GEOMETRICAL PROPERTIES OF THE EULER ANGLES

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The Euler angles are shown to provide a simple means for understanding some of the fundamental results of Spherical Trigonometry. Spherical Trigonometry, in turn, is used to develop a compact expression for the composition of two sets of Euler angles.

INTRODUCTION

The Euler angles and spherical trigonometry are generally treated as separate subjects. However, treating the two together will lead to new insights and some new results. Of particular interest for us has been the development of an expression for the composition of two rotations, each parameterized by the Euler angles. Generally, to accomplish this composition, the two rotations described by Euler angles sequences must first be expressed as direction-cosine matrices, the two direction-cosine matrices multiplied together, and then the rules for extracting Euler angles applied. It turns out that a much simpler procedure exists but is not generally known. In developing such a procedure, a better understanding of Spherical Trigonometry is also obtained.

SPHERICAL TRIGONOMETRY

Let $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{C}}$ be three unit vectors, which we may represent as three points on the unit circle[1] as shown in Fig. 1.

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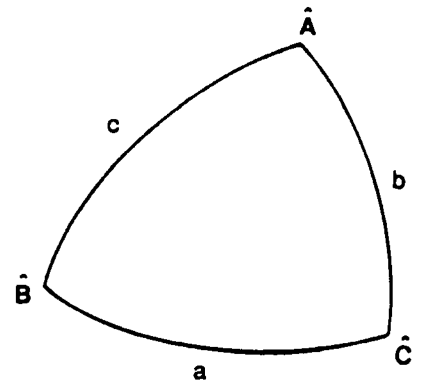

Fig. 1 A Spherical Triangle

Here a, b, and c are the lengths of the arcs of great circles connecting the three points. If O (not shown) is the center of the unit sphere, then these three arc lengths are equal to the three angles BOC, COA, and AOB, respectively. In addition to these three arc lengths, we may define the three dihedral angles, which are the angles between the planes defined by the center and two of the vertices. Thus, the dihedral angle A is the angles between the planes AOB and AOC. If we think of the angle as the angle of rotation of one plane into the other, then the axis of rotation is $\hat{\mathbf{A}}$. The complete description of the spherical triangle in terms of arc lengths and dihedral angles is shown in Fig. 2.

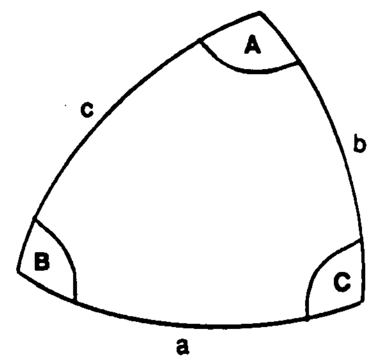

Fig. 2 A Spherical Triangle

To represent the three vertices of the triangle analytically, let us choose the z-axis to coincide with $\hat{\mathbf{A}}$, and let $\hat{\mathbf{B}}$ lie in the xz-plane. Then

$$\hat{\mathbf{A}} = \hat{\mathbf{z}} , \tag{1a}$$

$$\hat{\mathbf{B}} = \cos c\,\hat{\mathbf{z}} + \sin c\,\hat{\mathbf{x}} , \tag{1b}$$

$$\hat{\mathbf{C}} = \cos b\,\hat{\mathbf{z}} + \sin b \cos A\,\hat{\mathbf{x}} + \sin b \sin A\,\hat{\mathbf{y}} , \tag{1c}$$

Thus,

$$\cos a = \hat{\mathbf{B}} \cdot \hat{\mathbf{C}} = \cos b \cos c + \sin b \sin c \cos A\,. \tag{2a}$$

Likewise,

$$\cos b = \cos c \cos a + \sin c \sin a \cos B\,, \tag{2b}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C\,, \tag{2c}$$

obtained by choosing $\hat{\mathbf{z}}$ and $\hat{\mathbf{x}}$ appropriately in each case.

