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The Geometry of the Euler Angles

John L. Junkins[1] and Malcolm D. Shuster[2]

Abstract

The Euler angles are shown to provide a simple means for understanding some of the fundamental results of spherical trigonometry. Spherical trigonometry, in turn, is used to develop a compact expression for the composition of two sets of Euler angles and other relations.

Introduction

The Euler angles and spherical trigonometry are generally treated as separate subjects. However, treating the two together will lead to new insights and some new results. Of particular interest for us has been the development of an expression for the composition of two rotations, each parameterized by the Euler angles. Generally, to accomplish this composition, the two rotations described by Euler angle sequences must first be expressed as direction-cosine matrices, the two direction-cosine matrices multiplied together, and then the rules for extracting Euler angles applied. It turns out that a much simpler procedure exists but is not generally known. In developing such a procedure, a better understanding of spherical trigonometry is also obtained. Spherical trigonometry also elucidates the connection between closely related sets of Euler angles.

Spherical Trigonometry

Let $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{C}}$ be three unit vectors, which we may represent as three points on the unit sphere [1] as shown in Fig. 1. Here a, b, and c, are the lengths of the arcs of great circles connecting the three points. If O (not shown) is the center of the unit sphere, then these three arc lengths are equal to the three angles BOC,

[1]George J. Eppright Professor of Aerospace Engineering, Department of Aerospace Engineering, Texas A & M University, College Station, TX 77843.
[2]Senior Professional Staff, Space Department, The Johns Hopkins University Applied Physics Laboratory, Laurel, MD 20723-6099.

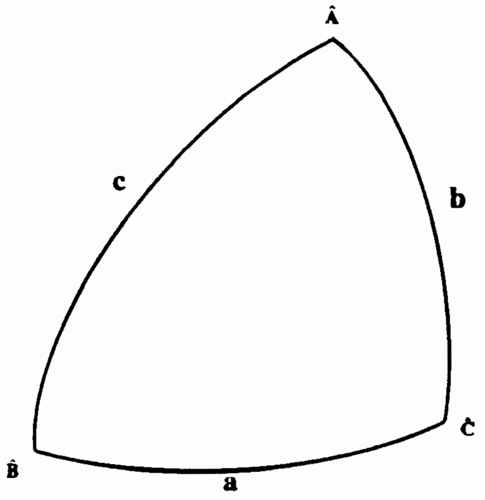


FIG. 1. A Spherical Triangle.

COA, and *AOB*, respectively. In addition to these three arc lengths, we may define the three dihedral angles, which are the angles between the planes defined by the center and two of the vertices. Thus, the dihedral angle *A* is the angle between the planes *AOB* and *AOC*. If we think of the angle as the angle of rotation of one plane into the other, then the axis of rotation is $\hat{\mathbf{A}}$. The complete description of the spherical triangle in terms of arc lengths and dihedral angles is shown in Fig. 2.

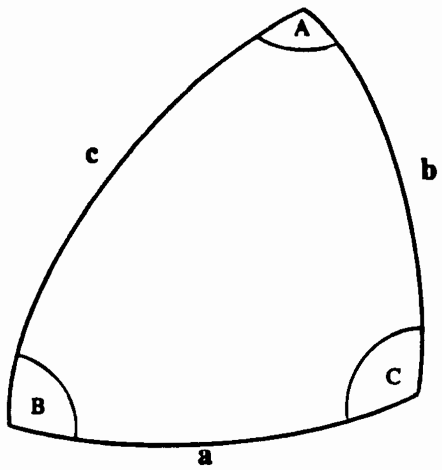


FIG. 2. A Spherical Triangle.

To represent the three vertices of the triangle analytically, let us choose the z-axis to coincide with $\hat{\mathbf{A}}$, and let $\hat{\mathbf{B}}$ lie in the xz-plane. Then

$$\hat{\mathbf{A}} = \hat{\mathbf{z}}, \tag{1a}$$

$$\hat{\mathbf{B}} = \cos c\,\hat{\mathbf{z}} + \sin c\,\hat{\mathbf{x}}, \tag{1b}$$

$$\hat{\mathbf{C}} = \cos b\,\hat{\mathbf{z}} + \sin b \cos A\,\hat{\mathbf{x}} + \sin b \sin A\,\hat{\mathbf{y}}, \tag{1c}$$

Thus,

$$\cos a = \hat{\mathbf{B}} \cdot \hat{\mathbf{C}} = \cos b \cos c + \sin b \sin c \cos A. \tag{2a}$$

Likewise, by cyclic permutation of the vertices

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \tag{2b}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{2c}$$

obtained by choosing $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$ appropriately in each case. It follows also that

$$(\hat{\mathbf{A}} \times \hat{\mathbf{B}}) \cdot \hat{\mathbf{C}} = \sin A \sin b \sin c. \tag{3}$$

