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Batch Estimation of Spacecraft Sensor Alignments
I. Relative Alignment Estimation

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Abstract

Simple and statistically correct algorithms are developed for batch estimation of spacecraft sensor relative misalignments without the need to compute the spacecraft attitude or angular velocity. These algorithms permit the estimation of sensor alignments in a framework free of unknown dynamical variables. In actual mission implementation, algorithms such as those presented here are often better behaved and more efficient than those which must compute sensor alignments simultaneously with the spacecraft attitude, say, by means of a Kalman filter. In particular, these algorithms are less sensitive to data dropouts of long duration, and the derived measurements used in the attitude-independent algorithm usually make data checking and editing of outliers much simpler than would be the case in the filter. A factorized estimator for the alignments, which is better behaved numerically and in some ways simpler to apply than the unfactorized algorithm, is also developed. Prelaunch alignment estimation is treated in detail as an example of relative alignment estimation. A very efficient approximation for this algorithm is developed which relies on the QUEST measurement model. The algorithms are applied to a realistic simulated example which approximates an actual mission.

Introduction

This is the first of two papers which will cover all aspects of batch estimation of spacecraft sensor alignments including both prelaunch and postlaunch calibration. Part I of this work is concerned largely with the inflight estimation of relative alignments. In many cases the relative alignments are all that are needed to support the mission. To estimate the relative alignments inflight it is helpful to know the prelaunch values of the alignments as a first approximation in the

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inflight estimation process. Due to the effects of launch shock, however, the prelaunch estimates of the relative alignments often have much less validity inflight than the estimates from inflight data alone. Thus, the estimators we study here are all "prior-free." For the inflight estimation of absolute alignments the opposite is true, as will be seen in Part II [1], which also treats the estimation of the launch-shock error levels. Interestingly enough, the prelaunch alignment calibration procedure is an example of relative alignment estimation and is treated here also.

While re-estimation inflight of the alignment of spacecraft attitude sensors is a part of nearly every mission, this area has not generally received much serious attention. The earliest published work in this area seems to be that of desJardins [2], who developed an unweighted least-squares method for estimating the sensor alignments of the Orbiting Astronomical Observatory (OAO). This analysis did not take account of any statistical information about the sensors nor of the pre-launch alignment estimates, except as a linearization point for the estimates. DesJardins did not use the inflight data directly, which would have made the estimates dependent on an unknown attitude, but instead considered the cosines of the angles between pairs of sensor measurements as effective measurements. Clearly, the absolute alignments of the sensors are not observable from this relative data. For this reason desJardins was forced to consider one of the six OAO star trackers as perfectly aligned and compute the alignments of the five remaining star trackers relative to it. Thus, the six equivalent star trackers were treated unsymmetrically. A similar approach was followed by Niebur *et al.* [3] for the Applications Explorer Mission.

For the Magsat mission Abshire *et al.* originally followed a different procedure [4]. For this mission A_k, the attitude matrix at time t_k, was computed by minimizing the cost function [5]

$$L(A_k) = \frac{1}{2} \sum_{i=1}^{n_k} a_i \, |\hat{\mathbf{W}}_{i,k} - A_k \hat{\mathbf{V}}_{i,k}|^2 \quad , \tag{1}$$

where the sum is over the n_k attitude sensors which are active in frame k. Abshire *et al.* reasoned that the misalignment matrices M_i could be estimated by minimizing

$$L_i(M_i) = \frac{1}{2} \sum_{k=1}^{N} |M_i \hat{\mathbf{W}}_{i,k} - A_k \hat{\mathbf{V}}_{i,k}|^2 \quad , \tag{2}$$

summing over time and with assuming the A_k known (calculated with $M_i = I$). In this way the M_i could also be calculated by means of the same algorithm as was used for attitude determination. Thus, rather than compute relative alignments among the attitude sensors, Abshire *et al.* compute absolute alignments (i.e., relative to the spacecraft body frame). Such an estimation scheme is imperfect, because these absolute alignment matrices and the attitude matrix are not simultaneously observable from inflight data. Thus, if the sensor alignments are recomputed frequently in this way throughout the mission, they should display a (divergent) random walk in their values. For this reason, only relative alignments, which were not affected by this random-walk divergence, were reported.

An earlier work [6], which may be regarded as a forerunner of the work presented here, attempted to overcome the restriction to estimating only relative alignments by including the prelaunch statistics heuristically. That work had several drawbacks. In particular, it neglected correlations and redundancy in the derived measurements (which were the cosines introduced by desJardins). Also, the statistical modeling of the prelaunch alignment estimates was very crude. This crudeness manifested itself in the resulting estimates of the postlaunch alignments, all of which were well within the computed 1σ error brackets, as noted at the time. Nonetheless, this was the first work which attempted to make any statistical characterization of the inflight alignment estimates or to take proper account of the inflight alignment estimates. The algorithm was efficient and robust and has been used since in nearly every mission supported by NASA Goddard Space Flight Center (for a recent example see Snow *et al.* [7] for the alignment algorithms proposed for the UARS spacecraft).

Part I of the present work is concerned chiefly with the estimation inflight of *relative* sensor alignments, that is, the inflight estimation of the alignments of the attitude sensors relative to one of these sensors. There are several important reasons for considering relative alignment estimation as a separate problem. First, it is more convenient to estimate the relative alignments first without recourse to prelaunch results (except as a first approximation) and only afterwards use the prelaunch estimates to calculate the absolute alignments from these. Secondly, it is often the case that the spacecraft payload can function as an attitude sensor and, therefore, the alignments relative to this payload are all that is needed (or it may be known that the alignment of one attitude sensor relative to the payload should not be greatly affected by launch shock). Since the prelaunch data when used inflight is not generally of high quality (due to the errors induced by launch shock), there is little quantitative gain in computing absolute rather then relative alignments in this case. Also, the estimation of absolute alignments is an important problem in itself and is treated in Part II of this work [1].

The present work is concerned only with batch estimation methods. These methods, in order to avoid the estimation of a prohibitively large number of attitude parameters ($3N$ parameters for N frames of data), must necessarily remove the attitude dependence from the estimation problem by using pseudo-measurements (the measured cosines). In a sequential estimator, pseudo-measurements can be avoided, and the complete vector data used to estimate attitude and alignments (either relative or absolute) directly. A comparison of the two approaches is given in [8].

The present work begins by defining the alignments in a manner which has proven to be well adapted to alignment estimation problems and leads to the simplest expressions. For simplicity, the alignment estimation problem is formulated initially in terms of absolute alignments. Following this, the dependence of the measurements, both the complete vectors and the derived cosines, is presented, and the problem of the lack of observability of the attitude and the alignments made manifest. In particular, the redundancy of the derived measurements is discussed and an algorithm presented for obtaining a non-redundant subset of these. The statistical character of the derived measurements is also developed. Given the statistical properties of the measurements the maximum likelihood estimate is derived for the relative alignments. When a spacecraft has many sensors, the simple algorithm for obtaining a non-redundant set of alignments may not be convenient and for poorly conditioned problems may also be inaccurate. For these cases a factorized algorithm [9] is presented whose implementation in software will require fewer decisions and which should be better behaved numerically. Finally, the algorithms are tested with realistically simulated data. The QUEST model is used to represent the errors in the sensor measurements and the very simple expressions which result for this model are presented. Appendix A gives the details on treating coplanar measurements. Appendix B presents a sketch of an implementation of the algorithms.

Definitions

Sensor Referenced Measurements

A spacecraft line-of-sight sensor such as a vector Sun sensor or star tracker measures a direction, $\hat{\mathbf{U}}_{i,k}$, in sensor coordinates, defined to be directed outward from the sensor, which can be described statistically as

$$\hat{\mathbf{U}}_{i,k} = \hat{\mathbf{U}}_{i,k}^{\text{true}} + \Delta\hat{\mathbf{U}}_{i,k} \quad , \tag{3}$$

where $\hat{\mathbf{U}}_{i,k}^{\text{true}}$ is the true value of the direction and $\Delta\hat{\mathbf{U}}_{i,k}$ is the measurement noise. Here i is the sensor index, $i = 1, \ldots, n_k$, and k is the temporal index, $k = 1, \ldots, N$. We assume that $\Delta\hat{\mathbf{U}}_{i,k}$ is Gaussian, zero-mean, and white, with covariance $R_{\hat{\mathbf{U}}_{i,k}}$. In more compact notation

$$\Delta\hat{\mathbf{U}}_{i,k} \sim \mathcal{N}(\mathbf{0}, R_{\hat{\mathbf{U}}_{i,k}}) \quad . \tag{4}$$

We assume more generally, in fact, that the measurements from different sensors are statistically independent, thus

$$E\{\Delta\hat{\mathbf{U}}_{i,k}\,\Delta\hat{\mathbf{U}}_{i'k'}^T\} = \delta_{ii'}\,\delta_{kk'}\,R_{\hat{\mathbf{U}}_{i,k}} \quad . \tag{5}$$

Here, $E\{\cdot\}$ denotes the expectation operator. Because the observations are constrained to be unit vectors, $R_{\hat{\mathbf{U}}_{i,k}}$ must be singular. In particular,

$$R_{\hat{\mathbf{U}}_{i,k}}\,\hat{\mathbf{U}}_{i,k}^{\text{true}} = \mathbf{0} \quad . \tag{6}$$

Clearly, equations (4) through (6) can be true only to lowest order in R. Since R is generally quite small, this level of approximation will be adequate for the purpose of alignment estimation.

Body-Referenced Vectors and Alignments

If $\hat{\mathbf{W}}_{i,k}$ denotes the measured direction in the spacecraft body frame, then the alignment matrix, S_i, is the proper orthogonal matrix defined by

$$\hat{\mathbf{W}}_{i,k} = S_i\,\hat{\mathbf{U}}_{i,k} \quad , \tag{7}$$

and, therefore,

$$\hat{\mathbf{W}}_{i,k} = S_i\,\hat{\mathbf{U}}_{i,k}^{\text{true}} + S_i\,\Delta\hat{\mathbf{U}}_{i,k} \quad , \tag{8}$$

$$\equiv \hat{\mathbf{W}}_{i,k}^{\text{true}} + \Delta\hat{\mathbf{W}}_{i,k} \quad . \tag{9}$$

Thus, the body-referenced observations have an error model given by

$$\Delta\hat{\mathbf{W}}_{i,k} \sim \mathcal{N}(\mathbf{0}, R_{\hat{\mathbf{W}}_{i,k}}) \quad , \tag{10}$$

with

$$R_{\hat{\mathbf{W}}_{i,k}} = S_i \; R_{\hat{\mathbf{U}}_{i,k}} \; S_i^T \quad , \tag{11}$$

and

$$R_{\hat{\mathbf{W}}_{i,k}} \hat{\mathbf{W}}_{i,k}^{\text{true}} = \mathbf{0} \quad . \tag{12}$$