It follows also that

$$(\hat{\mathbf{A}} \times \hat{\mathbf{B}}) \cdot \hat{\mathbf{C}} = \sin A \sin b \sin c,. \tag{3}$$

From the cyclic invariance of the scalar triple product,

$$(\hat{\mathbf{A}} \times \hat{\mathbf{B}}) \cdot \hat{\mathbf{C}} = (\hat{\mathbf{B}} \times \hat{\mathbf{C}}) \cdot \hat{\mathbf{A}} = (\hat{\mathbf{C}} \times \hat{\mathbf{A}}) \cdot \hat{\mathbf{B}}\,, \tag{4}$$

whence

$$\sin A \sin b \sin c = \sin B \sin c \sin a = \sin C \sin a \sin b\,. \tag{5}$$

Dividing all three members by $\sin a \sin b \sin c$ leads to

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}\,. \tag{6}$$

Equations (2) are the *spherical law of of cosines for sides* and Eq. (6) is the *spherical law of sines*. There are additional useful and important results which we will obtain in conjunction with the Euler angles.

THE EULER ANGLES

Any rotation matrix may be parameterized in terms of a symmetric sequence of Euler angles[2], which we write in the form

$$R_{\ell m \ell}(\varphi, \vartheta, \psi) \equiv R(\hat{\mathbf{u}}_\ell, \psi)\, R(\hat{\mathbf{u}}_m, \vartheta)\, R(\hat{\mathbf{u}}_\ell, \varphi)\,, \tag{7}$$

where $\hat{\mathbf{u}}_\ell$ and $\hat{\mathbf{u}}_m$ are two distinct unit column vectors which are chosen from the set

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{8}$$

Thus,

$$R(\hat{\mathbf{u}}_1, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix}, \quad R(\hat{\mathbf{u}}_2, \theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix}, \tag{9ab}$$

$$R(\hat{\mathbf{u}}_3, \theta) = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9c}$$

Fig. 3 Displacement of the x-axis under a rotatin about the z-axis

where $c\theta \equiv \cos\theta$ and $s\theta \equiv \sin\theta$. The three angles, φ, ϑ, ψ, are usually restricted to the intervals

$$0 \leq \varphi < 2\pi\,, \quad 0 \leq \vartheta \leq \pi\,, \quad \text{and} \quad 0 \leq \psi < 2\pi\,. \tag{10}$$

We call the sequence of Euler angles appearing in the parameterization of the rotation matrix in Eq. (1) *symmetric* in order to distinguish it from an asymmetric sequence of Euler angles in which no two axis column-vectors are identical. We particularize our discussion for the 3–1–3 sequence of Euler angles. However, the results will be true for any symmetric sequence.

SPHERICAL-TRIGONOMETRIC DEPICTION OF THE EULER ANGLES

Consider the action on the x-axis of a rotation described by 3–1–3 sequence of Euler angles $(\varphi,\ \vartheta,\ \psi)$. Imagine that the rotations correspond to a sequence of physical rotations such that

$$R(t) = \begin{cases} R(\hat{\mathbf{u}}_3,\ \varphi t)\,, & \text{for } 0 \leq t < 1\,, \\ R(\hat{\mathbf{u}}_1,\ \vartheta(t-1))\, R(\hat{\mathbf{u}}_3\, \varphi)\,, & \text{for } 1 \leq t < 2\,, \\ R(\hat{\mathbf{u}}_3,\ \psi(t-2)) R(\hat{\mathbf{u}}_1,\ \vartheta)\, R(\hat{\mathbf{u}}_3\, \varphi)\,, & \text{for } 2 \leq t \leq 3\,. \end{cases} \tag{11}$$

Under the first rotation, the x-axis is displaced in the y-direction by an arc length φ as shown in Fig. 3. Note that we describe the displacement of the physical x-axis and not the representation.

The second rotation being about the x-axis does not displace the x-axis. However, it rotates the y-direction by an angle ϑ, so that the further movement of the x axis is along another direction. The third rotation, about the new z-axis, displaces the x-axis again. The combined action of all three rotations on the x-axis is depicted in Fig. 4. Thus, the two outer angles correspond to arc lengths, and the medial angle to a dihedral angle. Many of the elements of the direction-cosine matrix have the appearance of the spherical law of cosines. An examination of the rotations required to connect the two axes whose direction cosine is being computed will lead to a diagram much like Fig. 4.

