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SOME INTERESTING PROPERTIES OF THE EULER ANGLES

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ABSTRACT

The composition of Euler angles, often stated to be very complicated, simplifies immensely when the connection between the Euler angles and spherical trigonometry is exploited. This connection with the Euler angles permits the simple derivation of the fundamental theorems of spherical trigonometry as well.

INTRODUCTION

Any rotation matrix may be parameterized in terms of a symmetric sequence of Euler angles [1], which we write in the form

$$R_{\ell m \ell}(\varphi, \vartheta, \psi) \equiv R(\hat{\mathbf{u}}_\ell, \psi)\, R(\hat{\mathbf{u}}_m, \vartheta)\, R(\hat{\mathbf{u}}_\ell, \varphi) \,, \tag{1}$$

where $\hat{\mathbf{u}}_\ell$ and $\hat{\mathbf{u}}_m$ are two distinct unit column-vectors which are chosen from the set

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \quad \text{and} \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} . \tag{2}$$

Thus,

$$R(\hat{\mathbf{u}}_1, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & s\theta \\ 0 & -s\theta & c\theta \end{bmatrix} , \quad R(\hat{\mathbf{u}}_2, \theta) = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} , \quad R(\hat{\mathbf{u}}_3, \theta) = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{3}$$

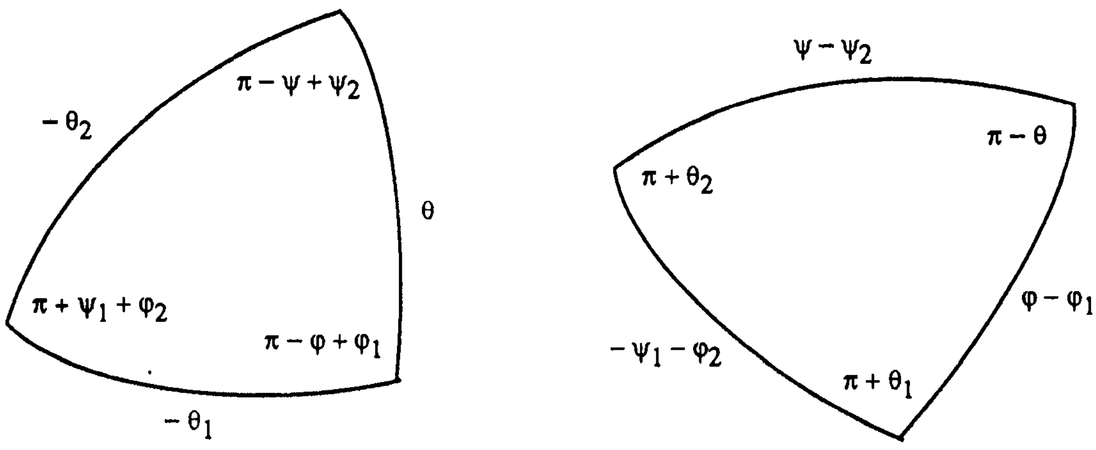
where $c\theta \equiv \cos\theta$ and $s\theta \equiv \sin\theta$. The three angles, φ, ϑ, ψ, are usually restricted to the intervals

$$0 \le \varphi < 2\pi \,, \quad 0 \le \vartheta \le \pi \,, \quad \text{and} \quad 0 \le \psi < 2\pi \,. \tag{4}$$

We call the sequence of Euler angles appearing in the parameterization of the rotation matrix in Eq. (1) *symmetric* in order to distinguish it from an asymmetric sequence of Euler angles in which no two axis column-vectors are identical. The problem we wish to address in particular in this note is as follows: Given two successive rotations parameterized by ℓ–m–ℓ Euler angles $(\varphi_1, \vartheta_1, \psi_1)$ and $(\varphi_2, \vartheta_2, \psi_2)$, respectively, how does one construct expressions for the ℓ–m–ℓ Euler angles $(\varphi, \vartheta, \psi)$ of the combined rotation as a function of $(\varphi_1, \vartheta_1, \psi_1)$ and $(\varphi_2, \vartheta_2, \psi_2)$ satisfying

$$R_{\ell m \ell}(\varphi, \vartheta, \psi) = R_{\ell m \ell}(\varphi_2, \vartheta_2, \psi_2)\, R_{\ell m \ell}(\varphi_1, \vartheta_1, \psi_1) \,? \tag{5}$$