Misalignments

In general, the alignment matrix S_i is not known exactly. Instead, what is known is S_i^o, the alignment matrix determined by the prelaunch alignment calibration. Thus, we are led to define the misalignment matrix, M_i, according to

$$S_i = M_i \, S_i^o \quad . \tag{13}$$

M_i is necessarily orthogonal. Therefore, we define the misalignment vectors, $\boldsymbol{\theta}_i$, according to

$$\begin{aligned} M_i &\equiv e^{[[\boldsymbol{\theta}_i]]} \quad , \\ &= I + \left(\frac{\sin|\boldsymbol{\theta}_i|}{|\boldsymbol{\theta}_i|}\right)[[\boldsymbol{\theta}_i]] + \left(\frac{1-\cos|\boldsymbol{\theta}_i|}{|\boldsymbol{\theta}_i|^2}\right)[[\boldsymbol{\theta}_i]]^2 \quad , \end{aligned} \tag{14}$$

where $e^{\{\cdot\}}$ denotes matrix exponentiation, and $[[\boldsymbol{\theta}]]$ denotes the usual antisymmetric matrix,

$$[[\boldsymbol{\theta}]] \equiv \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix} \quad . \tag{15}$$

Equation (14) is just Euler's formula for the rotation matrix as a function of the rotation vector. The angles $\theta_1, \theta_2, \theta_3$ are the misalignment angles or simply the misalignments (Do not confuse the subscripts on θ, which label components, with those on $\boldsymbol{\theta}$, which label sensors. To be more consistent, we should write $\boldsymbol{\theta}_i = [\theta_{i1}, \theta_{i2}, \theta_{i3}]^T$. Whenever possible, however, we will avoid such a cumbersome notation, which invites confusion of the component index with the temporal index.). Since the misalignment matrix is generally a very small rotation, the misalignments will be small, and we can write

$$M_i = I + [[\boldsymbol{\theta}_i]] + O(|\boldsymbol{\theta}_i|^2) \quad , \tag{16}$$

As a rule, we will keep only first-order terms. The measurement equation now becomes finally

$$\hat{\mathbf{U}}_{i,k} = S_i^{oT} M_i^T \hat{\mathbf{W}}_{i,k}^{\text{true}} + \Delta\hat{\mathbf{U}}_{i,k} \quad . \tag{17}$$

Alignment Measurements

Dependence of the Vector Measurements on the Attitude

If $\hat{\mathbf{V}}_{i,k}$ denotes the reference vector, i.e., the representation of the measured vector in the primary reference system, then the attitude matrix, A_k, is defined according to

$$\hat{\mathbf{W}}_{i,k}^{\text{true}} = A_k \, \hat{\mathbf{V}}_{i,k}^{\text{true}} \quad , \tag{18}$$

whence,

$$\hat{\mathbf{W}}_{i,k} = A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}_{i,k} - A_k \Delta\hat{\mathbf{V}}_{i,k} \quad , \tag{19}$$

where $\Delta\hat{\mathbf{V}}_{i,k}$ is the uncertainty in the reference vector, which we assume to be Gaussian and zero-mean. Hence,

$$E\{\Delta\hat{\mathbf{V}}_{i,k} \, \Delta\hat{\mathbf{V}}_{i,k}^T\} = R_{\hat{\mathbf{V}}_{i,k}} \quad . \tag{20}$$

The reference vector errors are not necessarily white, however, since the same reference object may be used repeatedly, and its direction is not remeasured (from the Earth) at each occurrence. We may, however, neglect the errors in the reference vectors compared to those in the observations, which are usually an order of magnitude larger. From this it follows that the actual sensor measurements are related to the reference vectors to good approximation by

$$\hat{\mathbf{U}}_{i,k} = S_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{U}}_{i,k} \quad . \tag{21}$$

We note immediately from equation (21) that the values of the measurement vectors are unchanged by the simultaneous transformations

$$S_i \to T\, S_i \quad , \qquad i = 1, \ldots, n \quad , \tag{22a}$$

$$A_k \to T\, A_k \quad , \quad k = 1, \ldots, N \quad , \tag{22b}$$

where T is an arbitrary proper orthogonal matrix. Thus, it is impossible from sensor measurements to distinguish a common misalignment of the sensors from a change in the attitude. It is, therefore, impossible to estimate the sensor alignments and the attitude unambiguously from the spacecraft sensor measurements alone.

In terms of the misalignments, equation (21) becomes

$$\hat{\mathbf{U}}_{i,k} = S_i^{oT} M_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{U}}_{i,k} \quad . \tag{23}$$

The Attitude-Independent Measurements

Equation (23) is the starting point for processing the inflight data. We begin by defining an "uncalibrated" body-referenced observation vector, $\hat{\mathbf{W}}_{i,k}^o$, according to

$$\hat{\mathbf{W}}_{i,k}^o \equiv S_i^o \, \hat{\mathbf{U}}_{i,k} = M_i^T \, \hat{\mathbf{W}}_{i,k} \quad , \tag{24}$$

so that

$$\hat{\mathbf{W}}_{i,k}^o = M_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{W}}_{i,k}^o \quad . \tag{25}$$

Equation (25) defines $\Delta\hat{\mathbf{W}}^o_{i,k}$. Thus, neglecting the random errors in the reference vectors,

$$\Delta\hat{\mathbf{W}}^o_{i,k} \sim \mathcal{N}(\mathbf{0}, R_{\hat{\mathbf{W}}^o_{i,k}}) \quad , \tag{26}$$

with

$$R_{\hat{\mathbf{W}}^o_{i,k}} = S^o_i \; R_{\hat{\mathbf{U}}_{i,k}} \; S^{oT}_i \quad . \tag{27}$$

If we now expand M_i to first order in $\boldsymbol{\theta}_i$, we obtain

$$\begin{aligned} \hat{\mathbf{W}}^o_{i,k} &\approx (I - [[\,\boldsymbol{\theta}_i\,]])\,\hat{\mathbf{W}}^{\text{true}}_{i,k} + \Delta\hat{\mathbf{W}}^o_{i,k} \quad , \\ &= \hat{\mathbf{W}}^{\text{true}}_{i,k} + [[\,\hat{\mathbf{W}}^{\text{true}}_{i,k}\,]]\,\boldsymbol{\theta}_i + \Delta\hat{\mathbf{W}}^o_{i,k} \quad . \end{aligned} \tag{28}$$

The uncalibrated body-referenced observation, $\hat{\mathbf{W}}^o_{i,k}$, as a function of the misalignments depends even to lowest order also on the attitude through $\hat{\mathbf{W}}^{\text{true}}_{i,k}$. We would prefer not to solve for $3N$ attitude parameters, which for $N = 1000$, a not unreasonable amount of data to process, would be in a batch framework quite overwhelming computationally. (Equation (23) supplemented by the prelaunch calibration estimates could be the basis for a Kalman filter/smoother solution of the problem [8], but this approach lies outside the context of the present work. Also, if the spacecraft is not equipped with three-axis gyros, which very often is the case, a Kalman filter/smoother solution may not even be feasible.) Thus, we look for a means of removing the attitude dependence of the measurements. To accomplish this we note that to first order in $\boldsymbol{\theta}_i$, $\boldsymbol{\theta}_j$, and the measurement noise terms

$$\begin{aligned} \hat{\mathbf{W}}^o_{i,k} \cdot \hat{\mathbf{W}}^o_{j,k} = \hat{\mathbf{V}}_{i,k} \cdot \hat{\mathbf{V}}_{j,k} &+ (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k}) \cdot (\boldsymbol{\theta}_i - \boldsymbol{\theta}_j) \\ &+ \hat{\mathbf{W}}^{\text{true}}_{i,k} \cdot \Delta\hat{\mathbf{W}}^o_{j,k} + \hat{\mathbf{W}}^{\text{true}}_{j,k} \cdot \Delta\hat{\mathbf{W}}^o_{i,k} \quad , \end{aligned} \tag{29}$$

which is independent of the attitude. Thus, we define for $i \neq j$

$$z_{ij,k} \equiv \hat{\mathbf{W}}^o_{i,k} \cdot \hat{\mathbf{W}}^o_{j,k} - \hat{\mathbf{V}}_{i,k} \cdot \hat{\mathbf{V}}_{j,k} \quad , \tag{30}$$

whence,

$$z_{ij,k} = (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k}) \cdot (\boldsymbol{\theta}_i - \boldsymbol{\theta}_j) + \Delta z_{ij,k} \quad , \tag{31}$$

with

$$\Delta z_{ij,k} \approx \hat{\mathbf{W}}^o_{i,k} \cdot \Delta\hat{\mathbf{W}}^o_{j,k} + \hat{\mathbf{W}}^o_{j,k} \cdot \Delta\hat{\mathbf{W}}^o_{i,k} \quad . \tag{32}$$

In equation (32) we have replaced $\hat{\mathbf{W}}^{\text{true}}_{i,k}$ by $\hat{\mathbf{W}}^o_{i,k}$ since this leads to no errors to lowest order in the covariance. The derived measurements, which are the observed cosine errors, are independent of the attitude to first order in the misalignments. The derived measurement errors satisfy to lowest order in the misalignments and the measurement noise

$$E\{\Delta z_{ij,k}\} = 0 \quad , \tag{33a}$$

$$E\{\Delta z^2_{ij,k}\} = E_k(i \mid j \mid i) + E_k(j \mid i \mid j) \quad , \tag{33b}$$

$$E\{\Delta z_{ij,k}\, \Delta z_{i\ell,k}\} = E_k(j \mid i \mid \ell) \quad , \tag{33c}$$

$$E\{\Delta z_{ij,k} \Delta z_{\ell m,k}\} = 0 \quad , \tag{33d}$$

where i, j, ℓ, and m above denote distinct indices and $E_k(i \mid j \mid \ell)$ is given by

$$E_k(i \mid j \mid \ell) \equiv \hat{\mathbf{W}}^{oT}_{i,k} \, R_{\hat{\mathbf{W}}^o_{j,k}} \, \hat{\mathbf{W}}^o_{\ell,k} \quad . \tag{34}$$

Note that $z_{ij,k}$ is symmetric in the indices i and j.

The Redundancy Problem

The measurements $z_{ij,k}$, $i < j$, cannot all be independent. If there are n_k active sensors in frame k, each measuring a unit vector, then there are only $2n_k$ equivalent independent scalar measurements, while there are $n_k(n_k - 1)/2$ possible $z_{ij,k}$ with $i < j$. Since three combinations of the $\hat{\mathbf{W}}^o_{i,k}$ are necessary to determine the attitude, and the $z_{ij,k}$ are by explicit construction attitude-independent, there can only be $2n_k - 3$ statistically independent $z_{ij,k}$. Table 1 displays these numbers. We see that for more than three sensors the derived measurements, $z_{ij,k}$, become redundant and this redundancy grows disproportionately with the number of sensors. (If some of the sensors are capable of measuring more than one vector simultaneously, the redundancy will be increased further. It should be noted, however, that the angles between vectors measured by the same sensor have no alignment information.)