From the cyclic invariance of the scalar triple product,

$$(\hat{\mathbf{A}} \times \hat{\mathbf{B}}) \cdot \hat{\mathbf{C}} = (\hat{\mathbf{B}} \times \hat{\mathbf{C}}) \cdot \hat{\mathbf{A}} = (\hat{\mathbf{C}} \times \hat{\mathbf{A}}) \cdot \hat{\mathbf{B}}, \tag{4}$$

it follows that

$$\sin A \sin b \sin c = \sin B \sin c \sin a = \sin C \sin a \sin b. \tag{5}$$

Dividing all three members by $\sin a \sin b \sin c$ leads to

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{6}$$

Equations (2) are the *spherical law of cosines for sides* and equation (6) is the *spherical law of sines.* There are additional useful and important results which we will obtain in conjunction with the Euler angles.

The Euler Angles

Any rotation matrix may be parameterized in terms of a symmetric sequence of Euler angles [1–3], which we write in the form

$$R_{lml}(\varphi, \vartheta, \psi) \equiv R(\hat{\mathbf{u}}_l, \psi) R(\hat{\mathbf{u}}_m, \vartheta) R(\hat{\mathbf{u}}_l, \varphi), \tag{7}$$

where $\hat{\mathbf{u}}_l$ and $\hat{\mathbf{u}}_m$ are two distinct unit column vectors which are chosen from the set

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

Thus,

$$R(\hat{\mathbf{u}}_1,\theta)=\begin{bmatrix}1&0&0\\0&c\theta&s\theta\\0&-s\theta&c\theta\end{bmatrix},\qquad R(\hat{\mathbf{u}}_2,\theta)=\begin{bmatrix}c\theta&0&-s\theta\\0&1&0\\s\theta&0&c\theta\end{bmatrix},\tag{9ab}$$

$$R(\hat{\mathbf{u}}_3,\theta)=\begin{bmatrix}c\theta&s\theta&0\\-s\theta&c\theta&0\\0&0&1\end{bmatrix},\tag{9c}$$

where $c\theta \equiv \cos\theta$ and $s\theta \equiv \sin\theta$. The three angles, φ, ϑ, ψ, are usually restricted to the intervals

$$0 \le \varphi < 2\pi, \qquad 0 \le \vartheta \le \pi, \quad \text{and} \quad 0 \le \psi < 2\pi. \tag{10}$$

We call the sequence of Euler angles appearing in the parameterization of the rotation matrix in equation (7) *symmetric* in order to distinguish it from an *asymmetric* sequence of Euler angles in which no two axis column-vectors are identical. We particularize our discussion to the 3–1–3 sequence of Euler angles. However, the results will be true for any symmetric sequence.

We note first that the multiplication of the three rotation matrices of equation (7) leads for the 3–1–3 case to

$$R_{313}(\varphi,\vartheta,\psi)$$
$$=\begin{bmatrix}\cos\psi&\sin\psi&0\\-\sin\psi&\cos\psi&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\cos\vartheta&\sin\vartheta\\0&-\sin\vartheta&\cos\vartheta\end{bmatrix}\begin{bmatrix}\cos\varphi&\sin\varphi&0\\-\sin\varphi&\cos\varphi&0\\0&0&1\end{bmatrix}\tag{11a}$$

$$=\begin{bmatrix}c\psi c\varphi - s\psi c\vartheta s\varphi & c\psi s\varphi + s\psi c\vartheta c\varphi & s\psi s\vartheta\\ -s\psi c\varphi - c\psi c\vartheta s\varphi & -s\psi s\varphi + c\psi c\vartheta c\varphi & c\psi s\vartheta\\ s\vartheta s\varphi & -s\vartheta c\varphi & c\vartheta\end{bmatrix},\tag{11b}$$

where $s\psi$ has been written in place of $\sin\psi$, et cetera. Four of the terms have the appearance of the right member of the spherical law of cosines for sides. This can be understood from the spherical-trigonometric description of the Euler rotations, which we now develop.

Spherical-Trigonometric Depiction of the Euler Angles

Consider the action on the x-axis of a rotation described by a 3–1–3 sequence of Euler angles $(\varphi, \vartheta, \psi)$. Imagine that the rotations correspond to a sequence of physical rotations such that

$$R(t)=\begin{cases}R(\hat{\mathbf{u}}_3,\varphi t), & \text{for } 0\le t<1\,,\\ R(\hat{\mathbf{u}}_1,\vartheta(t-1))R(\hat{\mathbf{u}}_3,\varphi), & \text{for } 1\le t<2\,,\\ R(\hat{\mathbf{u}}_3,\psi(t-2))R(\hat{\mathbf{u}}_1,\vartheta)R(\mathbf{u}_3,\varphi) & \text{for } 2\le t\le 3\,.\end{cases}\tag{12}$$

Under the first rotation, the x-axis is displaced in the y-direction by an arc length φ as shown in Fig. 3. Note that we describe the displacement of the physical x-axis and not the representation.

