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The Kinematic Equation for the Rotation Vector

Different derivations of the kinematic equation for the rotation vector are discussed within a common framework. Simpler and more direct derivations of this kinematic equation are presented than are found in the literature. The kinematic equation is presented in terms of both the body-referenced angular velocity and the inertially referenced angular velocity. The kinematic equation is shown to have the same form in both the passive and active descriptions of attitude.

I. INTRODUCTION

A number of derivations [1–6] have been presented of the kinematic equation of the rotation vector defined as

$$\boldsymbol{\theta} = \theta \hat{\mathbf{n}} \tag{1}$$

where θ is the angle of rotation and $\hat{\mathbf{n}}$ is the column vector representing the axis of rotation. These different derivations have much in common and all follow a general pattern. Three of these derivations are essentially identical within a homomorphism.

There are likely no undiscovered magical derivations of this equation. However, two of the derivations already published can be considerably simplified. After discussing the general character of the published derivations of this equation, those simplifications are presented.

Manuscript received May 1, 1991; revised February 25, 1992.

IEEE Log No. T-AES/29/1/02259.

II. THE GENERAL METHOD

Each of the published derivations begins with some other representation of the attitude, denoted here by S, for which the kinematic equation is known and may be written as

$$\frac{d}{dt}S = G(S,\omega) \tag{2}$$

where ω is the body-referenced angular velocity [6–8]. The representation S can be parameterized as a function of the rotation vector and the angle of rotation as

$$S = S(\boldsymbol{\theta},\theta). \tag{3}$$

Thus, for example, the direction-cosine matrix can be written

$$C(\boldsymbol{\theta},\theta) = I + \frac{\sin\theta}{\theta}[[\boldsymbol{\theta}]] + \frac{1-\cos\theta}{\theta^2}[[\boldsymbol{\theta}]]^2 \tag{4}$$

where

$$[[\mathbf{u}]] \equiv \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix} \tag{5}$$

and the related kinematic equation is

$$\frac{d}{dt}C = [[\omega]]C. \tag{6}$$

It is, of course, possible to consider S as a function of $\boldsymbol{\theta}$ alone. However, it is more convenient for the discussion which follows to express separately the dependence on θ through θ.

Let $F(S)$ be some differentiable function of S. Differentiating $F(S(\boldsymbol{\theta},\theta))$ with respect to time leads to

$$\frac{d}{dt}F(S) = \frac{\partial F}{\partial S}\left[\frac{\partial S}{\partial \boldsymbol{\theta}}\dot{\boldsymbol{\theta}} + \frac{\partial S}{\partial \theta}\dot{\theta}\right] = \frac{\partial F}{\partial \boldsymbol{\theta}}\dot{\boldsymbol{\theta}} + \frac{\partial F}{\partial \theta}\dot{\theta}. \tag{7}$$

Comparing (2) and (7) yields the system of equations

$$\frac{\partial F}{\partial \boldsymbol{\theta}}\frac{d\boldsymbol{\theta}}{dt} = -\frac{\partial F}{\partial \theta}\dot{\theta} + \frac{\partial F}{\partial S}G(S,\omega) \tag{8}$$

which yields an identity in $\boldsymbol{\theta}$, $\dot{\boldsymbol{\theta}}$, θ, $\dot{\theta}$, and ω, which we can use to solve eventually for $\dot{\boldsymbol{\theta}}$ as a function of ω and $\boldsymbol{\theta}$. Note that the derivatives in (7) and (8) with respect to $\boldsymbol{\theta}$ and S are not simple, and the products must be understood to imply a sum over elements as in matrix multiplication.

In almost all of the derivations of the kinematic equation for $\dot{\boldsymbol{\theta}}$, in order to obtain an *explicit* expression for this quantity, the terms in $\dot{\theta}$ must be eliminated. All of the derivations accomplish this in the same manner which is as follows.