To determine $2n_k - 3$ independent measurements from among the $n_k(n_k - 1)/2$ possible $z_{ij,k}$ we remark that

$$2n_k - 3 = (n_k - 1) + (n_k - 2) \quad . \tag{35}$$

Thus, it is tempting to suggest that the set of measurements

$$\{z_{1j,k}\, , j = 2, \ldots, n\, ; \; z_{2j,k}\, , j = 3, \ldots, n_k\}$$

is the desired set provided that $\hat{\mathbf{W}}^o_{1,k}$ and $\hat{\mathbf{W}}^o_{2,k}$ are neither collinear nor are they coplanar nor nearly coplanar with any of the remaining measurements. That these $2n - 3$ measurements are indeed not statistically dependent (in the sense that no linear combination of them with nonvanishing coefficients can have vanishing variance) is easily seen by arranging the noise terms as

$$\Delta z_{12,k} = \hat{\mathbf{W}}^o_{1,k} \cdot \Delta\hat{\mathbf{W}}^o_{2,k} + \hat{\mathbf{W}}^o_{2,k} \cdot \Delta\hat{\mathbf{W}}^o_{1,k} \quad , \tag{36a}$$

$$\Delta z_{1j,k} = \hat{\mathbf{W}}^o_{1,k} \cdot \Delta\hat{\mathbf{W}}^o_{j,k} + \hat{\mathbf{W}}^o_{j,k} \cdot \Delta\hat{\mathbf{W}}^o_{1,k} \quad , \quad j = 3, \ldots, n_k \tag{36b}$$

$$\Delta z_{2j,k} = \hat{\mathbf{W}}^o_{2,k} \cdot \Delta\hat{\mathbf{W}}^o_{j,k} + \hat{\mathbf{W}}^o_{j,k} \cdot \Delta\hat{\mathbf{W}}^o_{2,k} \quad , \quad j = 3, \ldots, n_k \tag{36c}$$

Table 1. Number of Independent Attitude-Independent Measurements Compared with Number of Derived Measurements

n	2n-3	n(n-1)/2
2	1	1
3	3	3
4	5	6
5	7	10
6	9	15

The expressions in the second and third lines are clearly not statistically dependent, since they contain distinct components of $\Delta\hat{\mathbf{W}}^o_{j,k}$, $j = 3, \ldots, n_k$, and these are independent (in the same sense) of the expression in the first line since the components of $\Delta\hat{\mathbf{W}}^o_{1,k}$ and $\Delta\hat{\mathbf{W}}^o_{2,k}$ which appear in equation (36a) are linearly independent of those which appear in equations (36b) and (36c) according to our assumption that $\hat{\mathbf{W}}^o_{1,k}$ and $\hat{\mathbf{W}}^o_{2,k}$ are neither mutually parallel nor parallel to any of the other observations. Thus, the only linear combination of these $(2n_k - 3)$ $z_{ij,k}$ which has zero variance must have vanishing coefficients.

The condition that $\hat{\mathbf{W}}^o_{1,k}$, $\hat{\mathbf{W}}^o_{2,k}$ and $\hat{\mathbf{W}}^o_{j,k}$ for $j > 2$ not be coplanar is necessary because $\Delta\hat{\mathbf{W}}^o_{i,k}$ has non-vanishing components only in the plane normal to $\hat{\mathbf{W}}^o_{i,k}$. Thus, if all the measurements are roughly coplanar, the present construction will yield pseudo-measurements which are not sensitive to one "active" component of $\hat{\mathbf{W}}^o_{i,k}$ (namely, the component out of the plane) and will not yield an acceptable set of derived measurements. In this case we must examine a more complete set of measurements which, in addition to those given by equation (30), includes as well the differences in the reference and observed scalar triple products

$$z_{ij\ell,k} = \hat{\mathbf{W}}^o_{i,k} \cdot (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k}) - \hat{\mathbf{V}}_{i,k} \cdot (\hat{\mathbf{V}}_{j,k} \times \hat{\mathbf{V}}_{\ell,k}) \quad , \quad i < j < \ell \tag{37}$$

The changes which must be made in our estimation algorithms when using this extended set are described in Appendix A.

(Note that vector magnetometers supply *three* equivalent scalar measurements, not two. The additional attitude-independent measurement may be taken to be $|\boldsymbol{B}_k|$, the magnitude of the measured field, which is independent of the alignments but not of additive magnetometer biases, which are often significant.)

The Inflight Estimator

If we define

$$\mathbf{Z}_k \equiv [z_{12,k}, \ldots, z_{1n_k,k}, z_{23,k}, \ldots, z_{2n_k,k}]^T \quad , \tag{38}$$

we may write

$$\mathbf{Z}_k = H_k \boldsymbol{\Theta} + \Delta\mathbf{Z}_k \quad , \tag{39}$$

where

$$\boldsymbol{\Theta} \equiv [\, \theta_1^T, \theta_2^T, \ldots, \theta_n^T \,]^T \quad , \tag{40}$$

is the total (absolute) misalignment vector. Thus, $\Delta\mathbf{Z}_k$ is a white Gaussian sequence with some covariance matrix $P_{\mathbf{Z}_k}$. The matrices H_k and $P_{\mathbf{Z}_k}$ are obtained directly from equations (31) through (34).

We remark again that from equation (31) the derived measurements $\mathbf{Z}_k$, $k = 1, \ldots, N$, are sensitive only to the relative misalignments, and, therefore, only the $3n - 3$ relative misalignments can be estimated unambiguously from the inflight data. For this reason we examine now the set of relative misalignments defined as

$$\boldsymbol{\psi}_i \equiv \boldsymbol{\theta}_i - \boldsymbol{\theta}_1 \quad , \quad i = 2, \ldots, n \quad . \tag{41}$$

This reduced set of parameters is completely observable. The total relative misalignment vector is defined as

$$\boldsymbol{\Psi} \equiv [\,\boldsymbol{\psi}_2^T,\, \boldsymbol{\psi}_3^T,\, \ldots,\, \boldsymbol{\psi}_n^T\,]^T \quad , \tag{42a}$$

$$= F\,\boldsymbol{\Theta} \quad , \tag{42b}$$

where

$$F = \begin{bmatrix} -I_{3\times3} & I_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} \\ -I_{3\times3} & 0_{3\times3} & I_{3\times3} & \cdots & 0_{3\times3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I_{3\times3} & 0_{3\times3} & 0_{3\times3} & \cdots & I_{3\times3} \end{bmatrix} \quad . \tag{43}$$

We now write the measurement vector defined in equation (38) as

$$\mathbf{Z}_k = h_{1k}\,\boldsymbol{\theta}_1 + H_k'\,[\,\boldsymbol{\theta}_2^T,\, \ldots,\, \boldsymbol{\theta}_n^T\,]^T + \Delta\mathbf{Z}_k \quad , \tag{44}$$

where the measurement sensitivity matrix, H_k, has been partitioned as

$$H_k = [\; h_{1k} \;|\; H_k' \;] \quad . \tag{45}$$

From equation (31) we note that $\mathbf{Z}_k$ is unchanged by the substitution

$$\boldsymbol{\theta}_i \rightarrow \boldsymbol{\theta}_i - \boldsymbol{\lambda} \quad . \tag{46}$$

Setting $\boldsymbol{\lambda} = \boldsymbol{\theta}_1$, it follows that $\mathbf{Z}_k$ can be written as a function of $\boldsymbol{\Psi}$ alone and the measurement model becomes equivalently

$$\mathbf{Z}_k = H_k'\,\boldsymbol{\Psi} + \Delta\mathbf{Z}_k \quad . \tag{47}$$

The application of maximum likelihood estimation techniques [10, 11] is now straightforward. The negative-log-likelihood function for the prior-free maximum likelihood estimate of $\boldsymbol{\Psi}$ (*prior-free* because no prior statistical information about $\boldsymbol{\Psi}$ is assumed) is simply

$$J_\Psi^{\text{prior-free}}(\boldsymbol{\Psi}) = \tfrac{1}{2} \sum_{k=1}^{N} \Big[(\mathbf{Z}_k - H_k'\,\boldsymbol{\Psi})^T\, P_{\mathbf{Z}_k}^{-1}\, (\mathbf{Z}_k - H_k'\,\boldsymbol{\Psi}) + \log\det P_{\mathbf{Z}_k} + (2n_k - 3)\log 2\pi \Big] \tag{48}$$

Minimizing $J_\Psi^{\text{prior-free}}(\boldsymbol{\Psi})$ over $\boldsymbol{\Psi}$ leads to the normal equations

$$P_{\Psi\Psi}^{-1}(\text{prior-free})\,\boldsymbol{\Psi}^*(\text{prior-free}) = \sum_{k=1}^{N} H_k'^T\, P_{\mathbf{Z}_k}^{-1}\,\mathbf{Z}_k \quad , \tag{49}$$

and

$$P_{\Psi\Psi}^{-1}(\text{prior-free}) = \sum_{k=1}^{N} H_k'^T\, P_{\mathbf{Z}_k}^{-1}\, H_k' \quad , \tag{50}$$

with $P_{\Psi\Psi}^{-1}(\text{prior-free})$ the inverse of the prior-free estimate-error covariance matrix. These are the desired equations for the relative misalignment estimates. (In general, unit vectors will be denoted by a caret; estimates (and estimators) by an asterisk.) The different rôles played by prior-free, *a priori* and *a posteriori* estimates will be important in Part II of this work [1].

A Factorized Method

The methodology derived above for treating the inflight data suffers from two important drawbacks. First, the set of active sensors may be different in every frame (labeled by k) and a complicated logic may be required, therefore, to determine $\hat{\mathbf{W}}_{1,k}$ and $\hat{\mathbf{W}}_{2,k}$ in each frame. Also, if, perchance, one of these two vectors is nearly collinear with one of the remaining vectors, the derived measurements may suffer greatly from loss of numerical significance. The present section presents an algorithm [9] which relies on the singular value decomposition and is numerically superior to the one above.

We may write the covariance matrix of the uncalibrated measurements in the form

$$R_{\hat{\mathbf{W}}^o_{i,k}} = G_{\hat{\mathbf{W}}^o_{i,k}} G^T_{\hat{\mathbf{W}}^o_{i,k}} \quad , \tag{51}$$

where $G_{\hat{\mathbf{W}}^o_{i,k}}$ is a "square-root" of the covariance matrix $R_{\hat{\mathbf{W}}^o_{i,k}}$. This factor may be symmetric and positive semi-definite, or triangular, or have any structure, provided that it satisfy equation (51).

It follows that the uncalibrated measurement noise may be rewritten

$$\Delta\hat{\mathbf{W}}^o_{i,k} = G_{\hat{\mathbf{W}}^o_{i,k}} \boldsymbol{\varepsilon}_{i,k} \quad , \tag{52}$$

where

$$E\{\boldsymbol{\varepsilon}_{i,k}\} = \mathbf{0} \quad , \tag{53}$$

$$E\{\boldsymbol{\varepsilon}_{i,k}\, \boldsymbol{\varepsilon}^T_{j,k}\} = \delta_{ij}\, I_{3\times 3} \quad . \tag{54}$$

(Note that $R_{\hat{\mathbf{W}}^o_{i,k}}$ is only rank 2. Hence, $\boldsymbol{\varepsilon}_{i,k}$ should perhaps be modeled as a two-dimensional random vector with mean zero and covariance $I_{2\times 2}$. The larger dimension also works because $G_{\hat{\mathbf{W}}^o_{i,k}}$ discards the extra component, i.e., the unphysical component is a right null vector of $G_{\hat{\mathbf{W}}^o_{i,k}}$.) Therefore, we may write

$$z_{ij,k} = (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k}) \cdot (\boldsymbol{\theta}_i - \boldsymbol{\theta}_j) + B^i_{ij,k}\, \boldsymbol{\varepsilon}_{i,k} + B^j_{ij,k}\, \boldsymbol{\varepsilon}_{j,k} \quad , \tag{55}$$

with

$$B^i_{ij,k} = (\hat{\mathbf{W}}^o_{j,k})^T\, G_{\hat{\mathbf{W}}^o_{i,k}} \quad , \tag{56a}$$

$$B^j_{ij,k} = (\hat{\mathbf{W}}^o_{i,k})^T\, G_{\hat{\mathbf{W}}^o_{j,k}} \quad . \tag{56b}$$