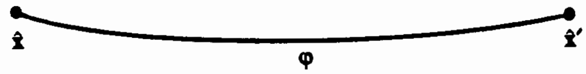


FIG. 3. Displacement of the x-Axis under a Rotation about the z-Axis.

The second rotation being about the x-axis does not displace the x-axis. However, it rotates the y-direction by an angle ϑ, so that the further movement of the x-axis is along another direction. The third rotation, about the new z-axis, displaces the x-axis again. The combined action of all three rotations on the x-axis is depicted in Fig. 4. Thus, the two outer angles correspond to arc lengths, and the medial angle to a dihedral angle.

Suppose now we consider a sequence of six rotations satisfying

$$R(\hat{\mathbf{u}}_1, \pi - B)R(\hat{\mathbf{u}}_3, c)R(\hat{\mathbf{u}}_1, \pi - A)R(\hat{\mathbf{u}}_3, b)R(\hat{\mathbf{u}}_1, \pi - C)R(\hat{\mathbf{u}}_3, a) = I. \qquad (13)$$

Because the complete sequence of rotations must be equivalent to the identity rotation (we will call such a sequence *closed*), it follows that the locus of any point on the unit sphere under this sequence of rotations, when the sequence of six rotations is written in a manner similar to equation (12), must be a closed curve. From the above discussion, the locus of the x-axis will be the spherical triangle shown in Fig. 2.

We can now understand the similarity of the expressions for the (1, 1), (1,2), (2, 1), and (2,2) elements of the direction-cosine matrix to the spherical law of cosines. Let us denote the original axes of the coordinate system by $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$ and the successive coordinate axes following the three Euler rotations of equation (11) by $\{\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'\}$, $\{\hat{\mathbf{i}}'', \hat{\mathbf{j}}'', \hat{\mathbf{k}}''\}$, and $\{\hat{\mathbf{i}}''', \hat{\mathbf{j}}''', \hat{\mathbf{k}}'''\}$, respectively, with $\hat{\mathbf{k}}' = \hat{\mathbf{k}}$, $\hat{\mathbf{i}}'' = \hat{\mathbf{i}}'$, and $\hat{\mathbf{k}}''' = \hat{\mathbf{k}}''$ for a 3–1–3 set of Euler angles. Then C_{ij}, the (i, j) element of the direction-cosine matrix, is given by

$$C_{ij} = \cos \nu_{ij} ,$$

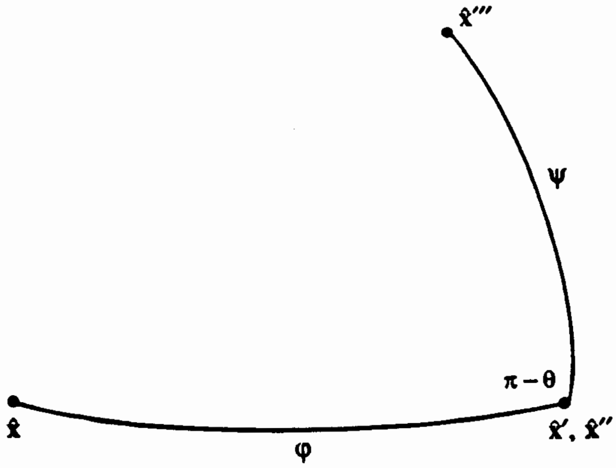


FIG. 4. Locus of the x-Axis in Response to a 3–1–3 Euler Sequence.