From (4) it follows that

$$\mathrm{tr}C = 1 + 2\cos\theta \tag{9}$$

where tr denotes the trace operation. Hence

$$\frac{d}{dt}\mathrm{tr}C = -2\sin\theta\dot{\theta}. \tag{10}$$

This is also equal to

$$\begin{aligned}\mathrm{tr}\frac{d}{dt}C &= \mathrm{tr}([[\boldsymbol{\omega}]]C)\\ &= -2\frac{\sin\theta}{\theta}\boldsymbol{\theta}\cdot\boldsymbol{\omega}.\end{aligned} \tag{11}$$

Comparing (10) and (11), one obtains

$$\dot{\theta} = \frac{1}{\theta}\boldsymbol{\theta}\cdot\boldsymbol{\omega}. \tag{12}$$

In all of the derivations, $\dot{\theta}$, when it appears, appears in the combination $\dot{\theta}\boldsymbol{\theta}$. Thus,

$$\frac{\partial F}{\partial \theta} = H(\theta,\theta)\boldsymbol{\theta}. \tag{13}$$

The combination $\dot{\theta}\boldsymbol{\theta}$ may be further manipulated to yield

$$\begin{aligned}\dot{\theta}\boldsymbol{\theta} &= \frac{1}{\theta}(\boldsymbol{\theta}\cdot\boldsymbol{\omega})\boldsymbol{\theta}\\ &= \frac{1}{\theta}(\theta^2\boldsymbol{\omega} + \boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}))\end{aligned} \tag{14}$$

and finally

$$\begin{aligned}\frac{\partial F}{\partial \boldsymbol{\theta}}\dot{\boldsymbol{\theta}} &= -H(\theta,\theta)\frac{1}{\theta}(\theta^2\boldsymbol{\omega} + \boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}))\\ &\quad + \frac{\partial F}{\partial S}G(S(\theta,\theta),\boldsymbol{\omega})\end{aligned} \tag{15}$$

which is now solved for $\dot{\boldsymbol{\theta}}$. All of the published derivations amount to applying these steps to a given function of some representation of the attitude.

It is interesting to note that the differentiation of

$$\theta = (\boldsymbol{\theta}\cdot\boldsymbol{\theta})^{1/2} \tag{16}$$

leads to

$$\dot{\theta} = \frac{1}{\theta}\boldsymbol{\theta}\cdot\dot{\boldsymbol{\theta}} \tag{17}$$

so that $\dot{\boldsymbol{\theta}}$ and $\boldsymbol{\omega}$ have identical components along $\boldsymbol{\theta}$. Consequently, it must be true that

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\omega} + f_1(\theta)\boldsymbol{\theta}\times\boldsymbol{\omega} + f_2(\theta)\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\boldsymbol{\omega}) \tag{18}$$

for some functions $f_1(\theta)$ and $f_2(\theta)$.

III. PUBLISHED DERIVATIONS

In the work of Bortz [1], the function studied is the vector

$$F(C) = (C - C^T)\mathbf{r} \tag{19}$$

where $\mathbf{r}$ is some arbitrary column vector. The specific value of $\mathbf{r}$ is unimportant since it is discarded in the course of the derivation. Other derivations predate that of Bortz. Stuelpnagel [9] gives a result for the kinematic equation expressed in terms of the antisymmetric matrix but omits the derivation as being too long and too complicated. Bortz acknowledges an unpublished derivation of Lanning [10]. Undoubtedly, a classical derivation exists but has been forgotten.

Gelman [2] works directly with (6), which he writes in terms of elements using the Levi-Civita symbol. The properties of the Levi-Civita symbol are then exploited to obtain relations for the temporal derivatives of the axis and angle of rotation in terms of the angular velocity and the inverse relation.

In the work of Nazaroff [3], the function is the Rodrigues vector $\mathbf{g}$, [6–8]

$$F(\mathbf{g}) = \mathbf{g} = (\tan(\theta/2))\hat{\mathbf{n}}. \tag{20}$$

Nazaroff actually derives kinematic equations for θ and $\hat{\mathbf{n}}$ and then determines $\dot{\boldsymbol{\theta}}$ from

$$\frac{d}{dt}\boldsymbol{\theta} = \left(\frac{d\theta}{dt}\right)\hat{\mathbf{n}} + \theta\left(\frac{d\hat{\mathbf{n}}}{dt}\right). \tag{21}$$

Savage [4] employs the quaternion $\overline{q}$ which he writes as

$$\overline{q} = \begin{bmatrix}\left(\frac{1}{\theta}\sin(\theta/2)\right)\boldsymbol{\theta}\\ \cos(\theta/2)\end{bmatrix}. \tag{22}$$

The function $F(\overline{q})$ consists of the vector components of $\overline{q}$. The derivations of Bortz, Nazaroff, and Savage, all follow rather closely the program outlined above.