Thus, we may construct a measurement vector, $\mathbf{Z}_k$, of dimension m_k, where $m_k = n_k(n_k - 1)/2$, which is the concatenation of *all* the cosine errors, as given by equation (30), and for which

$$\mathbf{Z}_k = H_k\, \boldsymbol{\Theta} + B_k\, \boldsymbol{\varepsilon}_k \quad , \tag{57}$$

where

$$\boldsymbol{\varepsilon}_k \equiv [\,\boldsymbol{\varepsilon}^T_{1,k}, \;\ldots\;, \boldsymbol{\varepsilon}^T_{n_k,k}\,]^T \sim \mathcal{N}(\mathbf{0},\, I_{3n_k\times 3n_k}) \quad , \tag{58}$$

and B_k is an $m_k \times 3n$-dimensional matrix whose nonvanishing submatrices are the $B^i_{ij,k}$ and $B^j_{ij,k}$ of equations (55) and (56). H_k now has dimension $m_k \times 3n$. Therefore,

$$P_{\mathbf{Z}_k} = B_k\, B^T_k \quad . \tag{59}$$

By the singular-value-decomposition (SVD) theorem [12], B_k may be factored as

$$B_k = U_k \, \mathcal{S}_k \, V_k^T \quad , \tag{60}$$

where U_k and V_k are orthogonal and $\mathcal{S}_k$ is a diagonal $m_k \times 3n_k$ matrix, and

$$(\mathcal{S}_k)_{11} \geq (\mathcal{S}_k)_{22} \geq \ldots \geq 0 \quad , \tag{61}$$

so that

$$P_{\mathbf{Z}_k} = U_k \, D_k \, U_k^T \quad , \tag{62}$$

with

$$D_k = \mathcal{S}_k \, \mathcal{S}_k^T \quad , \tag{63}$$

which is a diagonal and positive semi-definite $m_k \times m_k$ matrix.

If we now define

$$\boldsymbol{\zeta}_k \equiv U_k^T \, \mathbf{Z}_k \quad , \tag{64}$$

$$C_k \equiv U_k^T \, H_k \quad , \tag{65}$$

then

$$\boldsymbol{\zeta}_k = C_k \, \boldsymbol{\Theta} + \mathcal{S}_k \, \boldsymbol{\varepsilon}'_k \quad , \tag{66}$$

where

$$\boldsymbol{\varepsilon}'_k \equiv V_k^T \, \boldsymbol{\varepsilon}_k \sim \mathcal{N}(\mathbf{0}_{3n_k}, I_{3n_k \times 3n_k}) \quad . \tag{67}$$

If $\ell_{\max,k}$ is the largest index for which $(\mathcal{S}_k)_{\ell\ell} > 0$, then the first $\ell_{\max,k}$ components of $\boldsymbol{\zeta}_k$ contribute $\ell_{\max,k}$ independent measurements of $\boldsymbol{\Theta}$. The remaining components correspond simply to the constraints on $\mathbf{Z}_k$ and are eliminated by truncation. Thus, we write

$$\tilde{\boldsymbol{\zeta}}_k = \tilde{C}_k \, \boldsymbol{\Theta} + \tilde{\mathcal{S}}_k \, \boldsymbol{\varepsilon}'_k \quad , \tag{68}$$

where the tilde denotes the truncation (by row). The covariance matrix of $\tilde{\boldsymbol{\zeta}}_k$, $\tilde{D}_k$, is now non-singular. Note that for $n_k > 3$, B_k must have at least one vanishing singular value. (Care must be exercised in determining $\ell_{\max,k}$ because the vanishing of the singular value is always obscured by numerical error.)

In order to obtain the prior-free maximum likelihood estimate of the relative misalignments, the first three columns of $\tilde{C}_k$ can be eliminated in the same manner as H_k was truncated. Thus, we write

$$\tilde{\boldsymbol{\zeta}}_k = \tilde{C}'_k \, \boldsymbol{\Psi} + \tilde{\mathcal{S}}_k \, \boldsymbol{\varepsilon}'_k \quad , \tag{69}$$

from which it follows in the same way that

$$P_{\Psi\Psi}^{-1}(\text{prior-free}) \, \boldsymbol{\Psi}^*(\text{prior-free}) = \sum_{k=1}^{N} \tilde{C}_k'^T \, \tilde{D}_k^{-1} \, \tilde{\boldsymbol{\zeta}}_k \quad , \tag{70}$$

$$P_{\Psi\Psi}^{-1}(\text{prior-free}) = \sum_{k=1}^{N} \tilde{C}_k'^T \, \tilde{D}_k^{-1} \, \tilde{C}'_k \quad , \tag{71}$$

in analogy to equations (47), (49) and (50).

In general, the SVD operation will yield $2n_k - 3$ positive singular values and the remaining singular values will be zero. The $2n_k - 3$ columns of U_k which do not correspond to vanishing singular values provide the coefficients of those linear combinations of the $z_{ij,k}$ which are sensitive to the misalignments. The remaining column vectors of U_k provide the coefficients of the linear combinations of the $z_{ij,k}$ which are insensitive to the misalignments. These are the coefficients of the constraints on the $z_{ij,k}$. Since the covariance matrix of these new derived measurements is $\tilde{D}_k$, these new derived measurements are statistically independent as well. Thus, the SVD provides a maximal set of derived measurements which have the maximum sensitivity to the misalignments. When $n_k = 3$, the $z_{ij,k}$ are a maximal non-redundant set of derived measurements, and there is no advantage to using the factorized algorithm unless $P_{\mathbf{Z}_k}$ is ill-conditioned due to poor geometry.

Note that equation (69) is in a form which is suitable for square-root filtering [13], should the relative alignments turn out to be poorly observable and require a more cautious numerical treatment.

Relative Attitudes as Relative Alignments

Frequently a single sensor is able to calculate its attitude at a given instant independently of other sensors. This is particularly the case for imaging devices which can sense the directions of stars and are often part of the spacecraft scientific payload. The CCD star camera, sometimes a component of the spacecraft attitude determination system, is also such a device. For such sensors it is possible to compute relative alignments directly from the relative attitudes.

To develop an estimator for this case we define the uncalibrated single-sensor-based attitude for sensor i at time t_k as $A^o_{i,k}$, where

$$\hat{\mathbf{W}}^{o\,\text{true}}_{i,k} = A^o_{i,k}\, \hat{\mathbf{V}}^{\text{true}}_{i,k} \quad , \tag{72}$$

with $\hat{\mathbf{W}}^o_{i,k}$ is given by equation (24). Clearly,

$$A^o_{i,k} = M_i^T A_k \quad . \tag{73}$$

If $A^{o*}_{i,k}$ is the estimate of $A^o_{i,k}$ based on data from sensor i alone at time t_k (not corrected for the unknown misalignment), then

$$A^{o*}_{i,k} = (\delta A^o_{i,k})\, A^o_{i,k} \quad , \tag{74}$$

and $\delta A^o_{i,k}$ is with large probability a small rotation, so that

$$\delta A^o_{i,k} = I + [[\, \boldsymbol{\xi}_{i,k} \,]] + O(|\boldsymbol{\xi}_{i,k}|^2) \quad , \tag{75}$$

and

$$|\boldsymbol{\xi}_{i,k}| \ll 1 \tag{76}$$

with very large probability, and

$$E\{\, \boldsymbol{\xi}_{i,k} \,\} = \mathbf{0} \quad , \tag{77a}$$

$$E\{\, \boldsymbol{\xi}_{i,k}\, \boldsymbol{\xi}^T_{j,k'} \,\} = \delta_{ij}\, \delta_{kk'}\, P_{i,k} \quad . \tag{77b}$$

Thus,

$$\mathcal{Z}_{ij,k} \equiv A^{o*}_{i,k}\, A^{o*T}_{j,k} = (\delta A^o_{i,k})\, M_i^T M_j\, (\delta A^o_{j,k})^T \tag{78}$$

is also with large probability a small rotation and we may write

$$\mathcal{Z}_{ij,k} = I + [[\, \mathbf{z}^{(2)}_{i,k}\,]] + O(|\mathbf{z}^{(2)}_{i,k}|^2) \quad , \tag{79}$$

and $|\mathbf{z}^{(2)}_{i,k}| \ll 1$ with very large probability. Thus, from the definition of the quantities appearing in equation (78) it follows to lowest nonvanishing order that

$$\mathbf{z}^{(2)}_{ij,k} = \boldsymbol{\theta}_j - \boldsymbol{\theta}_i + \boldsymbol{\xi}_{i,k} - \boldsymbol{\xi}_{j,k} \quad , \tag{80a}$$

$$= \boldsymbol{\theta}_j - \boldsymbol{\theta}_i + \Delta\mathbf{z}^{(2)}_{ij,k} \quad . \tag{80b}$$

The superscript on $\mathbf{z}^{(2)}_{ij,k}$ serves to distinguish this derived measurement from the one defined earlier. From equation (80) we can construct

$$\mathbf{Z}^{(2)}_k = [\, \mathbf{z}^{(2)T}_{12,k}, \; \ldots \; , \mathbf{z}^{(2)T}_{1n_k,k}\,]^T \quad , \tag{81}$$

for which

$$\mathbf{Z}^{(2)}_k = H^{(2)}_k\, \boldsymbol{\Psi} + \Delta\mathbf{Z}^{(2)}_k \quad , \tag{82}$$

with

$$\Delta\mathbf{Z}^{(2)}_k \sim \mathcal{N}(\mathbf{0}, P^{(2)}_k) \quad . \tag{83}$$

The quantities $H^{(2)}_k$ and $P^{(2)}_k$ are easily constructed from the quantities appearing in equations (77) and (80). Note that $P^{(2)}_k$ will not be block diagonal owing to the correlations introduced by $\boldsymbol{\xi}_{1,k}$. In the special case that all the sensors are able to determine their attitude instantaneously and independently of all the other sensors, and all sensors are present in a given frame, then $H^{(2)}_k$ is simply the $3n_k \times 3n_k$ identity matrix.

In general, one will have some sensors which measure a complete attitude (that is, they measure more than one direction) instantaneously and others which measure only a single direction. We require, therefore, a derived measurement which uses data both from a sensor which measures a single vector and from a sensor which measures a complete attitude. For this case we define

$$\hat{\mathbf{W}}^{o*}_{ij,k} \equiv A^{o*}_{i,k}\, \hat{\mathbf{V}}_{j,k} \quad , \tag{84}$$

that is, the estimated value of $\hat{\mathbf{W}}^o_{j,k}$ based on the uncalibrated spacecraft attitude determined by the complete attitude sensor i at time t_k and ignoring the unknown misalignments. It is straightforward then to show that to lowest nonvanishing order the desired effective measurement of the alignments is

$$\mathbf{z}^{(3)}_{ij,k} \equiv \mathcal{P}(\hat{\mathbf{W}}^o_{j,k})\hat{\mathbf{W}}^{o*}_{ij,k} \times \hat{\mathbf{W}}^o_{j,k} \tag{85a}$$

$$= \mathcal{P}(\hat{\mathbf{W}}^o_{j,k})(\boldsymbol{\theta}_j - \boldsymbol{\theta}_i) + \Delta\mathbf{z}^{(3)}_{ij,k} \tag{85b}$$

with

$$\Delta\mathbf{z}^{(3)}_{ij,k} \approx -\mathcal{P}(\hat{\mathbf{W}}^o_{j,k}) \left[\boldsymbol{\xi}_{i,k} + [[\, \hat{\mathbf{W}}^o_{j,k}\,]]\, \Delta\hat{\mathbf{W}}^o_{j,k} \right] \quad . \tag{86}$$

Here $\mathcal{P}(\hat{\mathbf{W}}^o_{j,k})$ is a 2×3 matrix which projects three-dimensional vectors onto the two-dimensional space normal to $\hat{\mathbf{W}}^o_{j,k}$. Thus, if $\{\hat{\mathbf{W}}^o_{j,k}, \hat{\mathbf{a}}_k, \hat{\mathbf{b}}_k\}$ is a right-handed orthonormal triad, then we may choose

$$\mathcal{P}(\hat{\mathbf{W}}^o_{j,k}) = [\,\hat{\mathbf{a}}_k \mid \hat{\mathbf{b}}_k\,]^T \quad , \quad \mathcal{P}(\hat{\mathbf{W}}^o_{j,k})[[\,\hat{\mathbf{W}}^o_{j,k}\,]] = [\,\hat{\mathbf{b}}_k \mid -\hat{\mathbf{a}}_k\,]^T \quad . \tag{87}$$

This projection is necessary since by explicit construction the covariance of $\hat{\mathbf{W}}^{o*}_{ij,k} \times \hat{\mathbf{W}}^o_{j,k}$ can only be rank 2.