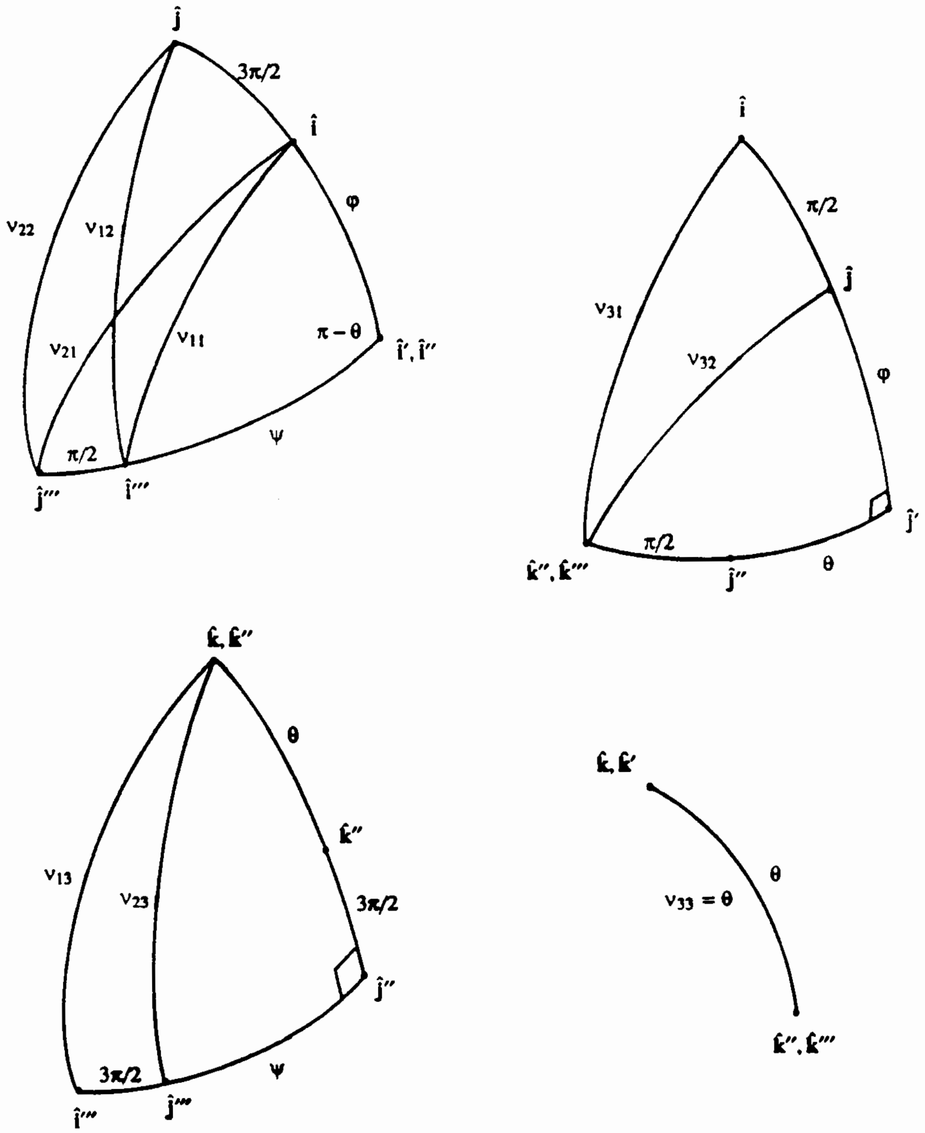


FIG. 5. The Spherical Trigonometric Representation of the Rotation Matrix for the 3–1–3 Set of Euler Angles.

with the arc length ν_{ij} given by the appropriate arc of Fig. 5. Applying the spherical law of cosines to the appropriate triangles of Fig. 5 leads immediately to the desired expressions for the elements of the direction-cosine matrix in terms of the Euler angles. Note that the arc lengths in the figures are not drawn to scale. This is particularly true for the arcs of length $\pi/2$ and $3\pi/2$. Note also that the spherical triangle for computing ν_{33} is completely degenerate. The spherical triangles for the 3–1–3 set of Euler angles are equally simple and are shown in Fig. 6

