The Journal of the Astronautical Sciences, Vol. 38, No. 3, July-September 1990, pp. 377-393

Kalman Filtering of Spacecraft Attitude and the QUEST Model

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Abstract

Efficient algorithms are presented for processing vector measurements within a Kalman Filter assuming a simple but realistic form for the measurement covariance matrix. The results are sufficiently general to be applicable to any spacecraft attitude system for which the vector measurements depend directly on the attitude alone or on the attitude and unknown sensor alignments. When simultaneous non-collinear vector measurements are available, it is shown that the prior-free maximum-likelihood estimate of the attitude based on these simultaneous measurements (the single-frame QUEST attitude) provides a sufficient statistic which can greatly reduce the computational burden of the filter.

Introduction

The present work completes a series of papers concerned with attitude estimation based on a simple but realistic measurement model [1] for vector observations. The first paper [2] showed that the Wahba problem [3] for the attitude was equivalent to a maximum-likelihood estimation problem given this measurement model. The second paper [4] showed that the QUEST solution [1] to the Wahba problem could be mechanized as a sequential filter which was equivalent term by term to the usual Kalman filter for spacecraft attitude [5]. This filter QUEST was efficient, however, only in cases where process noise could be neglected or where the prediction step of the filter could be represented effectively as a fading-memory filter. More important, however, was the restriction that the angular velocity be known exactly. Thus, in practical situations, filter QUEST becomes at best a suboptimal filter relying on the exactness of a gyro-based angular velocity or on the exactness of a dynamical model. Nonetheless, this level of optimality and exactness has proven sufficient for at least three spacecraft (which, however, relied on a batch rather than a filter implementation [6]).

The present work shows how the QUEST measurement model [1] can be incorporated within realistic Kalman filters (and smoothers). Since the QUEST at-

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titude estimation algorithm has proved very useful in batch attitude estimation for several spacecraft, this is certainly of interest. The form of the measurement model effects only the update step of the Kalman filter. Therefore, no discussion is presented here of the initialization and prediction steps (for these see, e.g., [5]). An example of the implementation of these methods in a complete Kalman filter for a proposed spacecraft has been completed recently [7].

The QUEST measurement model has been mentioned in connection with Kalman filters previously by Bar-Itzhack and his collaborators for the attitude matrix [8], the quaternion [9], and the Euler angles [10]. However, the results presented in those works do not take advantage of the symmetry of the QUEST model and simply present general Kalman filter results for an arbitrary linear measurement. In addition, those works [8, 9] use redundant attitude parameters in the state vector rather than taking the orthogonality or normality constraint of the attitude representation in account explicitly *ab initio*. Therefore, in order to compensate for the errors which arise from neglecting these constraints these methods must implement additional "fixes" in the update step and use an attitude covariance matrix which, despite the redundancy of the estimated variables, is not singular at every step. The present work, because it works entirely in terms of local infinitesimal rotations (rather than in terms of the arithmetic differences of the representations), avoids these complications entirely.

The present paper begins by comparing the QUEST measurement model with a model which has been inferred for an actual vector sensor in common use. The QUEST measurement model covariance is shown to be a reasonable approximation if the field of view of the sensor is not too large. The efficient implementation in the Kalman filter of the QUEST measurement model is then developed in detail. The Kalman filter implementation displays a simplicity which mirrors the simplicity of the batch QUEST algorithm. An important component in the derivation of the QUEST Kalman filter is to make a propitious choice of the measurement components to include in the filter, because the QUEST measurement model is necessarily singular. Efficient expressions are presented for both covariance and information filters.

It turns out that the singularity of the measurement covariance, which arises from the unit-normalization of the measurement, can be set aside, essentially because the attitude is insensitive to the length of the measured vector. The singular three-dimensional measurement covariance matrix can be replaced, therefore, with an equivalent measurement covariance matrix which is a multiple of the three-dimensional identity matrix. Thus, the need to construct independent measurement components is replaced by the inversion of a slightly larger matrix.

For the case where the attitude system furnishes a sequence of "frames" of data, each consisting of several simultaneous non-collinear measurements, the QUEST estimate of the attitude for each frame provides a sufficient statistic [11] for the attitude which can be used in place of the actual vector measurements in the Kalman filter. The reformulation of the Kalman filter in terms of this sufficient statistic leads to computational savings. It is in this form that the filter has been implemented in flight software destined for the Star Tracker mission [7].