Suppose now we consider a sequence of six rotations satisfying

$$R(\hat{\mathbf{u}}_1,\ \pi - B)\, R(\hat{\mathbf{u}}_3,\ c)\, R(\hat{\mathbf{u}}_1,\ \pi - A)\, R(\hat{\mathbf{u}}_3,\ b)\, R(\hat{\mathbf{u}}_1,\ \pi - C)\, R(\hat{\mathbf{u}}_3,\ a) = I\,. \tag{12}$$

Because the complete sequence of rotations must be equivalent to the identity rotation (we will call such a sequence *closed*), it follows that the locus of any point on unit sphere, when the

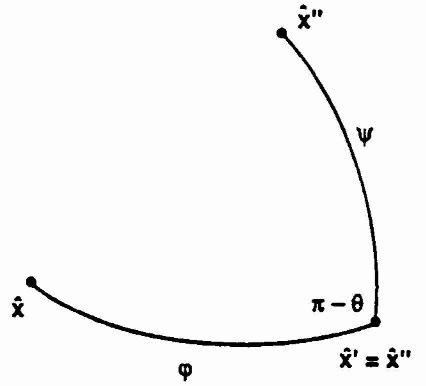

Fig. 4 Locus of the x-axis in response to a 3–1–3 Euler sequence

sequence of six rotations is written in a manner similar to Eq. (11), must be a closed curve. For the x-axis, the locus will be a spherical triangle as shown in Fig. 2. Examine now the effect of this sequence on the z-axis. The result must again be a spherical triangle, since only the three rotations about the x-axis cause a displacement of $\hat{z}$. However, now the arc lengths and the dihedral angles are reversed leading to the spherical triangle shown in Fig. 5. (The locus of the y-axis under this sequence of six rotations is an irregular right spherical hexagon.)

The spherical triangle of Fig. 5 is called the *polar complement* of the spherical triangle of Fig. 2. The existance of the spherical triangle of Fig. 5, given the spherical triangle of Fig. 2 is known as the *Polar Complement Theorem.*

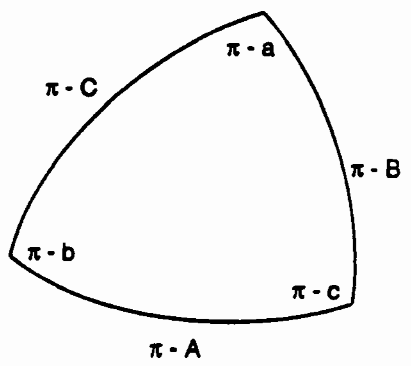

Fig. 5 Locus of the z-axis in response to a closed 1–3–1–3–1–3 Euler sequence

If we apply the law of cosines for sides to the complementary triangle we obtain

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a\,, \tag{13a}$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b\,, \tag{13b}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c\,, \tag{13c}$$

which is the *spherical law of cosines for angles* of the original spherical triangle of Fig. 2.

Thus, the Euler angles provide the vehicle for a very simple derivation of the Polar Complement Theorem and the Law of Cosines for Angles.

COMPOSITION OF THE EULER ANGLES

Suppose we are given two successive rotations, the first described by a 3–1–3 sequence of Euler angles $(\varphi_1, \vartheta_1, \psi_1)$, and the second described by a 3–1–3 sequence of Euler angles $(\varphi_2, \vartheta_2, \psi_2)$. What is the 3–1–3 sequence of Euler angles $(\varphi, \vartheta, \psi)$ of the combined rotation as given by

$$R_{313}(\varphi, \vartheta, \psi) = R_{313}(\varphi_2, \vartheta_2, \psi_2)\, R_{313}(\varphi_1, \vartheta_1, \psi_1)\,? \tag{14}$$