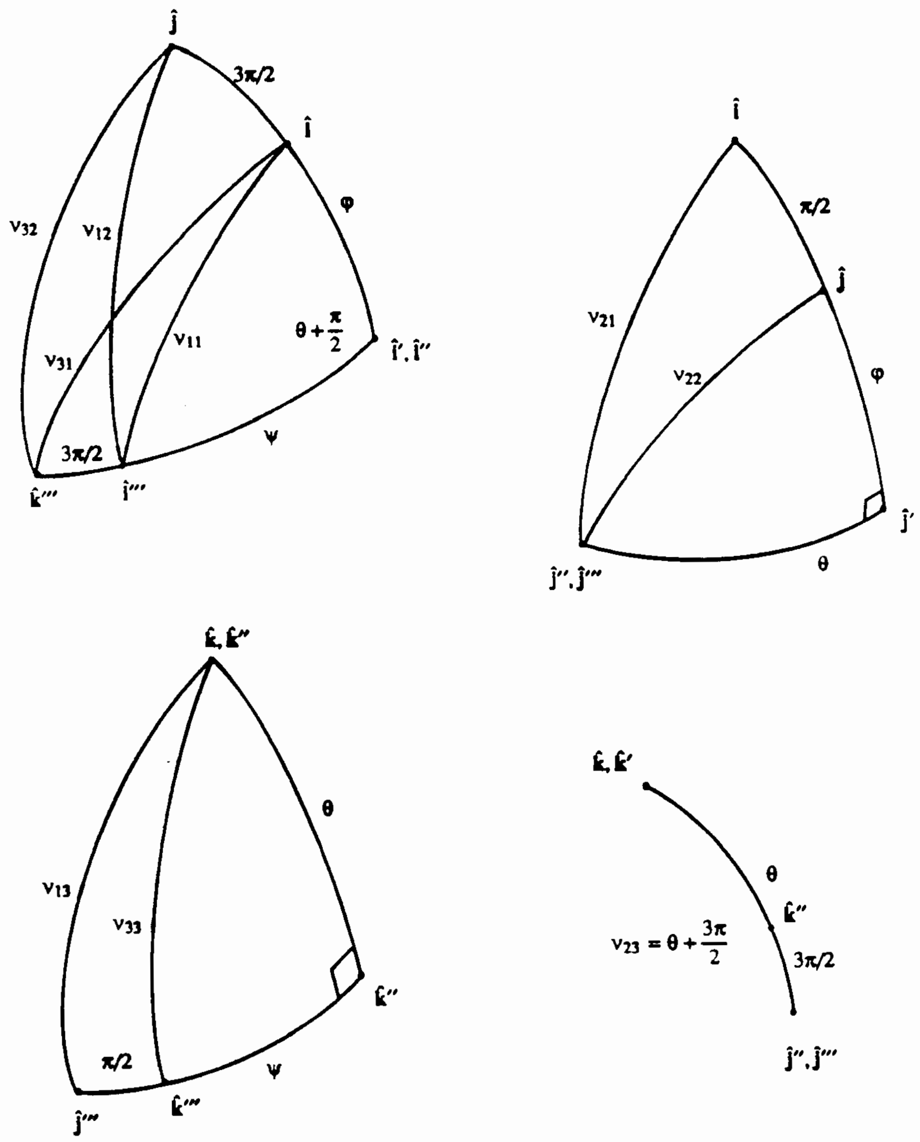


FIG. 6. The Spherical Trigonometric Representation of the Rotation Matrix for the 3–1–2 Set of Euler Angles.

One should compare the ease of computation of these expressions for the direction-cosine matrix with that from the Piograms [4]. The calculation of the spherical law of cosines may be said to be simpler than that of the Piograms. However, the construction of the appropriate spherical triangle generally takes a great deal more thought than the construction of the Piogram, which is fairly automatic. In addition, spherical trigonometry is much more difficult to apply to more than three rotations, which pose no particular problem for the Piograms.

Polar Complement Theorem

Examine now the effect of the 3–1–3 sequence of Euler angles on the z-axis. The result must again be a spherical triangle, since only the three rotations about the x-axis cause a displacement of $\hat{\mathbf{z}}$. However, now the arc lengths and the dihedral angles are reversed leading to the spherical triangle shown in Fig. 7. (The locus of the y-axis under this sequence of six rotations is an irregular right spherical hexagon and is of little obvious practical interest.)

The spherical triangle of Fig. 7 is called the *polar complement* of the spherical triangle of Fig. 2. The existence of the spherical triangle of Fig. 7, given the spherical triangle of Fig. 2, is known as the *Polar Complement Theorem*.

If we apply the law of cosines for sides to the complementary triangle we obtain

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \tag{14a}$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \tag{14b}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c, \tag{14c}$$

which is the *spherical law of cosines for angles* of the original spherical triangle of Fig. 2.

Thus, the Euler angles provide the vehicle for a very simple derivation of the Polar Complement Theorem and the Law of Cosines for Angles.

Composition of the Euler Angles

Suppose we are given two successive rotations, the first described by a 3–1–3 sequence of Euler angles $(\varphi_1, \vartheta_1, \psi_1)$, and the second described by a 3–1–3 sequence of Euler angles $(\varphi_2, \vartheta_2, \psi_2)$. What is the 3–1–3 sequence of Euler angles $(\varphi, \vartheta, \psi)$ of the combined rotation as given by

$$R_{313}(\varphi, \vartheta, \psi) = R_{313}(\varphi_2, \vartheta_2, \psi_2) R_{313}(\varphi_1, \theta_1, \psi_1)\,? \tag{15}$$

