) + b = d\,, \tag{63}$$

$$\hat{\boldsymbol{W}}_2 \cdot \hat{\boldsymbol{W}}_2 = a^2 + 2ab\,(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2) + b^2 + c^2 = 1\,. \tag{64}$$

The solution for a and b is immediate and is given by

$$a = \frac{(\hat{\boldsymbol{V}}_1 \cdot \hat{\boldsymbol{V}}_2) - (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2)\,d}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{S}}_2|^2}\,, \tag{65}$$

$$b = \frac{d - (\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2)(\hat{\boldsymbol{V}}_1 \cdot \hat{\boldsymbol{V}}_2)}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{S}}_2|^2}\,. \tag{66}$$

The solution for c is now given by

$$c = \pm\sqrt{1 - (a^2 + 2ab\,(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2) + b^2)}\,. \tag{67}$$

This last calculation can be simplified by noting that

$$a^2 + 2ab\,(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2) + b^2 = \frac{1}{|\hat{\boldsymbol{W}}_1 \times \hat{\boldsymbol{S}}_2|^2}\left[d^2 - 2d\,(\hat{\boldsymbol{V}}_1 \cdot \hat{\boldsymbol{V}}_2)(\hat{\boldsymbol{W}}_1 \cdot \hat{\boldsymbol{S}}_2) + (\hat{\boldsymbol{V}}_1 \cdot \hat{\boldsymbol{V}}_2)^2\right]. \tag{68}$$

The lack of a unique solution is now obvious from Equation (67). Although the triad algorithm[1] can now be used to calculate the attitude from the four vectors $\hat{\boldsymbol{V}}_1$, $\hat{\boldsymbol{V}}_2$, $\hat{\boldsymbol{W}}_1$, and $\hat{\boldsymbol{W}}_2$, the measured unit vectors are no longer uncorrelated, and the attitude covariance matrix is still that computed earlier (Equations (42) or (54)).

While the present algorithm is clearly more efficient than that developed in the main text, it also suffers from some numerical problems. Because of round-off error it is not guaranteed that $\hat{\boldsymbol{W}}_2$ is a unit vector. Worse still, large measurement errors may cause the argument of the square root in Equation (67) to be negative.

Discussion

We note with some dismay that for three data there is no single unambiguous solution to the attitude determination problem. On the other hand, for two vectors, which are equivalent to four data, the solution is generally overdetermined, so that no solution will exist. If we are given three angles, two of which are to the same body-fixed vector, then there will clearly by four possible attitude solutions, in general. We conjecture that for three angles, each one to a different body-fixed vector, there will be eight possible attitude solutions. It would appear, therefore, that the non-optimal attitude determination problem is always ambiguous or nonexistent, and only least-square solutions of the overdetermined problem yield unique results.

Acknowledgment

The author is grateful to J. Courtney Ray for suggesting this problem and to F. Landis Markley for a critical reading of the manuscript.

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