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# BATCH ESTIMATION OF SPACECRAFT SENSOR ALIGNMENTS

## II. Absolute Alignment Estimation

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### Abstract

The problem of estimating attitude sensor alignments inflight when these are not completely observable from the inflight data is investigated. The inflight relative misalignment estimates and the prelaunch estimates corrupted by launch shock are exploited to devise a "best" estimate of the absolute sensor misalignments in this case. The efficacy of this procedure is illustrated with realistically simulated data. Inferences obtained from actual mission experience are also presented. Simple solvable models are used to compare the absolute misalignments with the relative misalignments incorrectly interpreted as absolute and also with the pseudo-inverse solution. Comparisons are made also using the simulated data. The distribution of alignment estimation error levels as a function of the sensor field of view is studied within a simple model. The effect of alignment estimation errors, especially those arising from unobservable launch-shock effects, on eventual attitude estimation error levels is examined within the framework of the QUEST algorithm. To support the estimation of launch-shock effects a methodology for estimating the launch-shock error levels is developed and an estimator derived for a specific model for the launch shock and the prelaunch alignment covariance. This methodology is tested using the same simulated data.

## Introduction

In Part I of this work [1] a methodology was developed for estimating the relative misalignments of spacecraft attitude sensors from inflight data alone. The present article treats the problem of estimating the absolute alignments, that is the alignment of the spacecraft with regard to an arbitrary coordinate frame fixed in the spacecraft. This work builds on the results of [1].

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In general, if a spacecraft is equipped with $n$ attitude sensors (which for our discussion may be either part of the attitude determination system or payload sensors which happen to sense the attitude), the total *absolute* misalignment vector $\boldsymbol{\Theta}$ has dimension $3n$, the total *relative* misalignment vector, $\boldsymbol{\Psi}$, has dimension $3n-3$, and these satisfy

$$\boldsymbol{\Psi} = F\,\boldsymbol{\Theta} \quad , \tag{1}$$

where $F$ is a $(3n-3)\times 3n$ matrix of full rank. A specific form for $F$ was given in Part I [1] , but, in fact, any $(3n-3)\times 3n$ matrix of full rank may be used to define the relative misalignments. While $\boldsymbol{\Theta}$ is not observable from inflight data alone, $\boldsymbol{\Psi}$ can usually be estimated unambiguously from this data. This estimate of $\boldsymbol{\Psi}$, denoted $\boldsymbol{\Psi}^*$(prior-free) because it is not based on any information *prior* to the inflight data, satisfies

$$\boldsymbol{\Psi}^*(\text{prior-free}) = F\,\boldsymbol{\Theta} + \Delta\boldsymbol{\Psi}^*(\text{prior-free}) \quad , \tag{2}$$

where

$$\Delta\boldsymbol{\Psi}^*(\text{prior-free}) \sim \mathcal{N}(\mathbf{0},\, P_{\Psi\Psi}(\text{prior-free})) \quad . \tag{3}$$

Complete details for calculating $\boldsymbol{\Psi}^*$(prior-free) and $P_{\Psi\Psi}$(prior-free) were given in Part I [1] .

The estimate $\boldsymbol{\Psi}^*$(prior-free) is a sufficient statistic [2] for $\boldsymbol{\Theta}$ given the inflight data and may be used as an effective measurement for $\boldsymbol{\Theta}$. However, since the dimension of $\boldsymbol{\Psi}^*$(prior-free) is smaller than the dimension of $\boldsymbol{\Theta}$, knowledge of $\boldsymbol{\Psi}^*$(prior-free) alone is not adequate to estimate $\boldsymbol{\Theta}$ unambiguously. It is, however, possible, to obtain a postlaunch estimate of $\boldsymbol{\Theta}$ if the information contained in the prelaunch calibration is also used.

In general, the accuracy of the prelaunch calibration following launch is diminished greatly from that before launch owing to a variety of disturbances which we have labeled collectively *launch shock*. It is for this reason, of course, that the alignment calibration is repeated inflight. In general, launch shock has many causes: thermal flexure, zero-gravity, vibration, degradation of the sensor, etc., an exact characterization of which is not feasible. We can at best characterize only the statistical properties of the launch shock as a random process. We do this by determining the statistical properties of the inflight alignments and subtracting from these the statistical properties of the prelaunch alignments.

To describe launch-shock effects we write

$$\boldsymbol{\Theta}^{\text{inflight}} = \boldsymbol{\Theta}^{\text{prelaunch}} + \Delta\boldsymbol{\Theta}^{\text{launch-shock}} \quad , \tag{4}$$

and assume for the sake of simplicity that

$$\Delta\boldsymbol{\Theta}^{\text{launch-shock}} \sim \mathcal{N}(\mathbf{0},\, Q_{\Theta\Theta}^{\text{launch-shock}}) \quad . \tag{5}$$

Based on this model, the *a priori* maximum likelihood estimate of the inflight sensor misalignments and their covariance is given by

$$\boldsymbol{\Theta}^*(-) = \boldsymbol{\Theta}^*(\text{prelaunch}) = \mathbf{0} \quad , \tag{6}$$

$$P_{\Theta\Theta}(-) = P_{\Theta\Theta}(\text{prelaunch}) + Q_{\Theta\Theta}^{\text{launch-shock}} \quad . \tag{7}$$

Since the launch-shock introduces a change in the physical alignments we must now distinguish between $\mathbf{\Theta}^{\text{prelaunch}}$ and $\mathbf{\Theta}^{\text{inflight}}$. (If alignments change throughout the mission, we must write $\mathbf{\Theta}(t)$.) To simplify the notation, when $\mathbf{\Theta}$ appears without a verbal superscript, it generally denotes $\mathbf{\Theta}^{\text{inflight}}$.

Regarding $\mathbf{\Psi}^*$(prior-free) and $\mathbf{\Theta}^*(-)$ as two effective measurements of $\mathbf{\Theta}$, we may write the negative-log-likelihood function [2, 3] for $\mathbf{\Theta}$ given these two measurements as

$$\begin{aligned} J_\Theta(\mathbf{\Theta}) = \tfrac{1}{2}\left[\mathbf{\Theta}^T P_{\Theta\Theta}^{-1}(-)\,\mathbf{\Theta} + \log\det P_{\Theta\Theta}(-) + 3n\log 2\pi\right] \\ + \tfrac{1}{2}\left[(\mathbf{\Psi}^*(\text{prior-free}) - F\,\mathbf{\Theta})^T P_{\Psi\Psi}^{-1}(\text{prior-free})\,(\mathbf{\Psi}^*(\text{prior-free}) - F\,\mathbf{\Theta})\right. \\ \left. + \log\det P_{\Psi\Psi}(\text{prior-free}) + (3n-3)\log 2\pi\right] \quad . \end{aligned} \tag{8}$$

Carrying out the minimization leads to the normal equations

$$P_{\Theta\Theta}^{-1}(+)\ \mathbf{\Theta}^*(+) = F^T\ P_{\Psi\Psi}^{-1}(\text{prior-free})\,\mathbf{\Psi}^*(\text{prior-free}) \quad , \tag{9}$$

$$P_{\Theta\Theta}^{-1}(+) = P_{\Theta\Theta}^{-1}(-) + F^T\ P_{\Psi\Psi}^{-1}(\text{prior-free})\,F \quad , \tag{10}$$

where "$-$" denotes the *a priori* estimate, based only on data prior to launch, and "+" denotes the *a posteriori* estimate, based on data both prior and posterior to launch.

Thus, it is a simple manner to calculate the *a posteriori* estimate of the absolute once the prelaunch estimate, inflight prior-free estimate, and launch-shock error levels are known. Two questions remain: when should one calculate these *a posteriori* absolute misalignment estimates, and how should one calculate the launch-shock errors levels which figure in these calculations?

As was indicated in Part I, when the payload can be regarded as an attitude sensor, there is no need to go beyond the prior-free relative misalignments. If the body frame is defined in some arbitrary manner independently of the sensors, then the absolute misalignments may never be obtainable with any acceptable degree of accuracy while the attitude of the payload will be known very accurately. This is certainly an acceptable condition.

On the other hand let us imagine a mission in which the payload is a sensor with a very small field of view which is surveying the oceans. Although the mission data may depend critically on the attitude of the payload, the payload itself cannot be used much of the time to provide attitude information because there is no easily quantifiable attitude reference to use. In this case relative misalignment estimates with respect to one of the other sensors may not provide the means for determining the attitude of the payload. The absolute misalignments, then, provide the best estimate of the misalignments with respect to the body frame, which may have been coincident with the body frame before launch. Even if this frame is not fixed in the mission payload, it is still reasonable to expect that the launch-shock disturbances causing postlaunch misalignment of the payload from the body axes will be zero-mean so that these absolute misalignments still furnish the best guess of the payload attitude.

Since equations (4) through (10) provide all the theory that is necessary to calculate the *a posteriori* absolute misalignments, the remainder of this work will be devoted to the problem of determining the launch-shock errors and to understanding the nature of the absolute misalignment estimates.

We begin, thus, by developing an algorithm for computing the launch-shock error levels, testing the algorithm with the simulated data from [1] . We then explore the relationship of the *a posteriori* absolute misalignments to the *a posteriori* relative misalignments, which will provide important insights. Following this we investigate some qualitative properties of the absolute misalignments. We examine first the nature of the error which is obtained in naively using the relative misalignments as absolute misalignments when the payload cannot be used as an attitude sensor and show that this assumption will increase the payload attitude error by a factor of $\sqrt{n}$, where $n$ is the number of sensors. The pseudo-inverse estimate of the absolute misalignments (essentially an estimate which assumes infinite isotropic covariance for the prelaunch estimates) is also compared. Next, we explore the dependence of the alignment estimation accuracy on the field of view of the sensor and show that reasonable estimates can be obtained even when the sensors have very restricted fields of view. We then show how the alignment estimation errors enter the final attitude errors. The absolute alignment estimators are tested with simulated data from the same example used in [1]. Finally, we compare our experience with that from actual missions. This will complete our program on batch alignment estimation.