Expanding each of the rotations and rearranging terms leads to

$$R(\hat{\mathbf{u}}_1, -\vartheta_2)\, R(\hat{\mathbf{u}}_3, \psi - \psi_2)\, R(\hat{\mathbf{u}}_1, \vartheta)\, R(\hat{\mathbf{u}}_3, \varphi - \varphi_1)\, R(\hat{\mathbf{u}}_1, -\vartheta_1)\, R(\hat{\mathbf{u}}_3, -\psi_1 - \varphi_2) = I\,. \tag{15}$$

Thus, the nine Euler angles satisfy the two spherical triangles of Fig. 6. Of these two figures, the first gives the locus of the z-axis and the second the locus of the x-axis. We see immediately from the diagrams that singularities in the expressions must occur when any of the arc lengths or dihedral angles are 0 (or, equivalently, 2π) or π. Thus, while the computation of the Euler angles from the direction-cosine matrices is singular only for extreme values of the medial angle, ϑ, the analytical behavior of the composition rule for Euler angles is clearly much more diseased.

To compute an analytical form for the composition rule we note that the law of sines applied to Fig. 6 yields

$$\frac{\sin(\varphi - \varphi_1)}{\sin(\pi + \vartheta_2)} = \frac{\sin(\psi - \psi_2)}{\sin(\pi + \vartheta_1)} = \frac{\sin(-\varphi_2 - \psi_1)}{\sin(\pi - \vartheta)}\,, \tag{16}$$

which may be solved to yield

$$\sin(\varphi - \varphi_1) = \frac{\sin\vartheta_2}{\sin\vartheta}\sin(\varphi_2 + \psi_1)\,, \tag{17a}$$

$$\sin(\psi - \psi_2) = \frac{\sin\vartheta_1}{\sin\vartheta}\sin(\varphi_2 + \psi_1)\,. \tag{17b}$$

Likewise, applying the law of cosines for sides to the appropriate vertex of Fig. 6a yields

$$\cos(\vartheta) = \cos\vartheta_1 \cos(\vartheta_2) + \sin\vartheta_1 \sin(\vartheta_2)\cos(\pi + \psi_1 + \varphi_2)\,, \tag{18a}$$

$$\cos(-\vartheta_1) = \cos\vartheta \cos(-\vartheta_2) + \sin\vartheta \sin(-\vartheta_2)\cos(\pi - \psi + \psi_2)\,, \tag{18b}$$

$$\cos(-\vartheta_2) = \cos\vartheta \cos(-\vartheta_1) + \sin\vartheta \sin(-\vartheta_1)\cos(\pi - \varphi + \varphi_1)\,, \tag{18c}$$