It turns out that spherical trigonometric considerations lead to a simple expression for the Euler angles of the combined rotation.



a. Locus of the ℓ-axis   b. Locus of the m-axis

Figure 1. Loci of the body-coordinate axes

We note first that with some rearrangement Eq. (5) may be written as

$$R_{\ell m \ell}(\varphi - \varphi_1,\, \vartheta,\, \psi - \psi_2) = R_{m \ell m}(\vartheta_1,\, \varphi_2 + \psi_1,\, \vartheta_2)\,. \tag{6}$$

If we compare the (3, 3) elements of both members of Eq. (6) we find immediately that

$$\cos\vartheta = \cos\vartheta_1 \cos\vartheta_2 - \sin\vartheta_1 \sin\vartheta_2 \cos(\varphi_2 + \psi_1)\,. \tag{7}$$

Likewise, comparing the (3, 1) and (3, 2) elements leads to the two equations

$$\sin\vartheta \sin(\varphi - \varphi_1) = \sin\vartheta_2 \sin(\varphi_2 + \psi_1)\,, \tag{8a}$$

$$-\sin\vartheta \cos(\varphi - \varphi_1) = -\cos\vartheta_2 \sin\vartheta_1 - \sin\vartheta_2 \cos\vartheta_1 \cos(\varphi_2 + \psi_1)\,, \tag{8b}$$

which results in the expression

$$\varphi = \varphi_1 + \arctan_2\big(\sin\vartheta_1 \sin(\varphi_2 + \psi_1),\, \cos\vartheta_1 \sin\vartheta_2 + \sin\vartheta_1 \cos\vartheta_2 \cos(\varphi_2 + \psi_1)\big)\,, \tag{9a}$$

and a similar cumbersome expression for ψ from the comparison of the (1, 3) and (2, 3) elements

$$\psi = \psi_2 + \arctan_2\big(\sin\vartheta_2 \sin(\varphi_2 + \psi_1),\, \cos\vartheta_2 \sin\vartheta_1 + \sin\vartheta_2 \cos\vartheta_1 \cos(\varphi_2 + \psi_1)\big)\,. \tag{9b}$$

The function $\arctan_2(y,\, x)$ is the function whose value is $\arctan(y/x)$ and additionally lies in the proper quadrant. In FORTRAN this function has the name ATAN2.

AN ALTERNATE APPROACH

Let us instead rearrange Eq. (6) as

$$R(\hat{\mathbf{u}}_m,\, -\vartheta_2)\, R(\hat{\mathbf{u}}_\ell,\, \psi - \psi_2)\, R(\hat{\mathbf{u}}_m,\, \vartheta)\, R(\hat{\mathbf{u}}_\ell,\, \varphi - \varphi_1)\, R(\hat{\mathbf{u}}_m,\, -\vartheta_1)\, R(\hat{\mathbf{u}}_\ell,\, -\psi_1 - \varphi_2) = I\,, \tag{10}$$

and examine the locus of the ℓ- and m-axes through the various steps of the six rotations. For the ℓ-axis a rotation about $\hat{\mathbf{u}}_m$ will generate an arc of a great circle, which we usually call an arc in spherical trigonometry [2], while a rotation about $\hat{\mathbf{u}}_\ell$ will generate a dihedral angle, and contrarily for the locus of the m-axis. Since the product of all six rotations in Eq. (10) is the identity matrix, the locus of each of the axes must be a closed spherical polygon. The locus of the ℓ- or m-axes for each rotation has three arcs and three dihedral angles and is therefore a spherical triangle, as illustrated in Figure 1. (The locus for the third body-fixed axis is an irregular right spherical hexagon.)