## Estimation of Launch-Shock Error Levels

We develop now an algorithm [4] for estimating the launch-shock covariance parameters from inflight data. Since so little data is available to characterize the launch shock we might choose to make the simplest model possible for the launch-shock error covariance matrix, namely,

$$Q_{\Theta\Theta}^{\text{launch-shock}} = q\, I_{3n\times 3n} \quad . \tag{11}$$

In some cases, say when the primary reference cube and some of the sensors are mounted to an extremely rigid instrument plate (optical bench) while other sensors are scattered about the spacecraft, we might wish to propose a smaller value, $q_1$, for the sensors on the instrument plate and a larger value $q_2$, for the other sensors, or allow the launch shock effects of a number of sensors jointly mounted on some distant but rigid surface to be highly correlated. However, it should be considered first that not all of the apparent misalignment is due to geometric distortion of the spacecraft. Secondly, it must be kept in mind that the number of sensors in the spacecraft is limited, so that there are only $3n - 3$ quantities which may be used to estimate the parameters of the launch shock. Hence, while it may be reasonable to simulate detailed launch-shock effects before launch, it is a hopeless task to try to estimate the parameters of a very detailed model from inflight data. For the sake of generality, however, we write

$$Q_{\Theta\Theta}^{\text{launch-shock}} = Q_{\Theta\Theta}^{\text{launch-shock}}(\mathbf{q}) \quad , \tag{12}$$

where $\mathbf{q}$ is the vector of launch-shock parameters.

To estimate the launch-shock parameters we have at our disposal only the prior-free inflight misalignment estimate, $\mathbf{\Psi}^*$(prior-free), whose calculation has been described fully in [1]. We may regard $\mathbf{\Psi}^*$(prior-free) as an effective measurement of $\mathbf{\Theta}$ and write

$$\mathbf{\Psi}^*(\text{prior-free}) = F\,\mathbf{\Theta} + \Delta\mathbf{\Psi}^*(\text{prior-free}) \quad , \tag{13}$$

where

$$\Delta\mathbf{\Psi}^*(\text{prior-free}) \sim \mathcal{N}(\mathbf{0}, P_{\Psi\Psi}(\text{prior-free})) \quad . \tag{14}$$

The prelaunch estimate of $\mathbf{\Theta}$ propagated to postlaunch times is

$$\begin{aligned} \mathbf{\Theta}^*(-) &= \mathbf{\Theta} + \Delta\mathbf{\Theta}(-) \\ &= \mathbf{0} \quad , \end{aligned} \tag{15}$$

and

$$\Delta\mathbf{\Theta}(-) = \mathcal{N}(\mathbf{0}, P_{\Theta\Theta}(\text{prelaunch}) + Q_{\Theta\Theta}^{\text{launch-shock}}) \quad . \tag{16}$$

To determine $\mathbf{q}^*$ we note that

$$\mathbf{\Psi}^*(\text{prior-free}) = \mathbf{\Psi}^{\text{inflight}} + \Delta\mathbf{\Psi}^*(\text{prior-free}) \quad , \tag{17}$$

and

$$\mathbf{\Psi}^{\text{inflight}} = \mathbf{\Psi}^{\text{prelaunch}} + \Delta\mathbf{\Psi}^{\text{launch-shock}} \quad , \tag{18}$$

$$= F\,\mathbf{\Theta}^{\text{prelaunch}} + F\Delta\mathbf{\Theta}^{\text{launch-shock}} \quad . \tag{19}$$

For the prelaunch calibration, clearly,

$$\begin{aligned} \mathbf{\Theta}^{\text{prelaunch}} &= \mathbf{\Theta}^*(\text{prelaunch}) - \Delta\mathbf{\Theta}^*(\text{prelaunch}) \quad , \\ &= -\Delta\mathbf{\Theta}^*(\text{prelaunch}) \quad . \end{aligned} \tag{20}$$

Combining equations (17) through (20) leads to

$$\mathbf{\Psi}^*(\text{prior-free}) = -F\,\Delta\mathbf{\Theta}^*(\text{prelaunch}) + F\,\Delta\mathbf{\Theta}^{\text{launch-shock}} + \Delta\mathbf{\Psi}^*(\text{prior-free}) \quad , \tag{21}$$

and, therefore,

$$E\{\,\mathbf{\Psi}^*(\text{prior-free})\,\} = \mathbf{0} \quad , \tag{22a}$$

$$E\{\,\mathbf{\Psi}^*(\text{prior-free})\,\mathbf{\Psi}^{*T}(\text{prior-free})\,\} = P_{\Psi\Psi}(\text{total}) \quad , \tag{22b}$$

where

$$P_{\Psi\Psi}(\text{total}) \equiv P_{\Psi\Psi}(\text{prelaunch}) + Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q}) + P_{\Psi\Psi}(\text{prior-free}) \quad , \tag{23}$$

and

$$P_{\Psi\Psi} \equiv F\,P_{\Theta\Theta}F^T \quad , \tag{24a}$$

$$Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q}) \equiv F\,Q_{\Theta\Theta}^{\text{launch-shock}}\,F^T \quad . \tag{24b}$$

Note that the first two terms of equation (23) are just $P_{\Psi\Psi}(-)$. Thus, following the prescriptions of maximum likelihood estimation [2, 3] the negative-log-likelihood function for $\mathbf{q}$ given $\mathbf{\Psi}^*$(prior-free) is

$$J_q^{\text{prior-free}}(\mathbf{q}) = \tfrac{1}{2}\left[\mathbf{\Psi}^{*T}(\text{prior-free})P_{\Psi\Psi}^{-1}(\text{total})\,\mathbf{\Psi}^*(\text{prior-free}) + \log\det P_{\Psi\Psi}(\text{total}) + (3n-3)\log 2\pi\right] \quad , \tag{25}$$

which depends on $\mathbf{q}$ only through $Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q})$. Note that $\mathbf{\Psi}^*$(prior-free) and its covariance $P_{\Psi\Psi}$(prior-free) do not depend explicitly on $\mathbf{q}$.

Thus, $\mathbf{q}^*$ is a solution of

$$\frac{\partial J_q^{\text{prior-free}}}{\partial \mathbf{q}}(\mathbf{q}^*) = \tfrac{1}{2}\left\{\left[-\mathbf{\Psi}^{*T}(\text{prior-free})\,P_{\Psi\Psi}^{-1}(\text{total}) \times \frac{\partial Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q})}{\partial \mathbf{q}}\,P_{\Psi\Psi}^{-1}(\text{total})\,\mathbf{\Psi}^*(\text{prior-free})\right] + \operatorname{tr}\left(P_{\Psi\Psi}^{-1}(\text{total})\frac{\partial Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q})}{\partial \mathbf{q}}\right)\right\}\Bigg|_{\mathbf{q}=\mathbf{q}^*} = \mathbf{0} \quad , \tag{26}$$

which must be solved iteratively for $\mathbf{q}^*$. Asymptotically (i.e., as $n \to \infty$) the covariance matrix of the estimate error for $\mathbf{q}^*$ is given by [5]

$$\left[P_{qq}^{-1}\right]_{\varrho\varrho'} = \tfrac{1}{2}\operatorname{tr}\left[P_{\Psi\Psi}^{-1}(\text{total})\,\frac{\partial Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q})}{\partial q_\varrho}\,P_{\Psi\Psi}^{-1}(\text{total})\,\frac{\partial Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q})}{\partial q_{\varrho'}}\right] \quad . \tag{27}$$

As an example, consider the simplest parameterization of $Q_{\Theta\Theta}^{\text{launch-shock}}$ given by equation (11) and the parameterization of the prelaunch alignment estimate error covariance given by [1] . One then finds directly

$$Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q}) = q\begin{bmatrix} 2I & I & \cdots & I \\ I & 2I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & 2I \end{bmatrix} \quad , \tag{28}$$

and

$$P_{\Psi\Psi}(\text{prelaunch}) = \sigma_p^2\begin{bmatrix} 2I & I & \cdots & I \\ I & 2I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & 2I \end{bmatrix} \quad , \tag{29}$$

where $I$ signifies $I_{3\times 3}$ in the block representation. We shall assume that sufficient data has been collected before and after launch that

$$P_{\Psi\Psi}(\text{prelaunch}) << Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q}) \quad , \tag{30}$$

$$P_{\Psi\Psi}(\text{prior-free}) << Q_{\Psi\Psi}^{\text{ls}}(\mathbf{q}) \quad . \tag{31}$$

Then equations (26) and (27) reduce to

$$q^* \approx \frac{1}{3(n-1)} \left( \sum_{i=2}^{n} |\boldsymbol{\psi}_i^*(\text{prior-free})|^2 - \frac{1}{n} \left| \sum_{i=2}^{n} \boldsymbol{\psi}_i^*(\text{prior-free}) \right|^2 \right) \quad , \tag{32}$$

and

$$P_{qq} \approx \frac{2\,q^2}{3(n-1)} \quad , \tag{33}$$

with $\boldsymbol{\psi}_i^*$ as defined in [1]. Asymptotically, the prior-free estimates of $q$ and $\boldsymbol{\Psi}$ are uncorrelated (i.e., as $n \to \infty$, $P_{\Psi q}(\text{prior-free}) \to 0$).

The reliance on an asymptotic approximation is not overly restrictive. The relevant index is the dimension of $\boldsymbol{\Psi}^*(\text{prior-free})$, or $3n-3$. For agile spacecraft with five Sun sensors, dual horizon scanners, and a vector magnetometer, this number is already 21.

## A Numerical Example

To illustrate the algorithm for estimating the launch-shock parameters we examine again the numerical example of Part I. Consider a typical spacecraft equipped with three vector sensors each with an accuracy of 10. arc sec/axis and an effective (weighted) field of view of ±10. deg/axis. We assume the sensor errors to be well represented by the QUEST measurement model [6] . The details of the measurement model are given in [1] and will not be repeated here.