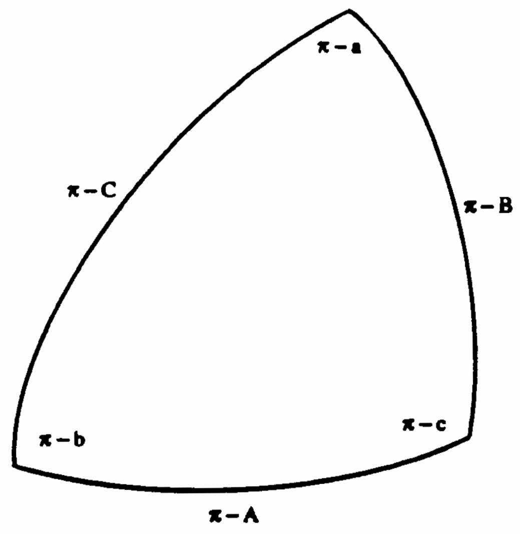


FIG. 7. Locus of the z-Axis in Response to a Closed 1–3–1–3–1–3 Euler Sequence.

Expanding each of the rotations and rearranging terms leads to

$$R(\hat{\mathbf{u}}_1, -\vartheta_2)R(\hat{\mathbf{u}}_3, \psi - \psi_2)R(\hat{\mathbf{u}}_1, \vartheta)R(\hat{\mathbf{u}}_3, \varphi - \varphi_1)R(\hat{\mathbf{u}}_1, -\vartheta_1) \times R(\hat{\mathbf{u}}_3, -\psi_1 - \varphi_2) = I. \quad (16)$$

Thus, the nine Euler angles satisfy the two spherical triangles of Fig. 8. Of these two figures, the first gives the locus of the z-axis and the second the locus of the x-axis. We see immediately from the diagrams that singularities in the expressions must occur when any of the arc lengths or dihedral angles are 0 (or, equivalently, 2π) or π. Thus, while the computation of the Euler angles from the direction-cosine matrices is singular only for extreme values of the media angle ϑ, the analytical behavior of the composition rule for Euler angles is clearly much more diseased, displaying a singularity at the critical value of any of the three media angles.

To compute an analytical form for the composition rule we note that the law of sines applied to Fig. 8 yields

$$\frac{\sin(\varphi - \varphi_1)}{\sin(\pi + \vartheta_2)} = \frac{\sin(\psi - \psi_2)}{\sin(\pi + \vartheta_1)} = \frac{\sin(-\varphi_2 - \psi_1)}{\sin(\pi - \vartheta)}, \quad (17)$$

which may be solved to yield

$$\sin(\varphi - \varphi_1) = \frac{\sin\vartheta_2}{\sin\vartheta}\sin(\varphi_2 + \psi_1), \quad (18a)$$

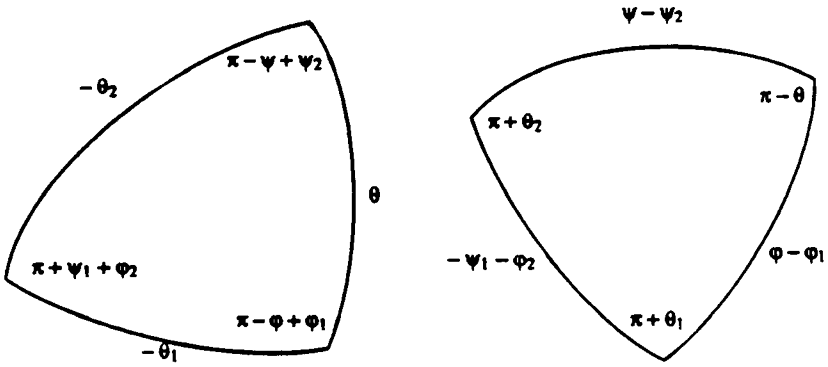
$$\sin(\psi - \psi_2) = \frac{\sin\vartheta_1}{\sin\vartheta}\sin(\varphi_2 + \psi_1). \quad (18b)$$

Likewise, applying the law of cosines for sides to the sides of Fig. 8a (or, equivalently, to the vertices of Fig. 8b) yields

$$\cos\vartheta = \cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos(\pi + \psi_1 + \varphi_2), \quad (19a)$$

$$\cos(-\vartheta_1) = \cos\vartheta\cos(-\vartheta_2) + \sin\vartheta\sin(-\vartheta_2)\cos(\pi - \psi + \psi_2), \quad (19b)$$

$$\cos(-\vartheta_2) = \cos\vartheta\cos(-\vartheta_1) + \sin\vartheta\sin(-\vartheta_1)\cos(\pi - \varphi + \varphi_1), \quad (19c)$$



a. Locus of the z-Axis. b. Locus of the x-Axis.