Hughes [6] uses the full direction-cosine matrix and determines $\boldsymbol{\omega}$ as a function of $\dot{\theta}$ and $\dot{\hat{\mathbf{n}}}$ by evaluating

$$[[\boldsymbol{\omega}]] = C^T(\theta,\hat{\mathbf{n}})\dot{C}(\theta,\hat{\mathbf{n}},\dot{\theta},\dot{\hat{\mathbf{n}}}). \tag{23}$$

This leads to three equations which are linear in $\dot{\theta}$ and $\dot{\hat{\mathbf{n}}}$, which Hughes then solves for these quantities. Equation (21) then yields $\dot{\boldsymbol{\theta}}$. Hughes' method is equivalent to the program outlined above with $F(C) = C$, but he simply solves first for $\boldsymbol{\omega}$ as a function of $\dot{\boldsymbol{\theta}}$ rather than *vice versa*. For lack of a better name, we have termed equations like (23) the inverse kinematic equation.

Jiang and Lin in the most recently published and rather novel derivation [5] deviate from this pattern by considering the equation

$$(C - I)\boldsymbol{\theta} = \mathbf{0} \tag{24}$$

which may be differentiated to obtain

$$(C - I)\dot{\boldsymbol{\theta}} + \dot{C}\boldsymbol{\theta} = \mathbf{0}. \tag{25}$$

With considerable effort this equation may be solved for $\dot{\boldsymbol{\theta}}$. While their starting point is very different from that of the other authors, most of the remaining steps of their derivation are very similar to those of the derivations which proceeded theirs. In the work of Jiang and Lin, ω is the space-referenced angular velocity, and the direction-cosine matrix is given in the active description. In the passive description with the body-referenced angular velocity, the more common convention, their equation becomes

$$(C^T - I)\dot{\theta} + \dot{C}^T\theta = \mathbf{0}. \tag{26}$$

(Note that θ is an eigenvector with eigenvalue +1 of both C and C^T, so that (25) and (26) hold simultaneously in both the active and passive descriptions.) Substituting

$$\dot{C}^T = C^T[[\omega]]^T = -C^T[[\omega]] \tag{27}$$

(26) may be transformed to

$$(I - C(\theta,\theta))\dot{\theta} = [[\omega]]\theta \tag{28}$$

a rather simpler equation than, in fact, appears in the note of Jiang and Lin. The virtue of the method of Jiang and Lin is that the quantity $\dot{\theta}$ does not appear and, therefore, need not be eliminated. In revenge for the loss of this chore, however, Jiang and Lin must perform a complicated matrix inversion, which they seem to accomplish by means of the Cayley-Hamilton Theorem [11].

The comparison of these derivations is complicated by the fact that some authors work in the active description, while others work in the passive description, and also some of the authors use the body-referenced angular velocity while others use the space-referenced angular velocity. The discussion of the work of Jiang and Lin above illustrates this confusion and the danger of taking a result out of the context of its derivation without exercising due caution.

IV. YET ANOTHER DERIVATION

The derivation by the author is similar to that of Bortz [1] but has been simplified considerably (and the need for a helping vector **r** has been eliminated) by taking advantage of the algebra of the 3×3 antisymmetric matrices [8]. These matrices satisfy

$$[[\mathbf{u}]]^T = -[[\mathbf{u}]] \tag{29}$$

$$[[\mathbf{u}]]\mathbf{v} = -\mathbf{u} \times \mathbf{v} \tag{30}$$

$$[[\mathbf{u}]]^2 = -|\mathbf{u}|^2 I + \mathbf{u}\mathbf{u}^T \tag{31}$$

$$[[\mathbf{u}]]^3 = -|\mathbf{u}|^2[[\mathbf{u}]] \tag{32}$$

$$[[\mathbf{u}]][[\mathbf{v}]] - [[\mathbf{v}]][[\mathbf{u}]] = -[[\mathbf{u} \times \mathbf{v}]] \tag{33}$$

$$\mathbf{u}\mathbf{u}^T[[\mathbf{v}]] + [[\mathbf{v}]]\mathbf{u}\mathbf{u}^T = -[[\mathbf{u} \times (\mathbf{u} \times \mathbf{v})]]. \tag{34}$$