The model for the prelaunch errors was as described by equation (96) of Part I [1] with $\sigma_p = 3.5$ arc sec and with launch-shock errors are described by equation (11) above with $q^{1/2} = 1.$ arc min. Thus, the model misalignments themselves have the form

$$\boldsymbol{\theta}_i = \mathbf{a}_i + \mathbf{b} \quad , \tag{34}$$

where

$$\mathbf{a}_i \sim \mathcal{N}(\mathbf{0}, (\sigma_p^2 + q)I_{3\times 3}) \quad , \quad \mathbf{b} \sim \mathcal{N}(\mathbf{0}, \sigma_p^2\, I_{3\times 3}) \quad , \tag{35}$$

and were sampled accordingly. The nominal alignments, expressed in terms of the Gibbs vector [7] , were taken to be

$$\mathbf{g}_1 = \mathbf{0} \quad , \quad \mathbf{g}_2 = (2.5,\, 0,\, 0)^T \quad , \quad \mathbf{g}_3 = (0,\, 2.5,\, 0)^T \quad , \tag{36}$$

which is, as has been noted, a typical set of alignments if sensor 1 is a Sun sensor and sensors 2 and 3 are star trackers.

One hundred samples of simulated data were generated. The results for the relative misalignment estimates are given in Table 2 of [1] . The estimate for the launch-shock standard deviation $q$ from equation (32) was

$$(q^*)^{1/2} = 50. \pm 28. \text{ arc sec} \quad , \tag{37}$$

in good agreement with the model value of 1. arc min. The large error level is typical of estimates of standard deviations.

## Absolute and Relative A Posteriori Misalignment Estimates

There is a direct and interesting connection between the estimates of the *a posteriori* absolute misalignments and those of the relative misalignments. The *a posteriori* absolute misalignment estimates are defined in equations (9) and (10) above. The *a posteriori* relative misalignment estimates are given likewise by

$$P_{\Psi\Psi}^{-1}(+)\,\mathbf{\Psi}^*(+) = P_{\Psi\Psi}^{-1}(\text{prior-free})\,\mathbf{\Psi}^*(\text{prior-free}) \quad , \tag{38}$$

$$P_{\Psi\Psi}^{-1}(+) = P_{\Psi\Psi}^{-1}(-) + P_{\Psi\Psi}^{-1}(\text{prior-free}) \quad , \tag{39}$$

where

$$P_{\Psi\Psi}(-) = F\,P_{\Theta\Theta}(-)F^T \quad , \tag{40}$$

It follows, in fact, from the general properties of the maximum likelihood estimate *and* the dependence of the inflight data on the relative misalignments alone that

$$\boldsymbol{\psi}_i^*(+) = \boldsymbol{\theta}_i^*(+) - \boldsymbol{\theta}_1^*(+) \quad , \tag{41}$$

which is equivalent to

$$\mathbf{\Psi}^*(+) = F\,\mathbf{\Theta}^*(+) \quad , \tag{42}$$

and, therefore,

$$P_{\Psi\Psi}(+) = F\,P_{\Theta\Theta}(+)F^T \quad . \tag{43}$$

It is, in fact, possible, as we shall now show, to determine $\mathbf{\Theta}^*(+)$ directly from $\mathbf{\Psi}^*(+)$, which lends additional insights.

Consider the parameter vector $\Xi$ defined by

$$\Xi \equiv [\,\boldsymbol{\theta}_1^T,\, \boldsymbol{\theta}_2^T - \boldsymbol{\theta}_1^T,\, \ldots\, ,\, \boldsymbol{\theta}_n^T - \boldsymbol{\theta}_1^T\,]^T \equiv [\,\boldsymbol{\theta}_1^T,\, \mathbf{\Lambda}^T\,]^T \quad . \tag{44}$$

Since $\Xi$ is an invertible function of $\mathbf{\Theta}$, it follows immediately [2] that

$$\Xi^*(+) = [\,\boldsymbol{\theta}_1^{*T}(+),\, \boldsymbol{\theta}_2^{*T}(+) - \boldsymbol{\theta}_1^{*T}(+),\, \ldots\, ,\, \boldsymbol{\theta}_n^{*T}(+) - \boldsymbol{\theta}_1^{*T}(+)\,]^T \quad . \tag{45}$$

However, if we calculate $\Xi^*(+)$ directly from the prior-free estimate, we have that $\Xi^*(+)$ must minimize the *a posteriori* negative-log-likelihood function

$$\begin{aligned} J_\Xi(\Xi) = \frac{1}{2} \begin{bmatrix} \boldsymbol{\theta}_1 \\ \mathbf{\Lambda} \end{bmatrix}^T & \begin{bmatrix} [P^{-1}(-)]_{\theta_1\theta_1} & [P^{-1}(-)]_{\theta_1\Lambda} \\ [P^{-1}(-)]_{\Lambda\theta_1} & [P^{-1}(-)]_{\Lambda\Lambda} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_1 \\ \mathbf{\Lambda} \end{bmatrix} \\ & + \frac{1}{2}\sum_{k=1}^{N} (\mathbf{Z}_k - H_k'\,\mathbf{\Lambda})^T\, P_{\mathbf{Z}_k}^{-1}\, (\mathbf{Z}_k - H_k'\,\mathbf{\Lambda}) \\ & + \text{terms independent of } \Xi \quad , \end{aligned} \tag{46}$$

where, for example, $[P^{-1}(-)]_{\Lambda\Lambda}$ denotes the $\Lambda\Lambda$ submatrix of $P_{\Xi\Xi}^{-1}$ . Thus,

$$[P^{-1}(-)]_{\Lambda\Lambda} = \left( P_{\Lambda\Lambda}(-) - P_{\Lambda\theta_1}(-)\, P_{\theta_1\theta_1}^{-1}(-)\, P_{\theta_1\Lambda}(-) \right)^{-1} \quad , \tag{47}$$

which is a standard result for partitioned inverses. The minimization of $J_\Xi(\Xi)$ over $\Xi$ is exactly equivalent to minimizing $J_\Theta(\mathbf{\Theta})$ over $\mathbf{\Theta}$, since the transformation from $\mathbf{\Theta}$ to $\Xi$ is invertible.

The second term of equation (46) can be recognized as the data-dependent part of the prior-free negative-log-likelihood for $\boldsymbol{\Psi}$. Thus, we can replace it with

$$\frac{1}{2}(\boldsymbol{\Psi}^*(\text{prior-free}) - \boldsymbol{\Lambda})^T P_{\Psi\Psi}^{-1}(\text{prior-free})\,(\boldsymbol{\Psi}^*(\text{prior-free}) - \boldsymbol{\Lambda}) \quad .$$

This follows from the fact that both these expressions are quadratic in $\boldsymbol{\Lambda}$ and both have the same first and second derivatives with respect to the components of $\boldsymbol{\Lambda}$. Hence, they can differ only by terms which are independent of $\boldsymbol{\Lambda}$, which will not effect the result for the optimum value of $\boldsymbol{\Lambda}$. We say that $\boldsymbol{\Psi}^*$(prior-free) is a sufficient statistic [2] for $\boldsymbol{\Lambda}$.

Calculating the gradient of $J_{\Xi}(\Xi)$ with respect to $\Xi$ and setting it equal to $\mathbf{0}$ leads to

$$[P^{-1}(-)]_{\theta_1\theta_1}\,\boldsymbol{\theta}_1^*(+) + [P^{-1}(-)]_{\theta_1\Lambda}\,\boldsymbol{\Lambda}^*(+) = \mathbf{0} \quad , \tag{48a}$$

$$[P^{-1}(-)]_{\Lambda\theta_1}\,\boldsymbol{\theta}_1^*(+) + [P^{-1}(-)]_{\Lambda\Lambda}\,\boldsymbol{\Lambda}^*(+)$$
$$- P_{\Psi\Psi}^{-1}(\text{prior-free})\,(\boldsymbol{\Psi}^*(\text{prior-free}) - \boldsymbol{\Lambda}^*(+)) = \mathbf{0} \quad . \tag{48b}$$

Substituting equation (48a) into equation (48b) results in

$$\Big([P^{-1}(-)]_{\Lambda\Lambda} - [P^{-1}(-)]_{\Lambda\theta_1}\,[P^{-1}(-)]_{\theta_1\theta_1}^{-1}\,[P^{-1}(-)]_{\theta_1\Lambda}\Big)\,\boldsymbol{\Lambda}^*(+)$$
$$- P_{\Psi\Psi}^{-1}(\text{prior-free})\,(\boldsymbol{\Psi}^*(\text{prior-free}) - \boldsymbol{\Lambda}^*(+)) = \mathbf{0} \quad . \tag{49}$$

The expression in parentheses on the first line of equation (49) is just $P_{\Lambda\Lambda}^{-1}(-)$ (compare a similar expression in equation (47)), which is identical by definition to $P_{\Psi\Psi}^{-1}(-)$. Thus, equation (29) is equivalent to finding $\boldsymbol{\Lambda}^*(+)$, the value of $\boldsymbol{\Lambda}$ which minimizes

$$J_{\Lambda}(\boldsymbol{\Lambda}) = \tfrac{1}{2}\,\left[\boldsymbol{\Lambda}^T\,P_{\Psi\Psi}^{-1}(-)\,\boldsymbol{\Lambda} + \log\det P_{\Psi\Psi}^{-1}(-) + (3n-3)\,\log 2\pi\right]$$
$$+ \Big[(\boldsymbol{\Psi}^*(\text{prior-free}) - \boldsymbol{\Lambda})^T P_{\Psi\Psi}^{-1}(\text{prior-free})\,(\boldsymbol{\Psi}^*(\text{prior-free}) - \boldsymbol{\Lambda})$$
$$+ \log\det P_{\Psi\Psi}(\text{prior-free}) + (3n-3)\,\log 2\pi\Big] \quad , \tag{50}$$

which is just the negative-log-likelihood function which defines $\boldsymbol{\Psi}^*(+)$. Hence, $\boldsymbol{\Lambda}^*(+)$ = $\boldsymbol{\Psi}^*(+)$, and equation (42) is proved. This is not a trivial result. In general, when a larger set of parameters is estimated starting from the estimate of a smaller set, even based on the same data, the estimate of *all* parameters will change. The crucial property which leads to equation (42) is the dependence of the measurements in equation (46) on the $\boldsymbol{\Psi}$ alone.