The Polar Complement Theorem

We remark first that the two spherical triangles are related, with the arcs of one corresponding to the polar compliment of the dihedral angles of the other and *vice versa.* (An angle and its polar complement sum to π.) Thus, we have derived the polar complement theorem: namely, that if a spherical triangle has dihedral angles, A, B, and C, with opposing sides, a, b, and c, then one can also construct a spherical triangle with dihedral angles, $\pi - a$, $\pi - b$, and $\pi - c$, and opposing sides, $\pi - A$, $\pi - B$, and $\pi - C$. Note also that the law of cosines for sides for a spherical triangle is identical to the law of cosines for the dihedral angles of the corresponding complementary spherical triangle. The law of cosines for sides was obtained essentially as Eq. (7).

The Construction of the Composite Angles

Returning to Figure 1, the law of sines for either triangle yields

$$\frac{\sin(\varphi - \varphi_1)}{\sin(\pi + \vartheta_2)} = \frac{\sin(\psi - \psi_2)}{\sin(\pi + \vartheta_1)} = \frac{\sin(-\varphi_2 - \psi_1)}{\sin(\pi - \vartheta)}, \tag{11}$$

which may be solved to yield

$$\sin(\varphi - \varphi_1) = \frac{\sin\vartheta_2}{\sin\vartheta}\sin(\varphi_2 + \psi_1), \tag{12a}$$

$$\sin(\psi - \psi_2) = \frac{\sin\vartheta_1}{\sin\vartheta}\sin(\varphi_2 + \psi_1). \tag{12b}$$

Likewise, applying the law of cosines for sides to the appropriate vertex of Figure 1a yields

$$\cos(-\vartheta_2) = \cos\vartheta\,\cos(-\vartheta_1) + \sin\vartheta\,\sin(-\vartheta_1)\,\cos(\pi - \varphi + \varphi_1), \tag{13}$$

which may be solved to yield

$$\cos(\varphi - \varphi_1) = \frac{\cos\vartheta_2 - \cos\vartheta\,\cos\vartheta_1}{\sin\vartheta\,\sin\vartheta_1}. \tag{14}$$

Similarly, one of the remaining vertices yields Eq. (7), while the other results in

$$\cos(\psi - \psi_2) = \frac{\cos\vartheta_1 - \cos\vartheta\,\cos\vartheta_2}{\sin\vartheta\,\sin\vartheta_2}. \tag{15}$$

Combining Eqs. (7), (12), (14) and (15) leads finally to

$$\vartheta = \arccos\big(\cos\vartheta_1\cos\vartheta_2 - \sin\vartheta_1\,\sin\vartheta_2\,\cos(\varphi_2 + \psi_1)\big), \tag{16a}$$

$$\varphi = \varphi_1 + \arctan_2\big(\sin\vartheta_1\,\sin\vartheta_2\,\sin(\varphi_2 + \psi_1),\ \cos\vartheta_2 - \cos\vartheta\,\cos\vartheta_1\big), \tag{16b}$$

$$\psi = \psi_2 + \arctan_2\big(\sin\vartheta_1\,\sin\vartheta_2\,\sin(\varphi_2 + \psi_1),\ \cos\vartheta_1 - \cos\vartheta\,\cos\vartheta_2\big). \tag{16c}$$

Similar results for a slightly restricted case were obtained previously by Lindberg [3].