In the beginning of the present work it is assumed that the vector measurements depend on the attitude alone. If the vector measurements depend on other quantities as well, such as on sensor misalignments or on hardware biases, the results presented here will need to be modified. For many spacecraft, for example, the Star Tracker Mission [12], the algorithms derived here assuming sensor alignments and biases are known are entirely adequate. The special structure of the QUEST measurement model makes the inclusion of alignment degrees of freedom in the Kalman filter particularly simple. The modification of the QUEST Kalman filter formalism to include simultaneous attitude and alignment estimation is therefore also presented.

The Measurement Model

Vector measurements are usually of two types: (1) those which measure all three components; and (2) those which measure only a direction (line of sight). The principal example of the first type is the vector magnetometer. The principal examples of the second type are vector Sun sensors, star cameras and star trackers and, to some extent, Earth sensors. For complete vector sensors, the observed vector at time t_k, $\mathbf{W}_k$, is related to the reference vector $\mathbf{V}_k$ according to

$$\mathbf{W}_k = A_k \mathbf{V}_k + \Delta\mathbf{W}_k , \tag{1}$$

where A_k is the attitude matrix at time t_k and $\Delta\mathbf{W}_k$ is the measurement noise, usually assumed to be Guassian and white with covariance $R_{\mathbf{W}_k}$. Normalizing both sides of the equation leads to

$$\hat{\mathbf{W}}_k = A_k \hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k , \tag{2}$$

where $\Delta\hat{\mathbf{W}}_k$ is to good approximation Gaussian and white and related to $\Delta\mathbf{W}_k$ according to

$$\Delta\hat{\mathbf{W}}_k = -\frac{1}{|\mathbf{V}_k|}[\![A_k\hat{\mathbf{V}}_k]\!]^2 \Delta\mathbf{W}_k . \tag{3}$$

Here, $[\![\mathbf{v}]\!]$ denotes the 3×3 antisymmetric matrix given by

$$[\![\mathbf{v}]\!] = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} . \tag{4}$$

Note that

$$\begin{aligned} R(\hat{\mathbf{n}},\theta) &= e^{[\![\theta\hat{\mathbf{n}}]\!]} \\ &= \cos\theta\, I + (1 - \cos\theta)\hat{\mathbf{n}}\hat{\mathbf{n}}^T + \sin\theta\, [\![\hat{\mathbf{n}}]\!] , \end{aligned} \tag{5}$$

which is Euler's formula for the rotation matrix for a rotation through an angle θ about an axis $\hat{\mathbf{n}}$, and

$$\begin{aligned} -[\![A_k\hat{\mathbf{V}}_k]\!]^2 &= I - (A_k\hat{\mathbf{V}}_k)(A_k\hat{\mathbf{V}}_k)^T , \\ &\equiv \mathcal{P}_{(A_k\hat{\mathbf{V}}_k)} , \end{aligned} \tag{6}$$

is the projection operator onto the space perpendicular to $A_k \hat{\mathbf{V}}_k$. Thus, the measurement covariance matrix of the unitized measurement is given by

$$R_{\hat{\mathbf{W}}_k} \equiv E\{\Delta\hat{\mathbf{W}}_k \Delta\hat{\mathbf{W}}_k^T\} = \frac{1}{|\mathbf{V}_k|^2} \mathcal{P}_{\{A_k \hat{\mathbf{V}}_k\}} R_{\mathbf{W}_k} \mathcal{P}_{\{A_k \hat{\mathbf{V}}_k\}}, \tag{7}$$

where $R_{\mathbf{W}_k}$ is the covariance matrix for the ununitized measurement. If it is further assumed that $R_{\mathbf{W}_k}$ is proportional to the identity matrix, then we obtain using equation (3) the covariance matrix of the QUEST measurement model,

$$R_{\hat{\mathbf{W}}_k}^{\mathrm{QUEST}} = \sigma_{\hat{\mathbf{W}}_k}^2 [I - (A_k \hat{\mathbf{V}}_k)(A_k \hat{\mathbf{V}}_k)^T], \tag{8}$$

which was the assumed covariance matrix for line-of-sight sensors in the QUEST model [1].

Line-of-sight sensors effectively measure the elevations of the line of sight as projected onto mutually perpendicular planes which contain the sensor boresight axis (here the z-axis). Denoting these two angles by α and β, the sensed line of sight is given by

$$\hat{\mathbf{W}} = \frac{1}{\sqrt{1 + \tan^2\alpha + \tan^2\beta}} \begin{bmatrix} \tan\alpha \\ \tan\beta \\ 1 \end{bmatrix} \equiv \frac{1}{\sqrt{1 + z_1^2 + z_2^2}} \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}. \tag{9}$$