A complete measurement vector, $\mathbf{Z}_k$, can now be constructed from the $\mathbf{z}_{ij,k}$, the $\mathbf{z}^{(2)}_{ij,k}$, and the $\mathbf{z}^{(3)}_{ij,k}$. The corresponding H_k and P_k are then readily constructed from the quantities appearing in equations (31)–(34), (77), (80), and (84)–(87). Usually, only two of the three derived measurement types will appear. (We choose not to consider the case in which the derived triplet measurements of equation (37) must also be taken into account.) The incorporation of the $\mathbf{z}^{(2)}_{ij,k}$ and the $\mathbf{z}^{(3)}_{ij,k}$ in the factorized algorithm is somewhat more complicated but straightforward.

Prelaunch Alignment Calibration

As an example of the use of the attitude matrix as a relative alignment, we may consider the method by which the alignments are determined on Earth prior to launch. This is accomplished by measuring the attitude of each sensor relative to a set of axes fixed in the laboratory. The attitude of each sensor is defined by a reference cube glued to that sensor. This cube is optically flat and its sides are perpendicular with great precision, typically 1. arc second. The attitude of each cube is determined by measuring the directions of the normals to two faces of each cube. These same cubes are then used as attitude references when the sensor itself is calibrated. These methods, as performed at the Optical Alignment Facility of NASA Goddard Space Flight Center, have been documented previously [14, 15].

Given the set of measured normals, which may not be exactly perpendicular, a set of exactly perpendicular directions is computed. Thus, for example, if $\hat{\mathbf{n}}_{i,1}$ and $\hat{\mathbf{n}}_{i,2}$, corresponding to the normals of the $+x$ and $-z$ faces of the ith cube, are measured, an exactly orthonormal set of "cube" axes can be constructed as

$$\hat{\mathbf{e}}_{i,1} \equiv \hat{\mathbf{n}}_{i,1} \quad , \tag{88a}$$

$$\hat{\mathbf{e}}_{i,2} \equiv \hat{\mathbf{n}}_{i,1} \times \hat{\mathbf{n}}_{i,2}/|\hat{\mathbf{n}}_{i,1} \times \hat{\mathbf{n}}_{i,2}| \quad , \tag{88b}$$

$$\hat{\mathbf{e}}_{i,3} \equiv \hat{\mathbf{e}}_{i,1} \times \hat{\mathbf{e}}_{i,2} \quad , \tag{88c}$$

and similarly for all other cubes, including the primary reference cube. The measured faces have been chosen so that the auxiliary axes will be close to the normals to the $+x$, $+y$, and $+z$ faces of the cubes, which simplifies the treatment which follows. A different choice of the measured surfaces would have required only that we relabel the indices of the auxiliary vectors in order to bring them into the received order. It should be noted that the choice of measured faces given above is a frequent impossibility because the $-z$ face is typically the glue joint. However, for the purpose of illustration this choice gives us a convenient order for the hierarchy of cross products (it makes them look like those of the TRIAD algorithm of [5, 16]) and makes $\hat{\mathbf{e}}_{i,1}$, $\hat{\mathbf{e}}_{i,2}$, and $\hat{\mathbf{e}}_{i,3}$ correspond to the orthogonalized x, y, and z axes of the cube.

From the auxiliary vectors so defined, the estimated elements of the prelaunch alignment matrices can be computed as

$$(S_i^{o*})_{\ell\ell'} = \hat{\mathbf{e}}_{o,\ell}^T \, \hat{\mathbf{e}}_{i,\ell'} \quad , \tag{89}$$

where the subscript o denotes the principal reference cube. Equation (89) is equivalent to equation (78) above. Note that for the prelaunch calibration the misalignments vanish by definition. This is the procedure by which the prelaunch alignments have been determined for the Solar Maximum Mission (SMM), Magsat, the International Ultraviolet Explorer (IUE), and numerous other spacecraft. Equation (89) is simply the TRIAD algorithm for attitude determination applied to alignment estimation.

To estimate the covariance of the prelaunch alignments, we write

$$\hat{\mathbf{n}}_{i,\ell} = \hat{\mathbf{n}}_{i,\ell}^{\text{true}} + \Delta\hat{\mathbf{n}}_{i,\ell} \quad , \tag{90}$$

where

$$\Delta\hat{\mathbf{n}}_{i,\ell} \sim \mathcal{N}(\mathbf{0}, R_{nn}(i, \ell)) \quad . \tag{91}$$

It is then straightforward to compute the covariance of $(S_i^{o*})_{\ell\ell'}$ from the results for the TRIAD algorithm [5, 16]. Since the prelaunch estimate of the alignment matrix is S_i^o (we discard the asterisk on this quantity henceforth), it follows that the prelaunch estimate of the misalignment matrix, M_i, is the identity matrix, and the prelaunch estimate of the misalignment, $\boldsymbol{\theta}_i$, is $\mathbf{0}$ by definition. Thus, we write in general,

$$\boldsymbol{\theta}_i^*(\text{prelaunch}) = \mathbf{0} = \boldsymbol{\theta}_i^{\text{prelaunch}} + \Delta\,\boldsymbol{\theta}_i^*(\text{prelaunch}) \quad , \tag{92}$$

where the $\Delta\,\boldsymbol{\theta}_i^*(\text{prelaunch})$, $i = 1, \ldots, n$, are the errors in the prelaunch estimates of the misalignment vectors, which are then also zero-mean and Gaussian. We write, therefore,

$$E\{\Delta\,\boldsymbol{\theta}_i^*(\text{prelaunch})\} = \mathbf{0} \quad , \tag{93a}$$

$$E\{\Delta\,\boldsymbol{\theta}_i^*(\text{prelaunch})\,\Delta\,\boldsymbol{\theta}_j^{*T}(\text{prelaunch})\} = P_{ij}(\text{prelaunch}) \quad , \tag{93b}$$

(Note that we have designated the physical random variable by a literal superscript while quantities associated with the estimator are indicated by a literal argument in parentheses.)

Since the orientation of the primary reference cube appears in the definition of every alignment, the alignment estimates are correlated. This is reflected in equation (80), which is true generally. Thus, the general structure of the alignment estimate covariance matrices is

$$P_{ij}(\text{prelaunch}) = \delta_{ij}\, P_i^{\text{lab}}(\text{prelaunch}) + P_o^{\text{lab}}(\text{prelaunch}) \quad , \tag{94}$$

where $P_i^{\text{lab}}(\text{prelaunch})$, $i = 0, \ldots, n$, denotes the covariance of the alignment of each cube relative to the laboratory. In particular, for the algorithm above and assuming that the measurement covariances are described well by the QUEST model [5],

$$R_{nn}(i,\ell) = \sigma_p^2 \left(I_{3\times3} - \hat{\mathbf{n}}_{i,\ell}^{\text{true}}\, \hat{\mathbf{n}}_{i,\ell}^{\text{true}\,T} \right) \quad , \tag{95}$$

we obtain the simple expression

$$P_{ij}(\text{prelaunch}) = \sigma_p^2 \left(1 + \delta_{ij} \right) I_{3\times3} \quad , \tag{96}$$

exhibiting explicitly the correlation in the prelaunch alignment estimates. From the observed repeatability of the measurements at NASA/GSFC we are led to assign to σ_p a value (in radian equivalent) consistent with

$$2\sigma_p^2 \approx (5.\ \text{arc sec})^2 \quad , \tag{97}$$

which holds approximately for all sensors. Equation (96) probably overestimates the correlation between alignments slightly since typically there will be fewer sources of measurement error for the normals of the primary reference cube. At the same time, sources of correlation between the measured normal vectors have been neglected. The expression and the degree of correlation are, thus, very reasonable and considering the uncertainties which are introduced by launch shock, and the degree of subjectiveness which exists in the prelaunch measurements, a more detailed model is probably not justified.

A few words should be said on the use of the TRIAD algorithm for the estimation of the prelaunch alignment matrix. The construction of an exactly orthonormal set as given by equations (88) is necessary simply to define the cube coordinate systems unambiguously in the absence of exactly orthogonal faces. If these orthonormalized axes are used deterministically as the estimated coordinate axes of the cube, the result is the TRIAD estimate of the attitude of the cube relative to the laboratory. Alternately, one may consider equations (88a) and (88b) as defining effective measurements of these axes. Thus

$$\hat{\mathbf{e}}_{i,\ell} = A^T(i)\, \hat{\mathbf{u}}_{i,\ell} + \Delta\hat{\mathbf{e}}_{i,\ell} \quad , \quad i = 0, \ldots, n \quad , \quad \ell = 1, 2 \quad , \tag{98}$$

where $\hat{\mathbf{u}}_{i,\ell}$ denotes the representation of the normal to face ℓ of cube i with respect to the coordinate axes of cube i. In the present example

$$\hat{\mathbf{u}}_{i,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \hat{\mathbf{u}}_{i,2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad , \tag{99}$$

The $\hat{\mathbf{e}}_{i,\ell}$, it will be recalled, are the representations of the corresponding cube normals with respect to laboratory axes. The cube attitude, $A(i)$, $i = 0, \ldots, n$ is defined as the transformation from the laboratory to the cube coordinate systems; hence, the transpose appearing in equation (98). The effective measurement noise is

$$\Delta\hat{\mathbf{e}}_{i,1} = \Delta\hat{\mathbf{n}}_{i,1} \quad , \tag{100a}$$

$$\Delta\hat{\mathbf{e}}_{i,2} = [[\, \hat{\mathbf{e}}_{i,2} \,]]^2 \left([[\, \hat{\mathbf{n}}_{i,2} \,]]\Delta\hat{\mathbf{n}}_{i,1} - [[\, \hat{\mathbf{n}}_{i,1} \,]]\Delta\hat{\mathbf{n}}_{i,2} \right) \quad , \tag{100b}$$

where we have used the fact that $\hat{\mathbf{n}}_{i,1} \cdot \hat{\mathbf{n}}_{i,2} \approx 0$.