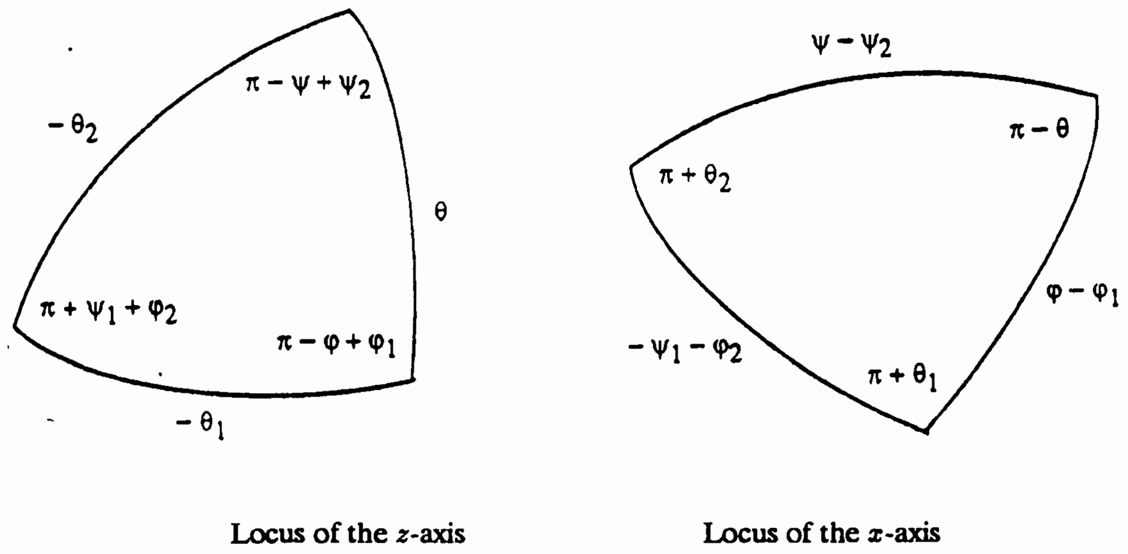

Fig. 6 Spherical Triangles for the Composition of 3–1–3 Euler Sequences

which may be solved to yield

$$\cos(\varphi - \varphi_1) = \frac{\cos\vartheta_2 - \cos\vartheta\,\cos\vartheta_1}{\sin\vartheta\,\sin\vartheta_1}, \tag{19a}$$

$$\cos(\psi - \psi_2) = \frac{\cos\vartheta_1 - \cos\vartheta\,\cos\vartheta_2}{\sin\vartheta\,\sin\vartheta_2}. \tag{19b}$$

Combining these results leads finally to

$$\vartheta = \arccos\big(\cos\vartheta_1\,\cos\vartheta_2 - \sin\vartheta_1\,\sin\vartheta_2\,\cos(\varphi_2 + \psi_1)\big), \tag{20a}$$

$$\varphi = \varphi_1 + \arctan_2\big(\sin\vartheta_1\,\sin\vartheta_2\,\sin(\varphi_2 + \psi_1),\,\cos\vartheta_2 - \cos\vartheta\,\cos\vartheta_1\big), \tag{20b}$$

$$\psi = \psi_2 + \arctan_2\big(\sin\vartheta_1\,\sin\vartheta_2\,\sin(\varphi_2 + \psi_1),\,\cos\vartheta_1 - \cos\vartheta\,\cos\vartheta_2\big). \tag{20c}$$

Similar results for a slightly restricted case were obtained previously by Lindberg[3].

The above results can be obtained analytically from the examination of the equations

$$R_{313}(\varphi - \varphi_1,\,\vartheta,\,\psi - \psi_2) = R_{131}(\vartheta_1,\,\varphi_2 + \psi_1,\,\vartheta_2), \tag{21b}$$

$$R_{313}(-\varphi + \varphi_1,\,\vartheta_1,\,\varphi_2 + \psi_1) = R_{131}(\vartheta,\,\psi - \psi_2,\,-\vartheta_2), \tag{21b}$$

$$R_{313}(\varphi_2 + \psi_1,\,\vartheta_2,\,-\psi + \psi_2) = R_{131}(-\vartheta_1,\,\varphi - \varphi_1,\,\vartheta). \tag{21c}$$

Calculating the (3, 3), (3, 1), (3, 2), (1, 3), and (2, 3) elements of these equations will, with some manipulation, furnish the above results. The singularity conditions, however, so clear from the spherical triangles, are difficult to extract from the equations.

A similarly simple result does not hold, apparently, for the asymmetric sequences of Euler angles (for example, the 3–1–2 sequence). For this case, the simplest loci for the compositon problem correspond to spherical quadrilaterals. Less efficient expressions than those developed here have been presented in an earlier work[4].

CONCLUSION

The Euler angles provide us with a mechanical understanding of the Polar Complement Theorem and a simple derivation of the spherical law of cosines for angles. Likewise, Spherical Trigonometry provides us with a simple and direct algorithm for combining two rotations described in terms of symmetric sequences of Euler angles. These results, which do not seem to be known generally, are satisfied by any symmetric set of Euler angles. Of more practical importance is that the spherical trigonometrical relations provide a much readier picture of the singularities involved in combining Euler angle sequences than would be obtained from a cursory inspection of the equations. Fig. 6 and Eqs. (20), the new results of this work, hold for any of the six symmetric Euler angle sequences. We feel that the spherical trigonometric approach provides the most efficient and elegant path to Equation (20).

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