From (4)

$$C - C^T = 2\frac{\sin\theta}{\theta}[[\theta]] \equiv a(\theta)[[\theta]] \tag{35}$$

and

$$\begin{aligned}\frac{d}{dt}(C - C^T) &= a(\theta)[[\dot{\theta}]] + a'(\theta)\dot{\theta}[[\theta]] \\ &= a(\theta)[[\dot{\theta}]] + a'(\theta)\frac{1}{\theta}[[\theta^2\omega + \theta \times (\theta \times \omega)]]\end{aligned} \tag{36}$$

where $a'(\theta)$ denotes the derivative of $a(\theta)$ with respect to θ. Likewise,

$$\begin{aligned}\frac{d}{dt}(C - C^T) &= [[\omega]]C - C^T[[\omega]]^T \\ &= [[\omega]]C + C^T[[\omega]].\end{aligned} \tag{37}$$

Substituting (4) into (37), this becomes

$$\begin{aligned}\frac{d}{dt}(C - C^T) =& 2[[\omega]] + \frac{\sin\theta}{\theta}([[\omega]][[\theta]] - [[\theta]][[\omega]]) \\ &+ \frac{1-\cos\theta}{\theta^2}([[\omega]][[\theta]]^2 + [[\theta]]^2[[\omega]]).\end{aligned} \tag{38}$$

Applying (31), (33), and (34), (38) becomes

$$\begin{aligned}\frac{d}{dt}(C - C^T) =& 2[[\omega]] + \frac{\sin\theta}{\theta}[[\theta \times \omega]] \\ &+ \frac{1-\cos\theta}{\theta^2}(-2\theta^2[[\omega]] - [[\theta \times (\theta \times \omega)]]).\end{aligned} \tag{39}$$

Equating the right members of (36) and (39) and evaluating the simple derivative $a'(\theta)$ leads to

$$\begin{aligned}&2\frac{\sin\theta}{\theta}\dot{\theta} - 2\frac{\sin\theta - \theta\cos\theta}{\theta^2}\frac{1}{\theta}(\theta^2\omega + \theta \times (\theta \times \omega)) \\ &= 2\omega + \frac{\sin\theta}{\theta}\theta \times \omega + \frac{1-\cos\theta}{\theta^2} \\ &\times (-2\theta^2\omega - \theta \times (\theta \times \omega)).\end{aligned} \tag{40}$$

Collecting terms,

$$\begin{aligned}2\frac{\sin\theta}{\theta}\dot{\theta} =& 2\frac{\sin\theta}{\theta}\omega + \frac{\sin\theta}{\theta}\theta \times \omega + \frac{1}{\theta^2} \\ &\times \left(2\frac{\sin\theta}{\theta} - (1 + \cos\theta)\right)\theta \times (\theta \times \omega).\end{aligned} \tag{41}$$

Solving for $\dot{\theta}$ and simplifying the one trigonometric expression which remains, one obtains finally

$$\dot{\theta} = \omega + \frac{1}{2}\theta \times \omega + \frac{1}{\theta^2}(1 - (\theta/2)\cot(\theta/2))\theta \times (\theta \times \omega) \tag{42}$$

which is the desired result.

V. SAVAGE'S DERIVATION

The simplest and most direct of all derivations is that of Savage [4]. Unfortunately, his work is not easily obtainable, and his presentation is clouded by the use of an older convention (which has existed since Hamilton) in which scalars and vectors dwell harmoniously in a common space in which they may be not only multiplied by but also added to one another. Savage's derivation is repeated here in more transparent notation.