Equations (48ab) contain other results. Solving equation (48a) gives $\boldsymbol{\theta}_1^*(+)$ as a function of $\boldsymbol{\Psi}^*(+)$. Using expressions for partitioned inverses again, this solution can be written equivalently as

$$\boldsymbol{\theta}_1^*(+) = -[P^{-1}(-)]_{\theta_1\theta_1}^{-1}\,[P^{-1}(-)]_{\theta_1\Lambda}\,\boldsymbol{\Lambda}^*(+) \quad ,$$
$$= P_{\theta_1\Psi}(-)\,P_{\Psi\Psi}^{-1}(-)\,\boldsymbol{\Psi}^*(+) \quad , \tag{51}$$

with

$$P_{\theta_1\Psi}(-) = \left[P_{\theta_2\theta_1}(-) - P_{\theta_1\theta_1}(-)\;\middle|\;\cdots\;\middle|\;P_{\theta_n\theta_1}(-) - P_{\theta_1\theta_1}(-)\right] \quad , \tag{52}$$

giving the partition of $P_{\theta_1\Psi}(-)$ in terms of $3 \times 3$ matrices.

If $P_{\theta_i\theta_j}(-)$ has the structure

$$P_{\theta_i\theta_j}(-) = P_a + \delta_{ij} P_b \quad , \tag{53}$$

that is, the *a priori* estimate errors are identically distributed, then equation (54) becomes

$$\boldsymbol{\theta}_1^*(+) = -\frac{1}{n} \sum_{i=2}^{n} \boldsymbol{\psi}_i^*(+) \quad , \tag{54}$$

which is equivalent to

$$\sum_{i=1}^{n} \boldsymbol{\theta}_i^*(+) = \mathbf{0} \quad . \tag{55}$$

As a consequence of these relations we may express the estimate of the launch-shock variance parameter of the simple model given by equation (11) approximately in terms of the *a posteriori* absolute misalignment estimates. If $P_{\Psi\Psi}(-) >> P_{\Psi\Psi}(\text{prior-free})$, the result becomes

$$q^* \approx \frac{1}{3(n-1)} \sum_{i=1}^{n} |\,\boldsymbol{\theta}_i^*(+)\,|^2 \quad . \tag{56}$$

## Pseudo-Inverse Estimates of the Misalignments

Equation (54) is especially interesting because it shows for identically distributed *a priori* errors that $\boldsymbol{\theta}_1^*(+)$ is directly obtainable from $\boldsymbol{\Psi}^*(+)$ independently of the nature of $P_{\Psi\Psi}(\text{prior-free})$ or the *detailed* structure of the *a priori* covariance. However, because the launch-shock errors will generally be much larger than the prelaunch alignment calibration errors or the errors in $\boldsymbol{\Psi}^*(\text{prior-free})$, it is also true that

$$\boldsymbol{\Psi}^*(+) \approx \boldsymbol{\Psi}^*(\text{prior-free}) \quad . \tag{57}$$

Hence,

$$\boldsymbol{\theta}_1^*(+) \approx -\frac{1}{n} \sum_{i=2}^{n} \boldsymbol{\psi}_i^*(\text{prior-free}) \quad , \tag{58a}$$

$$\boldsymbol{\theta}_i^*(+) \approx \boldsymbol{\psi}_i^*(\text{prior-free}) - \frac{1}{n} \sum_{i=2}^{n} \boldsymbol{\psi}_i^*(\text{prior-free}) \quad , \quad i = 2, \ldots, n \quad , \tag{58b}$$

so that the approximate *a posteriori* estimate of the absolute misalignments obtained under this assumption is independent of the *a priori* covariance matrix. Equivalently, it is the solution which assumes that

$$P_{\Theta\Theta}(-) = a\, I_{3n\times 3n} \quad , \tag{59}$$

in the limit that $a \to \infty$.

This solution is identical to the pseudo-inverse solution for the misalignments, defined as the minimum-length solution of

$$F\,\boldsymbol{\Theta} = \boldsymbol{\Psi}^*(\text{prior-free}) \quad . \tag{60}$$

The solution, assuming that $F$ is full rank, is

$$\mathbf{\Theta}^{\#} = F^{\#}\,\mathbf{\Psi}^{*}(\text{prior-free}) \quad , \tag{61}$$

where

$$F^{\#} \equiv F^{T}(F F^{T})^{-1} \tag{62}$$

is the pseudo-inverse of $F$. It is easy to show for the present example that the pseudo-inverse yields solutions which are identical to equations (58). From equations (9) and (10) the absolute *a posteriori* estimate and the pseudo-inverse solution are related by

$$\mathbf{\Theta}^{*}(+) = P_{\Theta\Theta}(+)\,F^{T}\,P_{\Psi\Psi}^{-1}(\text{prior-free})\,F\,\mathbf{\Theta}^{\#} \quad , \tag{63}$$

$$= \left(I - P_{\Theta\Theta}(+)\,P_{\Theta\Theta}^{-1}(-)\right)\mathbf{\Theta}^{\#} \quad . \tag{64}$$

It might appear that $\mathbf{\Theta}^{\#} \to \mathbf{\Theta}^{*}(+)$ as the amount of data becomes infinite. However, we note that the matrix $F^{T}P_{\Psi\Psi}^{-1}(\text{prior-free})\,F$ is singular. Hence, $P_{\Theta\Theta}(+)$ may have some eigenvalues which are of the same order (or even equal) to those of $P_{\Theta\Theta}(-)$. Whether or not this happens depends on the structure of $P_{\Theta\Theta}(-)$. However, in many cases $\mathbf{\Theta}^{\#}$ is often close to $\mathbf{\Theta}^{*}(+)$.

The assumption of identically distributed errors may be very wrong if one attitude sensor is mounted very close to the payload on the same *rigid* support while other attitude sensors are mounted on more distant and less rigid parts of the spacecraft structure. In such cases the assumption $\boldsymbol{\theta}_1^{*}(+) \approx \mathbf{0}$ may be more appropriate.

## Relative versus Absolute Alignment Estimation

It is part of the mythology of alignment estimation that it is more correct to estimate the relative misalignments (because they are completely observable from inflight data alone) than the absolute misalignments. Some reports even bolster this claim by presenting simulation results which demonstrate (quite correctly) that the variances of the relative misalignments are smaller than those of the absolute misalignments. Unfortunately, if we wish to estimate the attitude of a payload instrument is not itself usable as an attitude sensor, it is the absolute misalignments which we require in order to transform data from the sensor frames to the instrument (i.e., body) frame. Thus, if only the relative misalignments are estimated, some assumption must be made about the value of the absolute misalignment of one of the sensors. As noted in Part I, it has been the practice in these cases to set

$$\boldsymbol{\theta}_1^{*}(+) \equiv \mathbf{0} \quad , \tag{65}$$

independent of the nature of the spacecraft. As a result, works which estimate only relative misalignments in these cases have also tended to discard the prelaunch alignment calibration information, except to provide a reference point for the relative misalignment estimator. Thus, users of this naive approach make two serious approximations, which may not be justified in all cases. These assumptions *are* justified in the cases where the payload is also an attitude sensor or if it is known that the relative alignment of the payload to sensor 1 is small. In this section we will investigate the consequences of this approach when every sensor is expected to have the same misalignment error level with respect to the payload [8].

The inflight data can be represented by a single effective measurement, $\mathbf{Z}$, of the form

$$\mathbf{Z} = H\,\boldsymbol{\Theta} + \mathbf{v} \quad , \tag{66}$$

with

$$\mathbf{v} \sim \mathcal{N}(\mathbf{0},\ R) \quad . \tag{67}$$

The prior-free estimate of the relative misalignments, for example, is just such an effective measurement. $\mathbf{Z}$ can always be constructed as a sufficient statistic [2] for the misalignments. The prior-free estimate of the relative alignments is such a sufficient statistic. (In the present example, however, we shall want $R$ to have the same dimension as $\mathbf{Z}$. This can be accomplished only by increasing the dimension of $\mathbf{Z}$ and $\mathbf{v}$ by adding three components with nonvanishing variances. These additional components in $\mathbf{Z}$ do not effect the misalignment estimates, however, because the structure of the sensitivity matrix $H$ will make all misalignment estimates insensitive to them.) Thus, equations (66) and (67) entail no approximation. In terms of this sufficient statistic, the correctly computed maximum likelihood estimate of the misalignments will, therefore, have *a posteriori* covariance

$$P_{\Theta\Theta}(+) = \left[P_{\Theta\Theta}^{-1}(-) + H^T R^{-1} H\right]^{-1} \quad . \tag{68}$$

The naive approach to relative misalignment estimation sets

$$\boldsymbol{\theta}_1^{*\,\mathrm{naive}}(+) \equiv \mathbf{0} \quad , \tag{69}$$

and estimates $\boldsymbol{\Theta}' \equiv [\,\boldsymbol{\theta}_2^T, \ldots, \boldsymbol{\theta}_n^T]^T$ by minimizing

$$J^{\mathrm{rel}} \equiv \frac{1}{2}\left(\mathbf{Z} - H'\,\boldsymbol{\Theta}'\right)^T R^{-1}\left(\mathbf{Z} - H'\,\boldsymbol{\Theta}'\right) \quad , \tag{70}$$

where

$$H \equiv \left[\,h_1 \;\middle|\; H'\,\right] \quad , \tag{71}$$

in similar fashion to Part I of this work. Thus, this naive "relative" misalignment estimate is equivalent to the prior-free relative misalignment estimate presented earlier but wrongly interpreted as being the absolute misalignment vector. The naive estimate of $\boldsymbol{\Theta}'$ is, therefore,

$$\boldsymbol{\Theta}'^{*\,\mathrm{naive}}(+) = \left(H'^T R^{-1} H'\right)^{-1} H'^T R^{-1} \mathbf{Z} \quad . \tag{72}$$

supplemented by equation (69).