The measurements $\tan\alpha$ and $\tan\beta$ are largely uncorrelated. For a commercially available vector Sun sensor which has been used frequently in near-Earth spacecraft, the covariance matrix of these two quantities has been determined to have the form

$$R_z = \frac{\sigma^2}{1 + c(z_1^2 + z_2^2)} \begin{bmatrix} (1 + cz_1^2)^2 & (cz_1 z_2)^2 \\ (cz_1 z_2)^2 & (1 + cz_2^2)^2 \end{bmatrix}, \tag{10}$$

where c is a quantity of order unity. Note that the covariance matrix of equation (10) has been written in terms of the measurement vector itself, which is only approximately correct since it makes the covariance matrix a random quantity. However, since truth is never available to us, this is an approximation which ultimately will always have to be made. If equation (8) is written equivalently in terms of the quantities z_1 and z_2, the result for z_1 and z_2 not too large is

$$R_z^{\mathrm{QUEST}} = \sigma_{\hat{\mathbf{W}}_k}^2 \begin{bmatrix} (1 - z_1^2)(1 + z_1^2 + z_2^2) & z_1 z_2 (z_1^2 + z_2^2) \\ z_1 z_2 (z_1^2 + z_2^2) & (1 - z_2^2)(1 + z_1^2 + z_2^2) \end{bmatrix}. \tag{11}$$

For $z_1^2 + z_2^2 \ll 1$, the QUEST measurement model agrees well with the inferred measurement model for the real sensor. Thus, for both models the diagonal elements differ from the same constant by terms of order $|\mathbf{z}|^2$ and the off-diagonal terms differ from zero by terms of order $|\mathbf{z}|^4$.

For precise sensors the measurements are constrained typically to be within fifteen degrees or less of the sensor bore-sight. Hence, for very accurate sensors the QUEST approximation is very good indeed. Less precise sensors generally have much larger fields of view, the vector Sun sensor cited above being one example. However, for these sensors the accuracy requirements are also much more modest, and the approximation of using the QUEST measurement model is not significant except, perhaps in those cases where the measurement covariance matrix is radically anisotropic.

The Kalman Filter

True Measurement Model

The measurements z_1 and z_2, defined by equation (9), implemented straightforwardly furnish a not inefficient Kalman filter. Let $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3\}$ denote the three sensor axes with $\hat{\mathbf{u}}_3$ the sensor boresight. Then

$$z_{1,k} \equiv \tan \alpha_k = \frac{\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{W}}_k}{\hat{\mathbf{u}}_3 \cdot \hat{\mathbf{W}}_k}, \qquad z_{2,k} \equiv \tan \beta_k = \frac{\hat{\mathbf{u}}_2 \cdot \hat{\mathbf{W}}_k}{\hat{\mathbf{u}}_3 \cdot \hat{\mathbf{W}}_k}. \tag{12}$$

If, following the notation of [4], $A^*_{k|k-1}$ denotes the predicted[2] attitude matrix at time t_k, then the attitude error ξ_k is given by

$$A_k \equiv e^{[\![\xi_k]\!]} A^*_{k|k-1}, \tag{13}$$

or, to first order in ξ_k,

$$A_k = (I + [\![\xi_k]\!]) A^*_{k|k-1}. \tag{14}$$

It is easy to show then that

$$z_{1,k} = \tan \alpha_{k|k-1} + H_{1,k}\xi_k + \Delta z_{1,k}, \tag{15a}$$

$$z_{2,k} = \tan \beta_{k|k-1} + H_{2,k}\xi_k + \Delta z_{2,k}, \tag{15b}$$

where

$$\tan \alpha_{k|k-1} = \frac{\hat{\mathbf{u}}_{k,1} \cdot \hat{\mathbf{W}}_{k|k-1}}{\hat{\mathbf{u}}_{k,3} \cdot \hat{\mathbf{W}}_{k|k-1}}, \qquad \tan \beta_{k|k-1} = \frac{\hat{\mathbf{u}}_{k,2} \cdot \hat{\mathbf{W}}_{k|k-1}}{\hat{\mathbf{u}}_{k,3} \cdot \hat{\mathbf{W}}_{k|k-1}}, \tag{16}$$

$$H_{1,k} = \frac{1}{\hat{\mathbf{u}}_{k,3} \cdot \hat{\mathbf{W}}_{k|k-1}} [(\hat{\mathbf{u}}_{k,1} \times \hat{\mathbf{W}}_{k|k-1}) - \tan \alpha_{k|k-1}(\hat{\mathbf{u}}_{k,3} \times \hat{\mathbf{W}}_{k|k-1})]^T, \tag{17a}$$

$$H_{2,k} = \frac{1}{\hat{\mathbf{u}}_{k,3} \cdot \hat{\mathbf{W}}_{k|k-1}} [(\hat{\mathbf{u}}_{k,2} \times \hat{\mathbf{W}}_{k|k-1}) - \tan \beta_{k|k-1}(\hat{\mathbf{u}}_{k,3} \times \hat{\mathbf{W}}_{k|k-1})]^T, \tag{17b}$$

and $\Delta z_{1,k}$ and $\Delta z_{2,k}$ are sensor random noise. As in earlier work,

$$\hat{\mathbf{W}}_{k|k-1} \equiv A^*_{k|k-1} \hat{\mathbf{V}}_k. \tag{18}$$

Equations (15a) and (15b) assume that the measurements depend only on the attitude. Otherwise, additional terms must be added.