The cube attitude matrices, $A(i)$, $i = 0, \ldots, n$, may now be estimated using the attitude estimation algorithm of choice. The prelaunch alignment matrices are then given by

$$S_i^o = A^*(0)\, A^{*T}(i) \quad . \tag{101}$$

For the special case of the TRIAD algorithm, equation (101) reduces to equation (89). Equation (94) will hold generally provided that there is only one set of normals
$\{\hat{\mathbf{n}}_{i,1}, \hat{\mathbf{n}}_{i,2}\}$ measured for each cube. This is generally the case since these measurements are costly and the errors associated with a single set of measurements are much smaller than those which will be introduced later by launch shock. For this reason there is little motivation to improve the prelaunch alignment calibration procedures as they presently exist.

Transformations between Relative Alignment Representations

The above discussion assumed that it is sensor 1 to which the relative misalignments are referred. This need not be the case. It is simple, in fact, to transform relative misalignment estimates and their covariances between different representations originating in different choices of the reference sensor.

Let us define

$$\boldsymbol{\psi}_{mi} = \boldsymbol{\theta}_i - \boldsymbol{\theta}_m \quad , \quad i = 1, \ldots, n \quad , \tag{102}$$

which are the relative misalignments with respect to sensor m. Likewise we define

$$\boldsymbol{\Psi}_m \equiv [\, \boldsymbol{\psi}_{mi}^T, \ldots, \boldsymbol{\psi}_{mn}^T \,]^T \quad . \tag{103}$$

Thus, $\boldsymbol{\Psi}_m$ is a $3n$-dimensional vector for which three components (corresponding to $\boldsymbol{\psi}_{mm}$) vanish identically. We write also

$$\boldsymbol{\Psi}_m^* \equiv [\, \boldsymbol{\psi}_{mi}^{*T}, \ldots, \boldsymbol{\psi}_{mn}^{*T} \,]^T \quad , \tag{104}$$

with necessarily

$$\boldsymbol{\psi}_{mm}^* = \mathbf{0} \quad . \tag{105}$$

If $\Delta\boldsymbol{\Psi}_m$ is the estimate error of $\boldsymbol{\Psi}_m$, then

$$P_{\Psi\Psi}(m) \equiv E\{\, \Delta\boldsymbol{\Psi}_m \Delta\boldsymbol{\Psi}_m^T \} \tag{106}$$

is a $3n \times 3n$ matrix for which 3 rows and 3 columns are identically zero. We construct $\boldsymbol{\Psi}_m$ and $P_{\Psi\Psi}(m)$ from the $(3n-3)$-dimensional relative misalignment vector estimate and $(3n-3) \times (3n-3)$ estimate-error covariance matrix by inserting zeros in the appropriate places. To transform a set of relative misalignments referred to sensor m, together with the corresponding covariance matrix, to a set referred to sensor ℓ, we note that

$$\boldsymbol{\psi}_{\ell i} = \boldsymbol{\psi}_{mi} - \boldsymbol{\psi}_{m\ell} \quad . \tag{107}$$

Thus,

$$\boldsymbol{\psi}_{\ell i}^* = \boldsymbol{\psi}_{mi}^* - \boldsymbol{\psi}_{m\ell}^* \quad , \tag{108}$$

and

$$[\, P_{\Psi\Psi}(\ell)\,]_{ij} = [\, P_{\Psi\Psi}(m)\,]_{ij} - [\, P_{\Psi\Psi}(m)\,]_{i\ell} - [\, P_{\Psi\Psi}(m)\,]_{\ell j} + [\, P_{\Psi\Psi}(m)\,]_{\ell\ell} \quad . \tag{109}$$

By deleting now the rows and columns associated with $\boldsymbol{\psi}_{\ell\ell}$ the desired $(3n-3)$ relative misalignment vector relative to sensor ℓ and $(3n-3)\times(3n-3)$ covariance matrix are obtained.

A Numerical Example

We illustrate these methods with a numerical example. Consider a typical spacecraft equipped with three vector sensors each with an accuracy of 10. arc sec/axis and an effective (weighted) field of view of ±10. deg/axis. We assume the sensor errors to be well represented by the QUEST measurement model [5], which has been used in the attitude determination algorithm for several spacecraft. We may write this measurement model as

$$E\{\Delta\hat{\mathbf{U}}_{i,k}\,\Delta\hat{\mathbf{U}}_{i,k}^T\} = \sigma_{i,k}^2\left(I - \hat{\mathbf{U}}_{i,k}^{\rm true}\,\hat{\mathbf{U}}_{i,k}^{{\rm true}\,T}\right) \quad . \tag{110}$$

Thus, to lowest order in the misalignments and neglecting errors in the reference vectors

$$E\{\Delta\hat{\mathbf{W}}_{i,k}^o\,\Delta\hat{\mathbf{W}}_{i,k}^{oT}\} = \sigma_{i,k}^2\left(I - \hat{\mathbf{W}}_{i,k}^o\,\hat{\mathbf{W}}_{i,k}^{oT}\right) \quad . \tag{111}$$

Substituting these expressions into equation (34) it follows for the QUEST model that

$$E_k(i \mid j \mid \ell) = \sigma_{j,k}^2\,(\hat{\mathbf{W}}_{i,k}^o \times \hat{\mathbf{W}}_{j,k}^o)\cdot(\hat{\mathbf{W}}_{\ell,k}^o \times \hat{\mathbf{W}}_{j,k}^o) \quad . \tag{112}$$

Note that the approximation of the QUEST measurement model lies in the form of the covariance matrix of the measurement noise, not in the nonrandom part of the measurement.

Thus, the measurement is described by

$$\mathbf{Z}_k = \begin{bmatrix} z_{23,k} \\ z_{31,k} \\ z_{12,k} \end{bmatrix} = H_k\,\boldsymbol{\Theta} + \Delta\mathbf{Z}_k \quad , \tag{113}$$

where the reordering of the components and sign changes will serve to give the measurement vector a cyclic symmetry and simplify later calculations. The sensitivity matrix, H_k, is now given by

$$H_k = \begin{bmatrix} \mathbf{0}^T & (\hat{\mathbf{W}}_{2,k}^o \times \hat{\mathbf{W}}_{3,k}^o)^T & -(\hat{\mathbf{W}}_{2,k}^o \times \hat{\mathbf{W}}_{3,k}^o)^T \\ -(\hat{\mathbf{W}}_{3,k}^o \times \hat{\mathbf{W}}_{1,k}^o)^T & \mathbf{0}^T & (\hat{\mathbf{W}}_{3,k}^o \times \hat{\mathbf{W}}_{1,k}^o)^T \\ (\hat{\mathbf{W}}_{1,k}^o \times \hat{\mathbf{W}}_{2,k}^o)^T & -(\hat{\mathbf{W}}_{1,k}^o \times \hat{\mathbf{W}}_{2,k}^o)^T & \mathbf{0}^T \end{bmatrix} \quad , \tag{114}$$

and

$$P_{\mathbf{Z}_k} = \begin{bmatrix} d_{23,k} & f_{23,31,k} & f_{23,12,k} \\ f_{23,31,k} & d_{31,k} & f_{31,12,k} \\ f_{23,12,k} & f_{31,12,k} & d_{12,k} \end{bmatrix} \quad , \tag{115}$$

where

$$d_{ij,k} = (\sigma_{i,k}^2 + \sigma_{j,k}^2)|\hat{\mathbf{W}}_{i,k}^o \times \hat{\mathbf{W}}_{j,k}^o|^2 \quad , \tag{116a}$$

$$f_{23,31,k} = -\sigma_{3,k}^2\,(\hat{\mathbf{W}}_{2,k}^o \times \hat{\mathbf{W}}_{3,k}^o)\cdot(\hat{\mathbf{W}}_{3,k}^o \times \hat{\mathbf{W}}_{1,k}^o) \quad , \tag{116b}$$

$$f_{23,12,k} = -\sigma_{2,k}^2\,(\hat{\mathbf{W}}_{2,k}^o \times \hat{\mathbf{W}}_{3,k}^o)\cdot(\hat{\mathbf{W}}_{1,k}^o \times \hat{\mathbf{W}}_{2,k}^o) \quad , \tag{116c}$$

$$f_{31,12,k} = -\sigma_{1,k}^2\,(\hat{\mathbf{W}}_{3,k}^o \times \hat{\mathbf{W}}_{1,k}^o)\cdot(\hat{\mathbf{W}}_{1,k}^o \times \hat{\mathbf{W}}_{2,k}^o) \quad . \tag{116d}$$

For the QUEST model [5] we may factor the measurement covariance matrix as

$$R_{\hat{\mathbf{W}}^o_{i,k}} = \left(\sigma_{i,k}\,[[\,\hat{\mathbf{W}}^o_{i,k}\,]]\right)\,\left(\sigma_{i,k}\,[[\,\hat{\mathbf{W}}^o_{i,k}\,]]\right)^T \quad . \tag{117}$$

Thus, the total measurement sensitivity in the factorized algorithm becomes

$$B^i_{ij,k} = (\hat{\mathbf{W}}^o_{j,k})^T\,(\sigma_{i,k}\,[[\,\hat{\mathbf{W}}^o_{i,k}\,]])$$
$$= \sigma_{i,k}\,(\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k})^T \quad , \tag{118a}$$
$$B^j_{ij,k} = \sigma_{j,k}\,(\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{i,k})^T \quad , \tag{118b}$$

from which it follows that

$$B_k = H_k \begin{bmatrix} \sigma_{1,k}\,I_{3\times 3} & O_{3\times 3} & \cdots & O_{3\times 3} \\ O_{3\times 3} & \sigma_{2,k}\,I_{3\times 3} & \cdots & O_{3\times 3} \\ \vdots & \vdots & \ddots & \vdots \\ O_{3\times 3} & O_{3\times 3} & \cdots & \sigma_{n_k,k}\,I_{3\times 3} \end{bmatrix} \quad . \tag{119}$$

Thus, B_k in this special case is obtainable from H_k by simple scalar multiplication of the latter's columns.

The absolute misalignments were sampled from a Gaussian distribution with zero mean and covariance matrix given by equation (96) and (97) on which were superimposed launch-shock errors assumed to have mean zero and standard deviation 1. arc min. for each component of $\mathbf{\Theta}$. This is a reasonable value for the launch-shock error level, which overwhelms, in general, the prelaunch alignment estimation errors. The nominal alignments, expressed in terms of the Gibbs vector [17], were taken to be

$$\mathbf{g}_1 = \mathbf{0} \quad , \quad \mathbf{g}_2 = (2.5,\,0,\,0)^T \quad , \quad \mathbf{g}_3 = (0,\,2.5,\,0)^T \quad , \tag{120}$$

which is a typical set of alignments if sensor 1 is a Sun sensor and sensors 2 and 3 are star trackers.

One hundred samples of simulated data were generated. Attitude independent measurements were generated using both the factorized and unfactorized algorithms and the maximum likelihood estimate of $\mathbf{\Psi}$ and $P_{\Psi\Psi}$(prior-free), its estimate-error covariance matrix, were computed from equations (49) and (50) and equations (70) and (71), respectively.

Table 2 shows the comparison of the model values and the corresponding estimates. For the particular simulated data set, no difference was observed between the results for the factorized and the unfactorized algorithm. Note that two-thirds of the estimates fall approximately within one standard deviation of the model misalignments as expected.