From equations (66), (69), (71) and (72) it follows for the complete misalignment vector that

$$\boldsymbol{\Theta}^{*\,\mathrm{naive}}(+) = \boldsymbol{\Theta} + G_{\theta_1}\,\theta_1 + G_v\,\mathbf{v} \quad , \tag{73}$$

where

$$G_{\theta_1} = \begin{bmatrix} -I_{3\times 3} \\ \left(H'^T R^{-1} H'\right)^{-1} \left(H'^T R^{-1} h_1\right) \end{bmatrix} \quad , \tag{74a}$$

$$G_v = \begin{bmatrix} 0_{3\times 3} \\ \left(H'^T R^{-1} H'\right)^{-1} H'^T R^{-1} \end{bmatrix} \quad . \tag{74b}$$

The naive "relative" misalignment estimates are seen to be biased by terms linear in the true value of $\theta_1$, which is not surprising.

The true covariance matrix of the naive "relative" misalignment estimates is thus

$$P_{\Theta\Theta}^{\mathrm{naive}} = G_{\theta_1}\,P_{\theta_1\theta_1}(-)\,G_{\theta_1}^T + G_v\,R\,G_v^T \quad , \tag{75}$$

while the covariance which is incorrectly claimed for this naive method is

$$\text{“}P_{\Theta'\Theta'}^{\mathrm{naive}}(+)\text{”} = \left(H'^T R^{-1} H'\right)^{-1} \quad . \tag{76}$$

We speak of naive "relative" misalignment estimation, but in order to make a consistent comparison we compare absolute misalignments from all methods since it is the absolute misalignments which must be used in the attitude data processing after the alignment calibration is completed. The quotation marks in equation (76) remind us that the covariance is based on incorrect statistical assumptions.

We can evaluate all three expressions in a common model. We assume for the sake of simplicity that

$$R = \sigma^2\,I_{3n\times 3n} \quad , \tag{77}$$

and

$$H = I_{3n\times 3n} - \frac{1}{n}\,L_{3n\times 3n} \quad , \tag{78}$$

where

$$L_{3m\times 3n} \equiv \begin{bmatrix} I & I & \cdots & I \\ I & I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{bmatrix} \quad , \tag{79}$$

and $L_{3m\times 3n}$ has $3m$ rows and $3n$ columns. This is the simplest model for $\mathbf{Z}$ which is a function of the relative misalignments alone, as required by the general form of $\mathbf{Z}_k$. (Note that this $\mathbf{Z}$ has only $(3n-3)$ statistically independent components, as required.) Using this model and taking the *a priori* inflight covariance matrix to be given by equations (7), (11), and equation (96) of [1] leads to

$$P_{\Theta\Theta}(+) = \left((\sigma^{-2} + (\sigma_p^2 + q)^{-1}\right)^{-1} \left(I_{3n\times 3n} - \frac{1}{n} L_{3n\times 3n}\right) + \left((n+1)\sigma_p^2 + q\right) \frac{1}{n} L_{3n\times 3n} \quad , \tag{80}$$

$$\text{“}P_{\Theta\Theta}^{\text{naive}}(+)\text{”} = \begin{bmatrix} 0_{3\times 3} & 0 \\ 0 & \sigma^2(I_{3(n-1)\times 3(n-1)} + L_{3(n-1)\times 3(n-1)}) \end{bmatrix} \quad , \tag{81}$$

and

$$P_{\Theta\Theta}^{\text{naive}} = \text{“}P_{\Theta\Theta}^{\text{naive}}(+)\text{”} + (2\,\sigma_p^2 + q)\, L_{3n\times 3n} \quad . \tag{82}$$

Equation (82) states that the true covariance for the naive relative misalignment estimation procedure is equal to the one claimed for that method based on its incorrect statistical assumptions plus a correction term. Note that the correction term is roughly linear in $q$, which is large.

We can compare the three covariances by computing the average variances in each case (defined as $1/(3n)$ times the trace of the covariance matrix). The result for the true average variance of the correctly computed maximum likelihood estimate (from equation (80)) is

$$\langle \sigma_{\text{true}}^2 \rangle = (1 - (1/n))\,\sigma_p^2 + (1/n)\,q + (1 - (1/n))\,\left((\sigma^{-2} + (\sigma_p^2 + q)^{-1}\right)^{-1} \quad . \tag{83}$$

The average variance claimed by the naive relative misalignment estimation based on its own false statistical assumptions is

$$\langle \text{“}\sigma_{\text{naive}}^2\text{”} \rangle = 2\,(1 - (1/n))\,\sigma^2 \quad . \tag{84}$$

Finally the true typical variance for the naive relative estimates is

$$\langle \sigma_{\text{naive}}^2 \rangle = 2\,\sigma_p^2 + q + 2\,(1 - (1/n))\,\sigma^2 \quad . \tag{85}$$

Since the launch-shock covariance is the largest contributor to each of these expressions, the naive relative estimates clearly are the poorer result (by a factor $n$), although based on the incorrect statistical assumptions on which the naive estimators are based, they would seem (without closer scrutiny) to be the best.

Physically what is happening is that by setting $\boldsymbol{\theta}_1^*(+) \equiv \mathbf{0}$ in the naive approach, the entire prelaunch uncertainty is forced into that quantity, and each of the other misalignment vectors is shifted in the opposite sense by the same amount. The more consistent maximum likelihood estimate, which does not prejudice the estimation against one misalignment, spreads this uncertainty over all the misalignments and effectively reduces their effect by a factor $1/\sqrt{n}$. An eigenvalue analysis of the two covariances (calculated with realistic statistics) shows that both have $(3n - 3)$ of their eigenvalues equal roughly to $\sigma^2$, as we would expect intuitively, since $(3n - 3)$ misalignments should be determined accurately by the inflight data no matter almost what crimes are committed in constructing the estimators. For the consistent maximum likelihood approach the remaining three eigenvalues are $((n+1)\sigma_p^2 + q)$, while for the naive approach they are approximately $n(\sigma_p^2 + q)$, which is considerably larger.

In this same context we may consider also the pseudo-inverse solution, defined by equations (61) and (62). For the present example

$$P_{\text{pseudoinverse}} \equiv \text{Cov}(\boldsymbol{\Theta}^{\#} - \boldsymbol{\Theta})$$
$$= \sigma^2 I_{3n\times 3n} + \left(\sigma_p^2 + \frac{1}{n}(\sigma_p^2 + q) - \frac{1}{n}\sigma^2\right) L_{3n\times 3n} \quad , \tag{86}$$

whence,

$$\langle \sigma^2_{\text{pseudo-inverse}} \rangle = (1 + (1/n))\sigma_p^2 + (1/n)q + (1 - (1/n))\sigma^2 \quad , \tag{87}$$

which is almost as small as $\langle \sigma^2_{\text{true}} \rangle$.

## Alignment Estimation Accuracies for Narrow Fields of View

When the field of view of the sensor is small it becomes difficult to distinguish misalignments about the sensor boresights from those about the other sensor axes. Common usage has been to simply restrict allowable misalignments to the axes normal to the boresight. We will investigate the necessity and wisdom of such a procedure within a simple but realistic model.

Suppose that the spacecraft is equipped with three vector sensors, each with a limited field of view and with boresights nominally along the three spacecraft body axes. We will assume that each frame contains measurements for all three sensors and that the distribution of these measurements about each sensor boresight is axially symmetric with a root-mean-square angular radius of $\sqrt{2}\alpha$ (i.e., the root-mean-square spread of each of the components of $\hat{\mathbf{W}}_{i,k}$ about the boresight is $\alpha$). Thus, we may write the measurement equation as

$$\mathbf{Z}_k = \begin{bmatrix} z_{23,k} \\ z_{31,k} \\ z_{12,k} \end{bmatrix} = H_k\,\boldsymbol{\Theta} + \Delta\mathbf{Z}_k \quad , \tag{88}$$

where the reordering of the components and sign changes will serve to give the measurement vector a cyclic symmetry and simplify later calculations. The sensitivity matrix, $H_k$, is given now (as a function of the uncalibrated body-referenced observation vectors defined in [1]) by

$$H_k = \begin{bmatrix} \mathbf{0}^T & (\hat{\mathbf{W}}^o_{2,k} \times \hat{\mathbf{W}}^o_{3,k})^T & -(\hat{\mathbf{W}}^o_{2,k} \times \hat{\mathbf{W}}^o_{3,k})^T \\ -(\hat{\mathbf{W}}^o_{3,k} \times \hat{\mathbf{W}}^o_{1,k})^T & \mathbf{0}^T & (\hat{\mathbf{W}}^o_{3,k} \times \hat{\mathbf{W}}^o_{1,k})^T \\ (\hat{\mathbf{W}}^o_{1,k} \times \hat{\mathbf{W}}^o_{2,k})^T & -(\hat{\mathbf{W}}^o_{1,k} \times \hat{\mathbf{W}}^o_{2,k})^T & \mathbf{0}^T \end{bmatrix} \quad . \tag{89}$$

We will assume that

$$\Delta\mathbf{Z}_k \sim \mathcal{N}(\mathbf{0},\, P_{\mathbf{Z}_k}) \quad , \tag{90}$$

with

$$P_{\mathbf{Z}_k} = \sigma^2 I_{3\times 3} \quad . \tag{91}$$

Note from equation (89) that due to the narrow fields of view

$$\mathbf{Z}_k \approx \begin{bmatrix} \hat{\mathbf{e}}_1 \cdot (\theta_2 - \theta_3) \\ \hat{\mathbf{e}}_2 \cdot (\theta_3 - \theta_1) \\ \hat{\mathbf{e}}_3 \cdot (\theta_1 - \theta_2) \end{bmatrix} + \Delta \mathbf{Z}_k \quad , \tag{92}$$

so that only $\theta_{2x} - \theta_{3x}$ , $\theta_{3y} - \theta_{1y}$ , and $\theta_{1z} - \theta_{2z}$ will be determined with high accuracy. Three combinations of the misalignments will be determined not at all by the inflight data, and the remaining three will be determined poorly. For general $n$ (and not very undesirable geometry) we note that $\mathbf{Z}_k$ will have $(2n - 3)$ statistically independent components. Thus, we expect that $(2n - 3)$ combinations will be determined well from the inflight data, $n$ combinations relatively poorly, and 3 combinations not at all. The loss in alignment estimation accuracy might seem, therefore, to be close to our naive expectations.