Differentiating (22) leads to

$$\frac{d}{dt}\overline{q} = b(\theta)\begin{bmatrix}\dot{\theta}\\0\end{bmatrix} + \begin{bmatrix}b'(\theta)\dot{\theta}\theta\\c'(\theta)\dot{\theta}\end{bmatrix} \tag{43}$$

with

$$b(\theta) \equiv \frac{1}{\theta}\sin(\theta/2); \qquad c(\theta) \equiv \cos(\theta/2) \tag{44}$$

the prime indicating again differentiation with respect to θ. Recalling (14)

$$\frac{d}{dt}\overline{q} = b(\theta)\begin{bmatrix}\dot{\theta}\\0\end{bmatrix} + \begin{bmatrix}b'(\theta)\frac{1}{\theta}(\theta^2\omega + \theta\times(\theta\times\omega))\\c'(\theta)\dot{\theta}\end{bmatrix}. \tag{45}$$

The kinematic equation for $\overline{q}$ is

$$\frac{d}{dt}\overline{q} = \frac{1}{2}\Omega(\omega)\overline{q} = \frac{1}{2}\begin{bmatrix}[[\omega]] & \omega\\-\omega^T & 0\end{bmatrix}\begin{bmatrix}b(\theta)\theta\\c(\theta)\end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix}b(\theta)\theta\times\omega + c(\theta)\omega\\-b(\theta)\omega\cdot\theta\end{bmatrix}. \tag{46}$$

Comparing the vector components of (45) and (46) yields

$$b(\theta)\dot{\theta} = -b'(\theta)\frac{1}{\theta}(\theta^2\omega + \theta\times(\theta\times\omega)) + \tfrac{1}{2}b(\theta)\theta\times\omega + \tfrac{1}{2}c(\theta)\omega. \tag{47}$$

Carrying out the simple differentiation, solving for $\dot{\theta}$, and collecting terms leads directly to (42).

VI. INVERSE KINEMATIC EQUATION

The inverse kinematic equation for θ, i.e., the equation for ω as a function of $\dot{\theta}$, is simpler than the kinematic equation and has the form

$$\omega = \frac{d\theta}{dt} - \left(\frac{1-\cos\theta}{\theta^2}\right)\theta\times\frac{d\theta}{dt} + \left(\frac{\theta-\sin\theta}{\theta^3}\right)\theta\times\left(\theta\times\frac{d\theta}{dt}\right). \tag{48}$$

Hughes [6] derives this equation by developing (23), which requires considerable effort. A simpler approach is to use the rules for quaternion composition [6–8] to write

$$\begin{bmatrix}\omega\\0\end{bmatrix} = 2\frac{d\overline{q}}{dt}\otimes\overline{q}^{-1}. \tag{49}$$

Substituting (43) and noting

$$\overline{q}^{-1} = \begin{bmatrix}-b(\theta)\theta\\c(\theta)\end{bmatrix} \tag{50}$$

leads directly to

$$\omega = 2[bc\dot{\theta} + (b'c - bc')\dot{\theta}\theta + b^2\theta\times\dot{\theta}]. \tag{51}$$

Evaluating the derivatives and collecting terms yields

$$\omega = \frac{\sin\theta}{\theta}\dot{\theta} - \frac{1-\cos\theta}{\theta^2}\theta\times\dot{\theta} + \left(\frac{1}{\theta} - \frac{\sin\theta}{\theta^2}\right)\dot{\theta}\theta. \tag{52}$$

Noting finally the relation

$$\dot{\theta}\theta = \frac{1}{\theta}(\theta\cdot\dot{\theta})\theta = \frac{1}{\theta}(\theta^2\dot{\theta} + \theta\times(\theta\times\dot{\theta})) \tag{53}$$

and substituting this into (52) leads directly to (48).

VII. OTHER FORMS OF KINEMATIC EQUATION

The results above assumed that the direction-cosine matrix was defined passively, that is, representations of vectors are transformed under a change from an orthonormal basis $\mathcal{E} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ to an orthonormal basis $\mathcal{E}' = \{\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3\}$ according to

$$\mathbf{u}_{\mathcal{E}'} = C\mathbf{u}_{\mathcal{E}} \tag{54}$$

where $\mathbf{u}_{\mathcal{E}}$ denotes the column-vector representation of the abstract vector $\mathbf{u}$ with respect to the (abstract) basis $\mathcal{E}$. In most instances $\mathcal{E}$ is an inertial basis, frequently denoted by $\mathcal{I}$, and $\mathcal{E}'$ is a body-fixed basis, sometimes denoted by $\mathcal{B}$. Also, ω was taken to be the body-referenced angular velocity, (that is, $\omega_{\mathcal{B}}$) so that (6) is satisfied.