FIG. 8. Spherical Triangles for the composition of 3-1-3 Euler Sequences.

which may be solved to yield

$$\cos(\varphi - \varphi_1) = \frac{\cos\vartheta_2 - \cos\vartheta\cos\vartheta_1}{\sin\vartheta\sin\vartheta_1}, \tag{20a}$$

$$\cos(\psi - \psi_2) = \frac{\cos\vartheta_1 - \cos\vartheta\cos\vartheta_2}{\sin\vartheta\sin\vartheta_2}, \tag{20b}$$

Combining these results leads finally to

$$\vartheta = \arccos(\cos\vartheta_1\cos\vartheta_2 - \sin\vartheta_1\sin\vartheta_2\cos(\varphi_2 + \psi_1)), \tag{21a}$$

$$\varphi = \varphi_1 + \arctan_2(\sin\vartheta_1\sin\vartheta_2\sin(\varphi_2 + \psi_1), \cos\vartheta_2 - \cos\vartheta\cos\vartheta_1), \tag{21b}$$

$$\psi = \psi_2 + \arctan_2(\sin\vartheta_1\sin\vartheta_2\sin(\varphi_2 + \psi_1), \cos\vartheta_1 - \cos\vartheta\cos\vartheta_2), \tag{21c}$$

Similar results for a slightly restricted case were reported previously by Lindberg [5].

The above results can also be obtained analytically from the examination of the equations

$$R_{313}(\varphi - \varphi_1, \vartheta, \psi - \psi_2) = R_{131}(\vartheta_1, \varphi_2 + \psi_1, \vartheta_2), \tag{22a}$$

$$R_{313}(-\varphi + \varphi_1, \vartheta_1, \varphi_2 + \psi_1) = R_{131}(\vartheta, \psi - \psi_2, -\vartheta_2), \tag{22b}$$

$$R_{313}(\varphi_2 + \psi_1, \vartheta_2, -\psi + \psi_2) = R_{131}(-\vartheta_1, \varphi - \varphi_1, \vartheta), \tag{22c}$$

Calculating the (3,3), (3,1), and (1,3) elements of equation (22a), the (3,3) and (1,3) elements of equation (22b) and the (3,1) element of equation (22c) will, with some manipulation, furnish the above results. The singularity conditions, however, so clear from the spherical triangles, are difficult to extract from the equations.

A similarly simple result does not hold, apparently, for the asymmetric sequences of Euler angles (for example, the 3–1–2 sequence). For this case, the simplest loci for the composition problem correspond to spherical quadrilaterals as shown in Fig. 9.

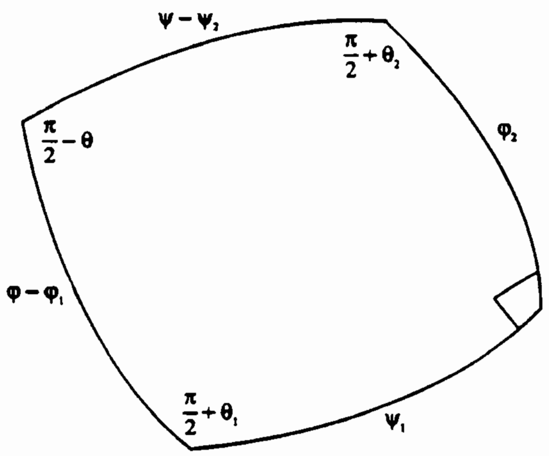


FIG. 9. Composition of the 3–1–2 Euler Angles.

From Fig. 9 we see that infinitesimal 3–1–2 Euler angles must simply add. The area of the spherical triangle in this case is obviously a function which is at least second order in the six initial Euler angles. Since the area of a spherical triangle is equal to the angular excess (in radians for a unit sphere), it follows for the spherical quadrilateral that

$$\sum_{i=1}^{4} (\text{dihedral angle})_i = 2\pi + \text{Area}. \tag{23}$$

Substituting the values of the four dihedral angles yields, therefore,

$$\vartheta = \vartheta_1 + \vartheta_2 + O(\text{angles}^2). \tag{24}$$

Since all four dihedral angles are close to $\pi/2$, it follows that the spherical quadrilateral is close to rectangular. Hence, opposite sides will be equal to second order in the angles, from which we have immediately

$$\varphi = \varphi_1 + \varphi_2 + O(\text{angles}^2), \quad \text{and} \quad \psi = \psi_1 + \psi_2 + O(\text{angles}^2). \tag{25}$$

Transformation of Angles

Spherical trigonometry also provides us with an easy means of connecting the 3–1–3 and 3–1–2 sequences of the Euler angles, which must satisfy

$$R_{313}(\varphi_{313}, \vartheta_{313}, \psi_{313}) = R_{312}(\varphi_{312}, \vartheta_{312}, \psi_{312}). \tag{26}$$