Discussion and Conclusions

We have developed a statistically consistent batch methodology for estimating spacecraft sensor relative misalignments from inflight data. The methodology begins with a general statistical model for the errors in the individual sensors and from these develops a statistical model for a set of derived measurements which are insensitive to the attitude. The derived measurements are just the ob-

Table 2. Comparison of Model and Estimated Relative Misalignments

Model Relative Misalignments	Estimated Relative Misalignments
−73. arc sec	−73. ± 1. arc sec
−40.	−50. ± 8.
63.	78. ± 13.
−14.	−7. ± 7.
−43.	−45. ± 1.
131.	146. ± 12.

served cosines between the measurements. This derived data set will be adequate provided that the data is not coplanar. In the extremely unlikely event that the data is coplanar, one must use instead a larger set of derived measurements consisting of these cosines and the scalar triple products of the measured directions. On the basis of this set of derived measurements we obtain a simple batch estimator of the relative misalignments. A factorized method, which is better behaved numerically is also developed. These algorithms have performed well in tests with realistic simulated data.

Some details of the implementation should be noted:

In the discussion above, sensor 1 was chosen both as the sensor to which the relative misalignments were referred and as the first of the two preferred sensors for computing the derived measurements. This choice was made to simplify the indices. There is no need for sensor 1 to have this double role, and, in fact, the relative misalignments may be referred to sensor "a" while the cosines are computed from the measurements of sensors "b" and "c" with the identity of sensors "a," "b," and "c" perfectly arbitrary except for the noncoplanarity restriction.

Because the calculation of the $z_{ij,k}$ as given by equation (30) requires the subtraction of nearly equal quantities, the numerical significance of the $z_{ij,k}$ is much less than that of the original vector measurements. If the misalignments are as small as an arc second, then nearly six significant figures (eighteen significant bits) are lost in the calculation of the $z_{ij,k}$. Thus, on many computers the calculations must be carried out in double precision.

The linearization of equation (31) discards truly negligible terms which are on the order of $|\Delta\hat{\mathbf{W}}_{i,k}|^2$ and terms on the order of $|\boldsymbol{\theta}_i|^2$, which may be less so. For $|\boldsymbol{\theta}_i|$ on the order of an arc minute, linearization errors will be on the order of .001 arc min, so that the $z_{ij,k}$ will possess only three significant digits in this case. The lack of numerical significance coupled with poor geometry can lead to inaccurate estimates. The errors due to the lack of numerical significance can be minimized by using the factorized techniques. (Note that in a Kalman filter mechanization, these same subtractions will occur in computing the innovation.)

The linearization errors proportional to $|\boldsymbol{\theta}_i|^2$ (i.e., $|\boldsymbol{\psi}_i|^2$) can be eliminated by evaluating the normal equations iteratively. In this case one uses the misalignments just calculated to update the alignment matrices and then repeats the calculation of a new $\boldsymbol{\Psi}^*$(prior-free) and $P_{\Psi\Psi}$(prior-free) for the corrections to these new alignment matrices. In general, this procedure will not be necessary.

The estimate is not optimal over all $2n$ sensor measurements since effectively three of the measurements have been removed in order to achieve attitude independence. Thus, the results of this algorithm will differ somewhat from the results that would obtain, say, from a complete Kalman filter treating both the attitude and the alignments. The difference in the relative alignment estimates that would arise from estimating the attitude simultaneously (say, in a Kalman filter) is not expected to be significant, however, provided that the data stream is not very defective (i. e., that the full complement of sensors appears in almost every frame). In the limiting case that no sensor data is simultaneous with any other, the above methodology cannot be used at all and one must have recourse to a Kalman filter in order to be able to estimate the misalignments.

Since the $z_{ij,k}$ are zero-mean and have easily calculable variances, the detection of outliers among these derived measurements is very direct. In general, these outliers will be due to an improper measurement of a vector $\hat{\mathbf{W}}_{i,k}$ or $\hat{\mathbf{W}}_{j,k}$ (due, for example, to the misidentification of a star). In this case, it is expected that most of the $z_{ij,k}$ for the given i and k or j and k will be outside the bounds. This makes the identification of outliers simple.

Very often, spacecraft carry sensors of widely different accuracies, some supplying attitude information with an accuracy of 10. arc sec while others are accurate to only 0.5 deg. If there are sufficient highly accurate sensors available most of the time, there is little to be gained by estimating the misalignment of the coarse sensors simultaneously with the fine. In this case it is preferable to estimate the misalignments of the fine sensors first using the attitude-independent algorithm presented here and then use simple regression techniques to estimate the misalignments of the coarser sensors using the computed spacecraft attitude from the fine sensors to remove the ambiguity. This approach replaces one large estimation problem by many much smaller ones.

The above algorithm is applicable to those spacecraft for which the payload can be treated as a vector sensor or for which it can be assumed that the payload will not be misaligned from one of the vector sensors during launch. For the Solar Maximum Mission (SMM) the payload was mounted near the fine pointing Sun sensors (FPSSs) on a rigid titanium plate. Thus, it might appear that the SMM satisfied this criteria. However, the misalignment of a sensor depends on more than just thermal distortion effects. In particular, sensors deteriorate in space due both to use and to the interaction with the space environment. Also, the internal distortion of the sensor will not be prevented necessarily by the manner in which it is mounted. Thus, it is not clear that this algorithm is applicable to missions for which the payload does not sense the attitude. For the SMM, however, where two FPSSs were mounted in close proximity, it was observed that the relative misalignment of these two sensors remained nearly zero throughout the mission [18] lending credibility to the notion that the payload remained aligned with the FPSSs at all times. (But see also the remarks at the end of Part II.)

Finally, it should be noted that the estimated relative misalignments cannot themselves be implemented in spacecraft mission support without some additional assumption. The alignment matrix of sensor i *relative* to sensor 1 is

$$S_{i\leftarrow 1} = S_i\, S_1^T = M(\Delta\boldsymbol{\theta}_i)\, S_i^o\, S_1^{oT}\, M^T(\Delta\boldsymbol{\theta}_1) \quad , \tag{121}$$

with $M(\Delta\boldsymbol{\theta})$ given by equation (14). Thus, one cannot combine an estimate of $(\Delta\boldsymbol{\theta}_i - \Delta\boldsymbol{\theta}_1)$ with the prelaunch alignments directly to obtain a corrected relative alignment matrix. The usual solution is to set

$$\Delta\boldsymbol{\theta}_1 = \mathbf{0} \quad , \tag{122}$$

which is equivalent to referring all alignments before and after the launch to the reference frame defined by the measurements made by sensor 1. If it is not known that the misalignment of sensor 1 relative to the payload is negligible, this prescription can lead to difficulties, which are discussed more completely in Part II of this work.

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Appendix A: A More Complete Set of Attitude-Independent Measurements

When the measurements are all nearly coplanar, then the construction

$$z_{ij,k} = \hat{\mathbf{W}}^o_{i,k} \cdot \hat{\mathbf{W}}^o_{j,k} - \hat{\mathbf{V}}_{i,k} \cdot \hat{\mathbf{V}}_{j,k} \quad , \quad 1 \le i < j \le n_k \tag{A1}$$

will only take account of projections of $\hat{\mathbf{W}}^o_{i,k}$ in that single plane, and one of the components of that projection will always be along $\hat{\mathbf{W}}^o_{i,k}$ itself. Thus, the construction given by equation (A1) does not provide a complete set of attitude independent measurements.

This lack of completeness stems from the fact that the construction of equation (A1) is not the only set of fundamental scalars (i.e., not a combination of other scalars) we can construct which is independent of the attitude. There is in addition one other choice, which is the scalar triple product. Thus, we define also

$$z_{ij\ell,k} = \hat{\mathbf{W}}^o_{i,k} \cdot (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k}) - \hat{\mathbf{V}}_{i,k} \cdot (\hat{\mathbf{V}}_{j,k} \times \hat{\mathbf{V}}_{\ell,k}) \quad , \quad 1 \le i < j < \ell \le n_k \tag{A2}$$

for which it can be shown that to first order in the $\boldsymbol{\theta}_i$ and the $\Delta\hat{\mathbf{W}}_{i,k}$

$$\begin{aligned} z_{ij\ell,k} = {} & [\hat{\mathbf{W}}^o_{i,k} \times (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k})] \cdot \boldsymbol{\theta}_i + [\hat{\mathbf{W}}^o_{j,k} \times (\hat{\mathbf{W}}^o_{\ell,k} \times \hat{\mathbf{W}}^o_{i,k})] \cdot \boldsymbol{\theta}_j \\ & + [\hat{\mathbf{W}}^o_{\ell,k} \times (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k})] \cdot \boldsymbol{\theta}_\ell + \Delta z_{ij\ell,k} \quad , \end{aligned} \tag{A3}$$

with

$$\begin{aligned} \Delta z_{ij\ell,k} = {} & (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k}) \cdot \Delta\hat{\mathbf{W}}^o_{i,k} + (\hat{\mathbf{W}}^o_{\ell,k} \times \hat{\mathbf{W}}^o_{i,k}) \cdot \Delta\hat{\mathbf{W}}^o_{j,k} \\ & + (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k}) \cdot \Delta\hat{\mathbf{W}}^o_{\ell,k} \quad . \end{aligned} \tag{A4}$$

Note that $z_{ij\ell,k}$ is symmetric in the indices i, j, and ℓ. Also, the value of $z_{ij\ell,k}$ is unchanged under the transformation

$$\boldsymbol{\theta}_i \to \boldsymbol{\theta}_i + \boldsymbol{\lambda} \quad , \quad i = 1, \ldots, n \quad . \tag{A5}$$

Hence $z_{ij\ell,k}$ is sensitive only to the relative misalignments, as expected.

When $\hat{\mathbf{W}}^o_{i,k}$, $\hat{\mathbf{W}}^o_{j,k}$ and $\hat{\mathbf{W}}^o_{\ell,k}$ are not coplanar, then it is easy to show that

$$z_{ij\ell,k} = \beta_{ij\ell,k}\, z_{ij,k} + \beta_{i\ell j,k}\, z_{i\ell,k} + \beta_{j\ell i,k}\, z_{j\ell,k} \quad , \tag{A6}$$

where

$$\beta_{ij\ell,k} \equiv \frac{(\hat{\mathbf{V}}_{i,k} \times \hat{\mathbf{V}}_{\ell,k}) \cdot (\hat{\mathbf{V}}_{\ell,k} \times \hat{\mathbf{V}}_{j,k})}{\left(\hat{\mathbf{V}}_{i,k} \cdot (\hat{\mathbf{V}}_{j,k} \times \hat{\mathbf{V}}_{\ell,k})\right)} \quad . \tag{A7}$$

As the three vectors approach a coplanar geometry, the coefficients become infinite, so that we cannot express the scalar triple products in terms of scalar products in that case. Thus, in this case, we must include the scalar triple products as a distinct measurement.