We can consider also the space-referenced angular velocity $\omega_{\mathcal{I}}$, which satisfies

$$\omega = C\omega_{\mathcal{I}}. \tag{55}$$

Hence,

$$\frac{d}{dt}C = C[[\omega_{\mathcal{I}}]] \tag{56}$$

and (42) and (48) become, with respect to this angular velocity,

$$\dot{\theta} = \omega_{\mathcal{I}} - \frac{1}{2}\theta\times\omega_{\mathcal{I}} + \frac{1}{\theta^2}\times(1-(\theta/2)\cot(\theta/2))\theta\times(\theta\times\omega_{\mathcal{I}}) \tag{57}$$

and

$$\omega_{\mathcal{I}} = \frac{d\theta}{dt} + \left(\frac{1-\cos\theta}{\theta^2}\right)\theta\times\frac{d\theta}{dt} + \left(\frac{\theta-\sin\theta}{\theta^3}\right)\theta\times\left(\theta\times\frac{d\theta}{dt}\right). \tag{58}$$

The space-referenced angular velocity is seldom used since the inertia tensor of a rigid body is not constant in time with respect to an inertial coordinate system.

A more important complication comes from the choice of an active or a passive description. In the active description, the direction-cosine matrix is the orthogonal matrix which transforms representations of the inertial basis into representations of the body basis, both respect to the inertial basis. Thus,

$$(\hat{\mathbf{e}}_i')_{\mathcal{E}} = C^{\text{active}}(\hat{\mathbf{e}}_i)_{\mathcal{E}} \tag{59}$$

where $(\hat{\mathbf{e}}_i)_{\mathcal{E}}$ is the representation of the body basis vector $\hat{\mathbf{e}}_i'$ with respect to the inertial basis.

Thus,

$$C^{\text{active}} = C^T. \tag{60}$$

In the active description, (4) becomes

$$C(\theta,\boldsymbol{\theta}) = I - \frac{\sin\theta}{\theta}[[\boldsymbol{\theta}]] + \frac{1-\cos\theta}{\theta^2}[[\boldsymbol{\theta}]]^2 \tag{61}$$

and (6) becomes

$$\frac{d}{dt}C = -[[\boldsymbol{\omega}]]C. \tag{62}$$

However, although (61) and (62) are different in form from the equivalent equations in the passive description, the rotation vector $\boldsymbol{\theta}$ and the (body-referenced) angular velocity vector $\boldsymbol{\omega}$ have the identical geometrical significance. Therefore, (42) and (48) remain true in the active description.

VIII. DISCUSSION

All of these derivations, including the one offered by the author, have much in common. In finding the simplest method there is an obvious tradeoff between two conflicting desires. On the one hand, one wishes to have a system of equations of the smallest possible dimension in order to minimize the number of intermediate terms which must be cancelled against one another. Thus, Bortz, Hughes, and Jiang and Lin, who choose the direction-cosine matrix as the intermediate representation, are forced *ipso facto* to treat a system of dimension nine. The naive consideration of dimension alone, however, would lead one to the Rodrigues vector or (gasp!) the Euler angles. On the other hand, one wants the kinematic equation of the intermediate representation to be as simple as possible, which means linear. This suggests either the quaternion or the direction-cosine matrix as possible candidates, of which the quaternion is the clear winner. Thus, Savage must be credited with the simplest derivation, which he has kept well hidden.

The author's derivation, despite its forbidding dimensionality, is nonetheless rather simple and direct. This is because he has chosen to work as much as possible with the 3×3 identity matrix and the 3×3 antisymmetric matrices. Thus, his operations are largely similar to those for the quaternion [8], although (29)–(34) must seem more complicated than the quaternion algebra. The algebra of these 3×3 antisymmetric matrices is an extremely valuable tool for developing attitude identities, a fact that has been recognized by many authors.

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