By transposing one member to obtain a sequence of rotations whose combined effect is the identity, we obtain the spherical triangle of Fig. 10. Solution of this right spherical triangle by Napier's rules [6] leads directly to the relations

$$\vartheta_{313} = \arccos(\cos\vartheta_{312}\cos\psi_{312}), \tag{27a}$$

$$\varphi_{313} = \varphi_{312} + \arctan_2(-\sin\vartheta_{312}\cos\psi_{312}, \sin\psi_{312}), \tag{27b}$$

$$\psi_{313} = \arctan_2(-\cos\vartheta_{312}\sin\psi_{312}, \sin\vartheta_{312}), \tag{27c}$$

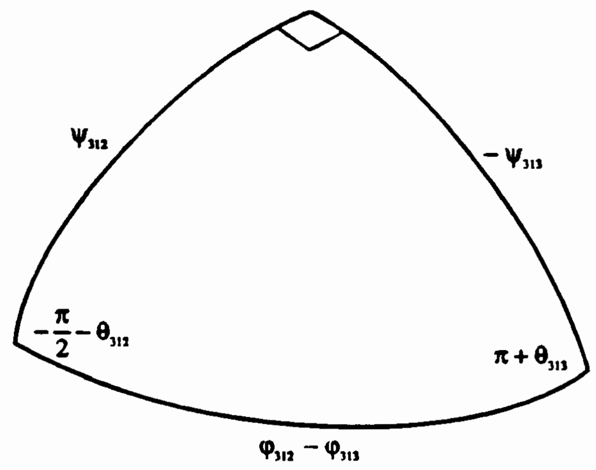


FIG. 10. Connection of the 3–1–3 to the 3–1–2 Euler Angles.

and

$$\vartheta_{312} = \arcsin(\sin \vartheta_{313} \cos \psi_{313}), \tag{28a}$$

$$\varphi_{312} = \varphi_{313} + \arctan_2(\sin \psi_{313}, \cos \vartheta_{313} \cos \psi_{313}), \tag{28b}$$

$$\psi_{312} = \arctan_2(-\sin \vartheta_{313} \sin \psi_{313}, \cos \vartheta_{313}), \tag{28c}$$

Similar relations can be derived for the connection to the 2–1–3 sequence. Note that Fig. 10 leads to a very unpleasant looking spherical triangle for $\varphi_{312}, \vartheta_{312}, \psi_{312}$ small, a consequence of the singular nature of the 3–1–3 Euler angles for infinitesimal rotations.

Conclusions

The Euler angles provide us with a mechanical understanding of the Polar Complement Theorem and a simple derivation of the spherical law of cosines for angles. Likewise, spherical trigonometry provides us with a simple and direct algorithm for combining two rotations described in terms of symmetric sequences of Euler angles. These results, which do not seem to be known generally, are satisfied by any symmetric set of Euler angles. Of more practical importance is that the spherical trigonometrical relations provide a much readier picture of the singularities involved in combining Euler angle sequences than would be obtained from a cursory inspection of the equations. Figure 8 and equations (21)–(25), the most important new results of this work, hold for any of the six symmetric Euler angle sequences. We feel that the spherical trigonometric approach provides the most efficient and elegant path to equation (21). Alternate expressions to those developed here have been presented in earlier works [7, 8].

Acknowledgments

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References

[1] JUNKINS, J. L., and TURNER, J. D. *Optimal Spacecraft Rotational Maneuvers*. Elsevier, Amsterdam, 1986.

[2] WERTZ, J. R. (editor) *Spacecraft Attitude Determination and Control*, Kluwer Academic, Dordrecht, 1978.

[3] SHUSTER M. D. "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[4] BAR-ITZHACK, I.Y. "The Piogram, an Easy Tool for Angular Position and Rate Computations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993 pp. 519–529.

[5] LINDBERG, R. E., Jr. "Two Impulse Large Angle Satellite Attitude Transfers," Master's Thesis, Department of Aerospace Engineering and Engineering Physics, The University of Virginia, Charlottesville, Virginia, 1976.

[6] BALLOU, D. H., and STEEN, F. H. *Plane and Spherical Trigonometry*, Ginn and Company, Boston, 1943.

[7] SHUSTER, M. D., and JUNKINS, J. L. "Some Interesting Properties of the Euler Angles," *Proceedings, Third Pan-American Congress of Applied Mechanics,* São Paulo, Brazil, January, 1993, pp. 523–526.

[8] SHUSTER, M. D., and JUNKINS, J. L. "Geometrical Properties of the Euler Angles," (Proceedings of the AAS/AIAA International Symposium on Spaceflight Dynamics, Greenbelt, Maryland, April 26–30, 1993), *Advances in the Astronautical Sciences,* Vol. 84, Univelt, San Diego, 1993, pp. 1165–1172.