To appreciate the magnitudes involved let us compute the estimate error covariance matrix in some detail. From equation (91) the Fisher information matrix for the inflight data is

$$\begin{aligned} P_{\Theta\Theta}^{-1}(\text{inflight}) &= \sum_{k=1}^{N} H_k^T \, P_{\mathbf{Z}_k}^{-1} \, H_k \\ &= \sigma^{-2} \sum_{k=1}^{N} H_k^T \, H_k \quad , \end{aligned} \tag{93}$$

and for $N$ very large,

$$P_{\Theta\Theta}^{-1}(\text{inflight}) = N \, \sigma^{-2} \, \left\langle H_k^T \, H_k \right\rangle \quad , \tag{94}$$

where $\langle \cdot \rangle$ denotes an average over the orientation of the observations within the field of view. Though we use the designation *inflight* in equations (93) and (94), in fact, this is nothing more than the prior-free information matrix associated with the estimation of $\mathbf{\Theta}$. Note, however, that $\mathbf{\Theta}^*(\text{inflight})$ does not exist, a consequence of the fact that $P_{\Theta\Theta}^{-1}(\text{inflight})$ must be singular. Evaluating the average in equation (94) leads to

$$P_{\Theta\Theta}^{-1}(\text{inflight}) = N \, \sigma^{-2} \times$$

$$\begin{bmatrix} 2\beta & 0 & 0 & -\beta & 0 & 0 & -\beta & 0 & 0 \\ 0 & 1+\beta & 0 & 0 & -\beta & 0 & 0 & -1 & 0 \\ 0 & 0 & 1+\beta & 0 & 0 & -1 & 0 & 0 & -\beta \\ -\beta & 0 & 0 & 1+\beta & 0 & 0 & -1 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 2\beta & 0 & 0 & -\beta & 0 \\ 0 & 0 & -1 & 0 & 0 & 1+\beta & 0 & 0 & -\beta \\ -\beta & 0 & 0 & -1 & 0 & 0 & 1+\beta & 0 & 0 \\ 0 & -1 & 0 & 0 & -\beta & 0 & 0 & 1+\beta & 0 \\ 0 & 0 & -\beta & 0 & 0 & -\beta & 0 & 0 & 2\beta \end{bmatrix} \quad , \tag{95}$$

with

$$\beta \equiv \alpha^2 \quad . \tag{96}$$

This matrix can be simplified by defining a new total misalignment vector, $\mathbf{\Phi}$ , by

$$\begin{aligned}\mathbf{\Phi} &\equiv [\,\Theta_1, \Theta_4, \Theta_7, \Theta_5, \Theta_8, \Theta_2, \Theta_9, \Theta_3, \Theta_6, \,]^T \\ &\equiv T\,\mathbf{\Theta} \quad . \end{aligned} \tag{97}$$

Since $\mathbf{\Phi}$ is simply a reordering of the components of $\mathbf{\Theta}$, it follows that $T$ is orthogonal. In terms of $\mathbf{\Phi}$

$$P_{\Phi\Phi}^{-1}(\text{inflight}) = T\,P_{\Theta\Theta}^{-1}(\text{inflight})\,T^T = N\,\sigma^{-2}\begin{bmatrix} \mathcal{M} & O_{3\times3} & O_{3\times3} \\ O_{3\times3} & \mathcal{M} & O_{3\times3} \\ O_{3\times3} & O_{3\times3} & \mathcal{M} \end{bmatrix} \quad , \tag{98}$$

with

$$\mathcal{M} = \begin{bmatrix} 2\beta & -\beta & -\beta \\ -\beta & 1+\beta & -1 \\ -\beta & -1 & 1+\beta \end{bmatrix} \quad . \tag{99}$$

Thus, the eigenvalues of $P_{\Phi\Phi}^{-1}(\text{inflight})$ (and, therefore, of $P_{\Theta\Theta}^{-1}(\text{inflight})$) each have a three-fold degeneracy. The eigenvalues of $\mathcal{M}$ are simply

$$\lambda_1 = 0 \quad , \quad \lambda_2 = 3\beta \quad , \quad \lambda_3 = 2+\beta \quad . \tag{100}$$

The vanishing of one of the eigenvalues of $\mathcal{M}$ is required by our earlier discussion of the singularity of $F^T P_{\Psi\Psi}^{-1}(\text{prior-free})F$.

If we assume the *a priori* covariance matrix for the misalignments to be

$$P_{\Theta\Theta}^{-1}(-) = \sigma_o^2\, I_{9\times9} \quad , \tag{101}$$

corresponding to $q >> \sigma_p^2$, then the three eigenvalues of $P_{\Theta\Theta}^{-1}(+)$ are

$$\sigma_1^2 = \sigma_o^2 \quad , \tag{102a}$$

$$\sigma_2^2 = \left(\frac{1}{\sigma_o^2} + \frac{3N\alpha^2}{\sigma^2}\right)^{-1} \quad , \tag{102b}$$

$$\sigma_3^2 = \left(\frac{1}{\sigma_o^2} + \frac{N(2+\alpha^2)}{\sigma^2}\right)^{-1} \quad , \tag{102c}$$

Choosing values

$$\sigma_o = 1.\ \text{arc min} \quad , \quad \sigma = 10.\ \text{arc sec} \quad , \quad \alpha = 5.\ \text{deg} \quad , \quad N = 100 \quad , \tag{103}$$

the three standard deviations become roughly

$$\sigma_1 = 1.\ \text{arc min} \quad , \quad \sigma_2 = 13.\ \text{arc sec} \quad , \quad \sigma_3 = 0.7\ \text{arc sec} \quad . \tag{104}$$

The restricted field of view is seen to be not a serious impediment to estimating misalignments accurately although the differences in alignment accuracies are substantial. The most serious deficiency, of course, is the complete lack of observability from inflight data of three combinations of the misalignments.

Note, however, that the three misalignment vectors which are poorly determined inflight, i.e., the three eigenvectors of $P_{\Theta\Theta}^{-1}(\text{inflight})$ with eigenvalue $3\beta$, are given by

$$\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} , \quad \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} , \quad \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} .$$

These are not the three boresight vectors. The loss in alignment estimation accuracy due to restricted fields of view is, therefore, somewhat different from not being able to estimate well the misalignments about the $n$ sensor boresights.

## Misalignment Estimation Accuracy and Attitude Estimation Accuracy

The error in the misalignment estimates necessarily translates into attitude error. To determine the degree to which this occurs we suppose that the attitude is determined using the QUEST algorithm [6]. Thus, we assume that the attitude algorithm will only use the estimates of the final inflight alignment calibration to correct the assumed alignments but not try to work the detailed covariance matrix of the inflight misalignment estimates into the attitude estimator (except perhaps for adjusting the values of the variances which are intrinsic to the QUEST algorithm). This is probably a reasonable, if suboptimal, approach.

Let $\Delta\boldsymbol{\theta}_i$, as usual, denote the misalignment vector estimate error for sensor $i$, and let $\xi_k^\theta$ be the additional attitude error arising from the misalignment errors. In the absence of misalignment errors, the optimal QUEST attitude matrix, $A_k^*$, minimizes [6] (as in [1], $\hat{\mathbf{V}}_{i,k}$ is the reference vector corresponding to $\hat{\mathbf{W}}_{i,k}$)

$$J_k \equiv \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma_{i,k}^2}|\hat{\mathbf{W}}_{i,k} - A_k\,\hat{\mathbf{V}}_{i,k}|^2 \quad . \tag{105}$$

In the presence of alignment errors, the correction, $\xi_k^\theta$, to the QUEST attitude (without correcting the weights for these alignment errors) minimizes

$$J_k' \equiv \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma_{i,k}^2}\left| e^{[[\Delta\boldsymbol{\theta}_i]]}\hat{\mathbf{W}}_{i,k} - e^{[[\xi_k^\theta]]}A_k^*\,\hat{\mathbf{V}}_{i,k}\right|^2 \quad . \tag{106}$$

$$\approx \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma_{i,k}^2}\left|\hat{\mathbf{W}}_{i,k} - e^{[[\xi_k^\theta - \Delta\boldsymbol{\theta}_i]]}A_k^*\,\hat{\mathbf{V}}_{i,k}\right|^2 \quad . \tag{107}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma_{i,k}^2}\left|\hat{\mathbf{W}}_{i,k} - A_k^*\,\hat{\mathbf{V}}_{i,k} + [[A_k^*\,\hat{\mathbf{V}}_{i,k}]]\,(\xi_k^\theta - \Delta\boldsymbol{\theta}_i)\right|^2 \quad , \tag{108}$$

and the equalities are true to $O(|\xi_k^\theta| + |\Delta\boldsymbol{\theta}_i|)$. Minimizing $J_k'$ over $\xi_k^\theta$ leads to

$$\xi_k^{\theta *} = F_k^{-1}\left(\sum_{i-1}^{n}F_{i,k}\,\Delta\boldsymbol{\theta}_i + \sum_{i=1}^{n}\frac{1}{\sigma_{i,k}^2}[[A_k^*\,\hat{\mathbf{V}}_{i,k}]]\,\hat{\mathbf{W}}_{i,k}\right) \quad , \tag{109}$$

where

$$F_{i,k} = \frac{1}{\sigma_{i,k}^2} \left[ I_{3\times 3} - \left( A_k^* \hat{\mathbf{V}}_{i,k} \right) \left( A_k^* \hat{\mathbf{V}}_{i,k} \right)^T \right] \quad , \tag{110}$$

which is the Fisher Information Matrix for the attitude associated with the single measurement $\hat{\mathbf{W}}_{i,k}$, and

$$F_k = \sum_{i=1}^{n} F_{i,k} \quad , \tag{111}$$

which is the Fisher information matrix for the attitude arising from all the measurements at time $t_k$.