The above results can be obtained analytically by using the same techniques as were used to obtain Eqs. (7) and (8) by applying these to the equivalent relations

$$R_{\ell m\ell}(-\varphi + \varphi_1, \vartheta_1, \varphi_2 + \psi_1) = R_{m\ell m}(\vartheta, \psi - \psi_2, -\vartheta_2), \tag{17a}$$

and

$$R_{\ell m\ell}(\varphi_2 + \psi_1, \vartheta_2, -\psi + \psi_2) = R_{m\ell m}(-\vartheta_1, \varphi - \varphi_1, \vartheta), \tag{17b}$$

which can be obtained by rearranging Eq. (6), although the derivation is far more cumbersome. A similarly simple result does not hold, apparently, for the asymmetric sequences of Euler angles (for example, the 3–1–2 sequence), the simplest loci of whose Euler axes correspond to spherical quadrilaterals.

ADDITION OF ANGULAR VELOCITIES AND EULER ANGLE RATES

If $R = R''R'$ is a composite rotation and

$$\frac{d}{dt}R = -[\omega\times]R\,, \quad \frac{d}{dt}R' = -[\omega'\times]R'\,, \quad \text{and} \quad \frac{d}{dt}R'' = -[\omega''\times]R''\,, \tag{18}$$

where $[\mathbf{v}\times]$ is the matrix representation of the cross-product operation, $\mathbf{v}\times$, [1], then the angular velocities combine according to the rule

$$\omega = \omega'' + R''\omega'\,. \tag{19}$$

The body-referenced angular velocity is related to the temporal derivatives of the Euler angles according to [4]

$$\omega = R(\hat{\mathbf{u}}_\ell,\, \psi)\,[\,R(\hat{\mathbf{u}}_m,\, \vartheta)\,\hat{\mathbf{u}}_\ell \mid \hat{\mathbf{u}}_m \mid \hat{\mathbf{u}}_\ell\,] \begin{bmatrix} \dot\varphi \\ \dot\vartheta \\ \dot\psi \end{bmatrix} \equiv M_{\ell m\ell}(\varphi,\, \vartheta,\, \psi) \begin{bmatrix} \dot\varphi \\ \dot\vartheta \\ \dot\psi \end{bmatrix}\,, \tag{20}$$

where the second factor in the central member of Eq. (20) is a 3×3 matrix labeled by its columns. The Euler angle rates from the two successive rotations can be combined then as

$$\omega = M_{\ell m\ell}(\varphi_2,\, \vartheta_2,\, \psi_2) \begin{bmatrix} \dot\varphi_2 \\ \dot\vartheta_2 \\ \dot\psi_2 \end{bmatrix} + R_{\ell m\ell}(\varphi_2,\, \vartheta_2,\, \psi_2)\, M_{\ell m\ell}(\varphi_1,\, \vartheta_1,\, \psi_1) \begin{bmatrix} \dot\varphi_1 \\ \dot\vartheta_1 \\ \dot\psi_1 \end{bmatrix}\,, \tag{21}$$

which leads to

$$\begin{bmatrix} \dot\varphi \\ \dot\vartheta \\ \dot\psi \end{bmatrix} = M_{\ell m\ell}^{-1}(\varphi,\, \vartheta,\, \psi) \left\{ M_{\ell m\ell}(\varphi_2,\, \vartheta_2,\, \psi_2) \begin{bmatrix} \dot\varphi_2 \\ \dot\vartheta_2 \\ \dot\psi_2 \end{bmatrix} + R_{\ell m\ell}(\varphi_2,\, \vartheta_2,\, \psi_2)\, M_{\ell m\ell}(\varphi_1,\, \vartheta_1,\, \psi_1) \begin{bmatrix} \dot\varphi_1 \\ \dot\vartheta_1 \\ \dot\psi_1 \end{bmatrix} \right\}\,. \tag{22}$$

CONCLUDING REMARKS

These results, which do not seem to be known generally, are satisfied by any symmetric set of Euler angles. (Equations (18) through (22) hold for any of the twelve Euler angle sequences.) We feel that the spherical trigonometric approach provides the most efficient and elegant path to these results.

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