The new means and covariances can now be written

$$E\{\Delta z_{ij\ell,k}\} = 0 \quad , \tag{A8a}$$

$$E\{\Delta z^2_{ij\ell,k}\} = E_k(i\,,\, j \mid \ell \mid i\,,\, j) + E_k(j\,,\, \ell \mid i \mid j\,,\, \ell) + E_k(\ell\,,\, i \mid j \mid \ell\,,\, i) \quad , \tag{A8b}$$

$$E\{\Delta z_{ij\ell,k}\, \Delta z_{ijm,k}\} = E_k(j\,,\, \ell \mid i \mid j\,,\, m) + E_k(\ell\,,\, i \mid j \mid m\,,\, i) \quad , \tag{A8c}$$

$$E\{\Delta z_{ij\ell,k} \Delta z_{imp,k}\} = E_k(j\,,\, \ell \mid i \mid m\,,\, p) \quad , \tag{A8d}$$

$$E\{\Delta z_{ij\ell,k} \Delta z_{mpq,k}\} = 0 \quad , \tag{A8e}$$

$$E\{\Delta z_{ij,k} \Delta z_{ij\ell,k}\} = E_k(j \mid i \mid j\,,\, \ell) + E_k(i \mid j \mid \ell\,,\, i) \quad , \tag{A8f}$$

$$E\{\Delta z_{ij,k} \Delta z_{i\ell m,k}\} = E_k(j \mid i \mid \ell\,,\, m) \quad , \tag{A8g}$$

$$E\{\Delta z_{ij,k} \Delta z_{\ell mp,k}\} = 0 \quad , \tag{A8h}$$

where i, j, ℓ, m, p, and q above denote distinct indices and $E_k(i\,,\, j \mid \ell \mid m\,,\, p)$ and $E_k(i \mid j, \mid \ell\,,\, m)$ are given by

$$E_k(i\,,\, j \mid \ell \mid m\,,\, p) \equiv (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k})^T R_{\hat{\mathbf{W}}^o_{\ell,k}} (\hat{\mathbf{W}}^o_{m,k} \times \hat{\mathbf{W}}^o_{p,k}) \quad , \tag{A9}$$

$$E_k(i \mid j \mid \ell\,,\, m) \equiv \hat{\mathbf{W}}^{oT}_{i,k} R_{\hat{\mathbf{W}}^o_{j,k}} (\hat{\mathbf{W}}^o_{\ell,k} \times \hat{\mathbf{W}}^o_{m,k}) \quad . \tag{A10}$$

For the QUEST measurement model, these take the form

$$E_k(i\,,\, j \mid \ell \mid m\,,\, p) = \sigma^2_{\ell,k} [\hat{\mathbf{W}}^o_{\ell,k} \times (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k})] \cdot [\hat{\mathbf{W}}^o_{\ell,k} \times (\hat{\mathbf{W}}^o_{m,k} \times \hat{\mathbf{W}}^o_{p,k})] \quad , \tag{A11}$$

$$E_k(i \mid j \mid \ell\,,\, m) = \sigma^2_{j,k} [\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{i,k}] \cdot [\hat{\mathbf{W}}^o_{j,k} \times (\hat{\mathbf{W}}^o_{\ell,k} \times \hat{\mathbf{W}}^o_{m,k})] \quad . \tag{A12}$$

The redundancy of the derived measurements in now much greater. If we define

$$N_p \equiv N_{\text{pair}} = \frac{n_k(n_k - 1)}{2} \quad , \tag{A13}$$

$$N_{p+t} \equiv N_{\text{pair}} + N_{\text{triplet}} = \frac{n_k(n_k - 1)}{2} + \frac{n_k(n_k - 1)(n_k - 2)}{6} = \frac{n_k(n_k + 1)(n_k - 1)}{6} \quad , \tag{A14}$$

Table A1. Number of Independent Attitude-Independent Measurements Compared with Number of Derived Measurements

n_k	$2n_k-3$	N_p	N_{p+t}
2	1	1	1
3	3	3	4
4	5	6	10
5	7	10	20
6	9	15	35
7	11	21	56
8	13	28	84
9	15	36	120
10	17	45	165

then we see that the redundancy of the derived measurements becomes much greater than for the cosine errors alone as shown in Table A1. If the measurements, while largely coplanar, are not nearly all parallel or antiparallel, then an effective measurement vector of length $2n - 3$ whose covariance matrix is full rank can be constructed from the set

$$\{z_{1j,k}\,, j = 2, \ldots, n\,;\; z_{12j,k}\,, j = 3, \ldots, n_k\} \quad .$$

The factorized algorithm works equally well with these additional measu rements. It is straightforward to show that

$$\begin{aligned} z_{ij\ell,k} = {} & [\hat{\mathbf{W}}^o_{i,k} \times (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k})] \cdot \boldsymbol{\theta}_i + [\hat{\mathbf{W}}^o_{j,k} \times (\hat{\mathbf{W}}^o_{\ell,k} \times \hat{\mathbf{W}}^o_{i,k})] \cdot \boldsymbol{\theta}_j \\ & + [\hat{\mathbf{W}}^o_{\ell,k} \times (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k})] \cdot \boldsymbol{\theta}_\ell \\ & + B^i_{ij\ell,k} \boldsymbol{\varepsilon}_{i,k} + B^j_{ij\ell,k} \boldsymbol{\varepsilon}_{j,k} + B^\ell_{ij\ell,k} \boldsymbol{\varepsilon}_{\ell,k} \quad , \end{aligned} \tag{A15}$$

with now

$$B^i_{ij\ell,k} = (\hat{\mathbf{W}}^o_{j,k} \times \hat{\mathbf{W}}^o_{\ell,k})^T \, G_{\hat{\mathbf{W}}^o_{i,k}} \quad , \tag{A16a}$$

$$B^j_{ij\ell,k} = (\hat{\mathbf{W}}^o_{\ell,k} \times \hat{\mathbf{W}}^o_{i,k})^T \, G_{\hat{\mathbf{W}}^o_{j,k}} \quad , \tag{A16b}$$

$$B^\ell_{ij\ell,k} = (\hat{\mathbf{W}}^o_{i,k} \times \hat{\mathbf{W}}^o_{j,k})^T \, G_{\hat{\mathbf{W}}_{\ell,k}} \quad . \tag{A16c}$$

Thus, we may still write

$$\mathbf{Z}_k = H_k \, \boldsymbol{\Theta} + B_k \, \boldsymbol{\varepsilon}_k \quad , \tag{A17}$$

and

$$P_{\mathbf{Z}_k} = B_k \, B_k^T \quad . \tag{A18}$$

and apply the singular-value decomposition as before. The dimension of $\mathbf{Z}_k$, however, is much greater. In practice, the need for this more extensive set of derived measurements will, hopefully, occur only rarely. If we consider, however, the case of a spacecraft with a very precise Sun sensor and two fixed-head star trackers, the two star trackers may have their boresights in a "V"-configuration with the plane of the "V" containing the boresight of the Sun sensor. Such a configuration is evidently a candidate for using the scalar triple products in addition to the scalar products as derived measurements.

Appendix B: Implementation

We give here the steps to implement both the factorized and the unfactorized algorithms in software. In general, the unfactorized algorithm will be used mostly with spacecraft having only a few sensors. For such spacecraft it is reasonable to process only those frames of data in which all the sensors have usable data. For spacecraft with many sensors, however, the fraction of complete frames may be too small and the number of logical decisions required to select two active sensors in each frame as cosine error references can easily outweigh the complexity of implementing the singular value decomposition, for which many published algorithms exist [12, 19, 20]. For simplicity, we assume that the sensors considered measure only a single vector or that for sensors which measure multiple vectors simultaneously it is the vectors which are used as measurements and not an intermediately computed attitude.

The Unfactorized Algorithm

- The index λ of the sensor to which the relative alignments will be referred is identified.
- The indices μ and ν of the sensors to which the cosines will be referred are identified.
- The uncalibrated sensor measurements, $\hat{\mathbf{W}}^o_{i,k}$, are computed from the sensor measurements in the sensor frame, $\hat{\mathbf{U}}_{i,k}$, using the prelaunch alignment matrices, S^o_i according to equation (24).
- The cosine-error measurements $z_{\mu j,k}$, $j \neq \mu$, and $z_{\nu j,k}$, $j \neq \mu$ or ν, are computed according to equation (30).
- For each frame of data the total measurement $\mathbf{Z}_k$, of dimension no greater than $2n_k - 3$, is constructed according to equation (38).
- The measurement sensitivity matrix, H_k, defined by equation (39) is computed according to the coefficients in equation (31).
- The total measurement covariance matrix, $P_{\mathbf{Z}_k}$, which is the covariance of $\Delta\mathbf{Z}_k$ of equation (39), is computed in terms of the individual expectations given by equations (33) and (34).
- The truncated measurement sensitivity matrix H'_k is computed by truncating the columns of H_k corresponding to sensor λ.
- These quantities are inserted into the right members of equations (49) and (50), which are solved to obtain $\mathbf{\Psi}^*$(prior-free) and $P_{\Psi\Psi}$(prior-free).
- The misalignment matrices, M_i, are calculated according to equation (14).
- The corrected alignment matrices, S_i, are calculated according to equation (13).
- The procedure is repeated until the computed misalignments are driven to zero (generally, a single iteration will suffice.)

The Factorized Algorithm

- The index λ of the sensor to which the relative alignments will be referred is identified.
- The uncalibrated sensor measurements, $\hat{\mathbf{W}}^o_{i,k}$, are computed from the sensor measurements in the sensor frame, $\hat{\mathbf{U}}_{i,k}$, using the prelaunch alignment matrices, S^o_i according to equation (24).

- The cosine-error measurements, $z_{ij,k}$, are calculated for all $i \neq j$ according to equation (30).
- For each frame of data the total measurement vector $\mathbf{Z}_k$ of dimension $n_k(n_k-1)/2$ is constructed from these cosine-error measurements.
- The measurement sensitivity matrix, H_k, of dimension $n_k(n_k-1)/2 \times 3n_k$, defined by equation (57), is computed according to the coefficients in equation (31).
- The square roots of the $R_{\hat{\mathbf{W}}^o_{i,k}}$ are computed.
- The matrix B_k, defined by equations (55) and (56), is computed.
- The SVD of B_k is computed to obtain U_k and $\mathcal{S}_k$ according to equation (60).
- The vector ζ_k and the matrix C_k are computed according to equations (64) and (65).
- The value of $\ell_{\max}$ beyond which the singular values of B_k are effectively zero is determined.
- All but the first $\ell_{\max}$ rows of ζ_k, C_k, and $\mathcal{S}_k$ are discarded to obtain $\tilde{\zeta}_k$, $\tilde{C}_k$, and $\tilde{\mathcal{S}}_k$.
- The truncated measurement sensitivity matrix $\tilde{C}'_k$ is computed by deleting the columns of $\tilde{C}_k$ corresponding to sensor λ.
- These quantities are inserted in the right members of equations (70) and (71), which are solved to obtain $\mathbf{\Psi}^*$(prior-free) and $P_{\Psi\Psi}$(prior-free).
- The misalignment matrices, M_i, are calculated according to equation (14).
- The corrected alignment matrices, S_i, are calculated according to equation (13).
- The procedure is repeated until the computed misalignments are driven to zero (generally, a single iteration will suffice.)

Coplanar Measurements

Unfactorized Algorithm: In cases where the measurements all lie in the same plane, and at least one measurement $\hat{\mathbf{W}}_{\mu,k}$ is far from being parallel to any of the other measurements, it should be possible to use the unfactorized algorithm with only the cosine errors $z_{\mu i,k}$, $i \neq \mu$, and the scalar-triple-product errors $z_{\mu\nu i,k}$, $i \neq \mu$ or ν, with ν any other sensor. The additional elements of the measurement sensitivity matrix H_k and the covariance matrix $P_{\mathbf{Z}_k}$ are computed in accordance with the results given in Appendix A.

Factorized Algorithm: The measurements consist now of all the $z_{ij,k}$ and all the $z_{ij\ell,k}$. The steps above are repeated using the results of Appendix A.

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