By definition, $\xi_k^\theta$ must vanish when the $\Delta\boldsymbol{\theta}_i$ all vanish. Therefore, the second term in equation (109) must vanish to first order, leading finally to

$$\xi_k^{\theta *} = F_k^{-1} \sum_{i-1}^{n} F_{i,k} \, \Delta\boldsymbol{\theta}_i \quad . \tag{112}$$

The misalignment errors, thus, lead to a random bias in the attitude. The attitude errors arising from the random sensor noise have covariance $F_k^{-1}$. Thus, the total covariance of the QUEST attitude solutions taking account of both sensor noise and misalignment estimate errors is

$$P_{\xi_k \xi_{k'}} = \delta_{kk'} \, F_k^{-1} + F_k^{-1} \left( \sum_{i,j}^{n} F_{i,k} \, P_{\theta\theta}^{ij} \, F_{j,k'} \right) F_{k'}^{-1} \quad . \tag{113}$$

Note that the attitudes are now autocorrelated due to the misalignment error.

How large are the two contributions to equation (113)? Consider the model of the last section, which assumed that the measurements are always close to the spacecraft body axes. Let us further assume that $N$, the number of frames of attitude data, is sufficiently large that the only important alignment errors come from the components of the misalignments that are unobservable from inflight data. Then

$$P_{\theta\theta}^{ij} = \frac{1}{3} \sigma_o^2 \, I_{3\times 3} \quad , \tag{114}$$

and

$$F_k^{-1} = \frac{1}{2} \sigma^2 \, I_{3\times 3} \quad . \tag{115}$$

Hence,

$$P_{\xi_k \xi_{k'}} = \left( \frac{1}{2} \sigma^2 \, \delta_{kk'} + \frac{1}{3} \sigma_o^2 \right) I_{3\times 3} \quad . \tag{116}$$

For the values assumed in the previous example the random sensor measurements contribute 7. arc sec/axis to the attitude error, while the unobservable (i.e., from inflight data) misalignments contribute 34. arc sec/axis. Thus, the unobservable misalignments can be the largest contributor to the attitude errors. If, however, the payload also functions as an attitude sensor, the contribution of the alignment errors to the attitude error can be made very small.

## The Numerical Example Revisited

We illustrate these methods with the numerical example described above and in Part I. Table 1 shows the comparison of the model values and the corresponding estimates. Note that two-thirds of the estimates fall within one standard deviation of the model misalignments as expected. The level of agreement is as much as would be expected. The large error brackets are due to launch-shock effects.

The level of agreement will be more manifest if we compare instead the components of $\mathbf{\Theta}$ along the eigenvectors of $P_{\Theta\Theta}(+)$. If we define the orthogonal matrix $C$ according to

$$C\, P_{\Theta\Theta}(+)\, C^T = P^D_{\Theta\Theta}(+) = \mathrm{diag}(p_1, p_2, \ldots, p_{3n}) \quad , \tag{117}$$

and

$$\mathbf{\Phi} = C\,\mathbf{\Theta} \quad , \tag{118}$$

then the estimates of the components of $\mathbf{\Phi}$ are uncorrelated and their estimate-error variances are given by the $p_i$, $i = 1, \ldots, 3n$. We call the components of $\mathbf{\Phi}$ the *eigenmisalignments*. If we compare these with the corresponding model values we find the result in Table 2. This shows the true level of agreement. Note that three of the estimates of the eigenmisalignments are exactly zero, a consequence of equation (55) above (because the prelaunch alignment estimation errors are identically distributed (although *not* independent)), and that the uncertainties in these estimates is given by the launch-shock error levels. Also, three of the remaining estimates of the eigenmisalignments are nearly an order of magnitude more accurate than the other remaining three, a phenomenon which has been noted in the section devoted to sensors having narrow fields of view. In the current example the three unobservable eigenmisalignments are quite large, on the order of $q^{1/2}$ (we have used the estimated value of $q^{1/2}$ in the computations). It is these three eigenmisalignments which dominate the differences between the model values and the estimates in Table 1. Because these particular

**Table 1. Comparison of Model and Estimated Absolute Misalignments**

| Model Misalignments | Estimated Misalignments |
|---|---|
| 37. arc sec | 27. ± 29. arc sec |
| −23. | 31. ± 29. |
| −58. | −72. ± 30. |
| −36. | −46. ± 29. |
| −63. | −18. ± 29. |
| 4. | 3. ± 29. |
| 22. | 18. ± 29. |
| −66. | −14. ± 29. |
| 73. | 69. ± 29. |

**Table 2. Comparison of Model and Estimated Eigenmisalignments**

| Model Eigenmisalignments | Estimated Eigenmisalignments |
|---|---|
| 60. arc sec | 0. ± 50. arc sec |
| −18. | 0. ± 50. |
| 63. | 0. ± 50. |
| 33. | 43. ± 10. |
| 85. | 90. ± 9. |
| − 8. | −4. ± 5. |
| 58. | 57. ± 1. |
| −31. | −32. ± 1. |
| −17. | −17. ± 1. |

three eigenmisalignments contribute to the actual misalignments with equal coefficients, the standard deviations of the actual misalignment estimates will be roughly the same and the most important correlations will be on the order of $1/\sqrt{n}$, where $n$ is the number of sensors. For the present example, in fact, had we chosen the three sensor boresights to be mutually orthogonal, we would have found nine of the correlations to be very nearly unity.

Let us investigate the performance of the naive "relative" misalignment estimation approach (which arbitrarily sets the misalignments of sensor 1 to zero and sets the "absolute" misalignments of the remaining sensors to the prior-free estimates of the corresponding relative misalignments) and the pseudo-inverse solution for the misalignments. These are given in Table 3, along with the model true values and the *a posteriori* maximum likelihood estimates of Table 1. The very poor performance of the naive "relative" procedure is evident. The pseudo-inverse solution, on the other hand, agrees rather well with the correctly calculated *a posteriori* estimates. This good agreement of the pseudo-inverse solution is not surprising, since the calculation of the pseudo-inverse is very much like the calculation of the *a posteriori* misalignments but with an extremely large diagonal *a priori* covariance matrix. This is approximately the nature of the assumed *a priori* covariance matrix. No error brackets are given for either the naive relative estimates or the pseudo-inverse estimates since the error brackets

**Table 3. Comparison of Model and Estimated Misalignments**

| Model Misalignments | A Posteriori Misalignments | "Relative" Misalignments | Pseudo-inverse Solution |
|---|---|---|---|
| 37. arc sec | 27. ± 29. arc sec | 0. arc sec | 27. arc sec |
| −23. | 31. ± 29. | 0. | 32. |
| −58. | −72. ± 30. | 0. | −75. |
| −36. | −46. ± 29. | −73. | −46. |
| −63. | −18. ± 29. | −50. | −18. |
| 4. | 3. ± 29. | 78. | 3. |
| 22. | 18. ± 29. | − 7. | 19. |
| −66. | −14. ± 29. | −45. | −13. |
| 73. | 69. ± 29. | 146. | 71. |

calculated for the former based on its own statistical assumptions are not meaningful in this context, and there is no statistical model associated with the pseudo-inverse solution. From the earlier heuristic discussion, however, we know that the "relative" misalignment estimates in this context should have a standard deviation on the order of $q^{1/2} \approx 50.$ arc sec, while the error associated with the pseudo-inverse for this example should not be much greater than that for the *a posteriori* estimates. (Note that the "naive relative misalignments" are the correct estimates of the relative misalignments and would provide a good approximation to the absolute alignments in other circumstances.)

## Discussion and Conclusions

We have developed a general methodology for estimating launch-shock error levels under that statistical assumption that the launch-shock errors are zero-mean and Gaussian. The methodology is general. However, for practical reasons one can estimate typically only a very few launch-shock parameters reliably. For the case of a spacecraft with three sensors we have estimated in our numerical example only a single launch-shock parameter, the standard deviation of the launch-shock induced alignment error per axis. Given the necessarily poor statistics for this estimation problem, the result is certainly acceptable. (At the same time it should be borne in mind that one can often tolerate large errors in the covariance estimates.)

There is, of course, no guarantee that the simple model of the present example would characterize the true launch shock error levels. In the case where some attitude sensors are mounted close to the payload and others much farther away on the spacecraft, it would be much more reasonable to estimate at least two launch-shock parameters. However, if only one attitude sensor is mounted near the payload, this may not be possible since every relative misalignment will contain at least one "far" sensor. One might argue that the combination of a "near" and "far" sensor would have a typical launch shock error covariance of $q_{\text{near}} + q_{\text{far}}$ while a combination involving two far sensors would have a typical covariance of $2q_{\text{far}}$. Unfortunately, the statistical significance of the launch-shock estimates are usually too poor to separate the two launch-shock parameters under these circumstances. The numerical example illustrates this. For a mission like the Solar Maximum Mission, however, where two fine pointing Sun sensors were located on the instrument support plate, such a separation was indeed possible [9].

One point which is obvious, however, is that there is little to be gained at present by improving the accuracy of the prelaunch alignment calibration procedures. Thus, the use of the simple estimator and covariance matrix for the prelaunch alignment, presented in Part I of this paper, will certainly be adequate until we have much more control over the delivery of spacecraft into space and the space environment.

Our detailed exposition of batch alignment estimation has developed a complete set of algorithms for evaluating the alignments of vector sensors, which have been shown to work well with realistically simulated data. For systems with poor geometries a factorized algorithm improves numerical accuracy and decreases the number of logical decisions which must be made in a software implementation. We have shown how to determine the level of launch-shock errors and how to include these in a more complete estimate of the misalignments than can be obtained from the inflight data alone. When the spacecraft payload is not also an attitude sensor or when it cannot be assumed that one sensor is not misaligned after launch with respect to the payload, the estimation of absolute misalignments as developed here yields a much more meaningful result. The algorithms developed here have all been tested with simulated data and with simple solvable models.

An important byproduct of this work has been to show what are achievable misalignment estimate error levels. In general, for a system of $n$ vector sensors three linear combinations of the misalignments are unobservable, and for sensors with narrow fields of view $2n-3$ misalignments will be estimated well and $n$ somewhat less well. Given sufficient data, however, it is only the three unobservable combinations which limit the accuracy of the alignment estimation procedure. Since these three combinations are just the average, they corrupt all other misalignments equally. Thus, if $\sigma_o$ is the standard deviation of the post-launch misalignments (i.e., corrupted by launch shock), the effect of the inflight calibration is to reduce this standard deviation to $\sigma_o/\sqrt{n}$. This is reflected in the numerical example. Additional accuracy is in general not possible unless the output of the payload also provides equivalent attitude information or it is known that the misalignment of one sensor from the payload is truly negligible. In many cases this is not the case. The example of the inflight determination of the misalignments of the Magsat sensors prior to the intervention of the Magsat scientist, as described below, is a good example.

It is interesting to follow the history of the alignment calibration of Magsat. Prior to launch the alignments of all the sensors and the scientific payload were determined at the Optical Alignment Facility at NASA Goddard Space Flight Center using the methods [10, 11] described in Part I of this work. The prelaunch alignment accuracy was, therefore, on the order of 5. arc sec. Inflight, the relative misalignments were calculated using the naive relative alignment approach. Because the scientific payload and the fine Sun sensor (FSS) on Magsat were mounted at the end of a long scissors boom and attitude measurements were translated from the experiment module to the spacecraft bus via an attitude transfer system (ATS) for which no previous flight experience existed, the relative misalignments were determined with respect to one of the fixed-head star trackers (FHSTs) rather than to the more logical FSS. The result of this inflight calibration [12] was that the two FHSTs had a relative misalignment of 11. arc sec, while the FSS was misaligned relative to one of the FHSTs by about 220. arc sec. The relative FSS-FHST2 misalignment was mostly in the roll component, for which the ATS was expected to have the largest error. (The relative misalignment of the two FHSTs, which were mounted on the same plate, is a good indicator of the level of uncompensated misalignment remaining between the FPSS and the scientific payload on the SMM spacecraft.) These relative alignment estimates are unassailable. However, they were also interpreted without alteration as absolute alignments for the purpose of computing the attitude of the fine vector magnetometer.

Misalignments were, as we said, recomputed by the experimenter [13] as part of the scientific data processing. He determined that the misalignment of the payload relative to the FSS was more on the order of 24. arc sec and that the 220. arc sec misalignment observed was associated more with the misalignment of the FHSTs relative to the FSS than with the FSS relative to the experiment. The misalignments did not remain constant in time due to degradation of the sensors and the changing environment of the spacecraft as its orbit decayed. Thus, the relative misalignment of the two FHSTs was observed to grow to 35. arc sec towards the end of the mission. The experimenter was able to estimate the alignment of the sensors relative to the payload to within 20. arc sec/axis, which was the total attitude error budget for the Magsat mission. Since the sensor measurement noise contributed about 7. arc sec/axis to the attitude error, this attitude accuracy requirement was well met. However, it is well to note that the attitude accuracy would have been closer to 4. arc min without the intervention of the experimenter. This statement is also true for the algorithms presented here, except that the uncompensated misalignments would have led to an attitude error of only 2. arc min.

The case of the Solar Maximum Mission would seem to afford an example in which prior-free relative misalignment estimation would be adequate. For that spacecraft the scientific payload and two fine pointing Sun sensors (FPSSs) were mounted on a single very rigid titanium instrument support plate. Post-launch analyses using the techniques developed here [9] revealed a relative misalignment of the two FPSSs of only one arc sec as well as a very pronounced temperature dependence. One would be lead to believe from our launch-shock analysis that the misalignment of the payload should be on this order of magnitude. However, the apparent misalignment is due not only to geometric displacement of the sensors but also to other effects as well. Adjustments to the alignments of the FPSSs of as much as 20. arc sec requested by the project scientist may be symptomatic of changes in the alignments of the FPSSs relative to the payload. This possibility has not been investigated.

The simplicity of the algorithms presented in this series of papers depends on the availability of simultaneous vector data from all sensors. If the data is not simultaneous or cannot be made simultaneous through the use of gyros or if it is not vectorial, then other means must be used to compute the alignments. In this later case one must usually rely on the Kalman filter. An example of the use of the Kalman filter with vector data assuming the QUEST measurement model is presented in [14]. A general comparison of batch, Kalman-filter, and hybrid methods, is sketched in [15]. An interesting comparison of batch and sequential methods has been reported recently by Krack, Lambertson, and Markley [16]. A very different approach to formulating alignment estimation problems underlies the work of Gray *et al.* [17].

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## References

[1] SHUSTER, M. D., PITONE, D. S., and BIERMAN, G. J., "Batch Estimation of Spacecraft Sensor Alignments, I. Relative Alignment Estimation, to appear in the *Journal of the Astronautical Sciences*.

[2] SORENSON, H. W., *Parameter Estimation*. Marcel Dekker, New York, 1980.

[3] NAHI, N. E., *Estimation Theory and Applications*, John Wiley & Sons, New York, 1969 (reprint: Robert K. Krieger Publishing Co., New York, 1976).

[4] SHUSTER, M. D., and PITONE, D. S., "Consistent Estimation of Spacecraft Sensor Alignments," *Proceedings, American Control Conference,* San Diego, California, pp. 1389–1395, May 23–25, 1990.

[5] GUPTA, N. K., and MEHRA, R. K., "Computational Aspects of Maximum Likelihood Estimation and Reduction in Sensitivity Function Calculations," *IEEE Transactions on Automatic Control,* Vol. AC-19, No. 1, pp. 359–367, 1969.

[6] SHUSTER, M. D., and OH, S. D., "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, pp. 70–77, 1981.

[7] WERTZ, J. R. (ed.), *Spacecraft Attitude Determination and Control*, D. Reidel Publishing Company, Dordrecht, the Netherlands, 1978.

[8] SHUSTER, M. D., "Fads and Fallacies in Spacecraft Sensor Alignment Estimation," *Symposium on Space Mechanics,* Toulouse, France, November 6–10, 1989, reprinted in *Mécanique Spatiale*, pp. 513–522, CEPADUES Editions, Toulouse, 1990.

[9] PITONE, D. S., and SHUSTER, M. D., "Attitude Sensor Alignment Calibration for the Solar Maximum Mission," *Proceedings, Flight Mechanics/Estimation Theory Symposium,* NASA Goddard Space Flight Center, Greenbelt, Maryland, pp. 21–39, May 22–24, 1990.

[10] APPLER, R. L., and HOWELL, B. J., "Optical Alignment of Multiple Components of a Common Coordinate System," *Applied Optics*, Vol. 7, pp. 1007–1015, 1968.

[11] EICHORN, W. L., "Optical Alignment Measurements at Goddard Space Flight Center," *Applied Optics*, Vol. 21, pp. 3891–3895, 1982.

[12] ABSHIRE, G., McCUTCHEON, R., SUMMERS, G., VANLANDINGHAM, F., and MEYERS, G., "High Precision Attitude Determination for Magsat," *Proceedings, ESA International Symposium on Spaceflight Dynamics,* Darmstadt, Federal Republic of Germany, April 1981.

[13] LANGEL, R., BERBERT, J., JENNINGS, T., and HORNER, R., "Magsat Data Processing: A Report for Investigators," NASA Technical Memorandum 82160, November 1981.

[14] SHUSTER, M. D., "Kalman Filtering of Spacecraft Attitude and the QUEST Model," *Journal of the Astronautical Sciences*, Vol. 38, No. 3, pp. 377–393, July–September, 1990.

[15] SHUSTER, M. D., "Inflight Estimation of Spacecraft Sensor Alignment," Paper No. AAS 90-041, AAS Guidance and Control Conference, Keystone, Colorado, February 1990, reprinted in *Advances in the Astronautical Sciences*, Vol. 72, pp. 253–274, 1990.

[16] KRACK. K., LAMBERTSON, M., and MARKLEY, F. L., "A Comparison of Two On-Orbit Attitude Sensor Alignment Methods," *Proceedings, Flight Mechanics/Estimation Theory Symposium,* NASA Goddard Space Flight Center, Greenbelt, Maryland, pp. 3–20, May 22–24, 1990.

[17] GRAY, C. W., HERMAN, L. K., KOLVE, D. I., and WESTERLUND, G. L., "On-Orbit Attitude Reference Alignment and Calibration," Paper No. AAS 90-042, AAS Guidance and Control Conference, Keystone, Colorado, February 1990, reprinted in *Advances in the Astronautical Sciences*, Vol. 72, pp. 275–292, 1990.