The QUEST Measurement Model

Geometrically, the QUEST measurement model is identical to the true measurement model. It differs only in the value of the measurement covariance, which we have seen above is a good approximation to the true measurement covariance. The simplicity of the QUEST measurement covariance should lead to

[2]In general, carets will denote unit vectors and asterisks estimates. However, when the subscript, such as $k|k-1$ or $k|k$, makes it obvious that the quantity in question is an estimate, the asterisk often will be left off to avoid an overly cumbersome notation.

simplifications of the Kalman filter. An idiosyncratic example of this was demonstrated in [4] for the special case that the state vector consisted of the attitude alone. More general implementations of the QUEST measurement model, however, require greater care, because the QUEST measurement covariance matrix is singular, and cannot, therefore, be inserted directly into the usual Kalman filter formulae. The implementation of this simple but singular covariance matrix in the Kalman filter equations is carried out by first projecting it onto a two-dimensional space on which it is non-singular. This projection turns out to be a purely formal device, which can be removed once suitable expressions for the Kalman filter quantities are generated.

We carry out the projection as follows: Let $\hat{\chi}_k$ be a unit vector which is very close to $A_k\hat{\mathbf{V}}_k$, and let $\{\hat{\chi}_k, \hat{\mathbf{a}}_k(\hat{\chi}_k), \hat{\mathbf{b}}_k(\hat{\chi}_k)\}$ be any right-handed orthonormal triad having $\hat{\chi}_k$ as the first vector. Define projected measurements

$$\zeta_{1,k} \equiv \hat{\mathbf{a}}_k(\hat{\chi}_k) \cdot \hat{\mathbf{W}}_k, \qquad \zeta_{2,k} \equiv \hat{\mathbf{b}}_k(\hat{\chi}_k) \cdot \hat{\mathbf{W}}_k, \tag{19}$$

and the projection matrix

$$U_k(\hat{\chi}_k) \equiv [\hat{\mathbf{a}}_k(\hat{\chi}_k) \vdots \hat{\mathbf{b}}_k(\hat{\chi}_k)]^T, \tag{20}$$

which satisfies

$$U_k(\hat{\chi}_k)U_k^T(\hat{\chi}_k) = I_{2\times2}, \tag{21a}$$

$$U_k^T(\hat{\chi}_k)U_k(\hat{\chi}_k) = I_{3\times3} - \hat{\chi}_k\hat{\chi}_k^T. \tag{21b}$$

Then, the projected measurement vector is

$$\boldsymbol{\zeta}_k \equiv [\zeta_{1,k}, \zeta_{2,k}]^T = U_k(\hat{\chi}_k)\hat{\mathbf{W}}_k, \tag{22a}$$

$$= U_k(\hat{\chi}_k)A_k\hat{\mathbf{V}}_k + \Delta\boldsymbol{\zeta}_k, \tag{22b}$$

where the projected measurement noise is

$$\Delta\boldsymbol{\zeta}_k = U_k(\hat{\chi}_k)\Delta\hat{\mathbf{W}}_k, \tag{23}$$

and satisfies

$$E\{\Delta\boldsymbol{\zeta}_k\} = \mathbf{0}. \tag{24a}$$

$$E\{\Delta\boldsymbol{\zeta}_k\Delta\boldsymbol{\zeta}_k^T\} = \sigma_k^2[I_{2\times2} - \boldsymbol{\varepsilon}_k\boldsymbol{\varepsilon}_k^T]. \tag{24b}$$

Now

$$\boldsymbol{\varepsilon}_k = U_k(\hat{\chi}_k)(A_k\hat{\mathbf{V}}_k - \hat{\chi}_k). \tag{25}$$

and because $\hat{\chi}_k$ is very close to $A_k\hat{\mathbf{V}}_k$, it follows that $|\boldsymbol{\varepsilon}_k| \ll 1$, and we may neglect the second term of equation (24b) and write to good approximation

$$R_{\zeta_k} \equiv E\{\Delta\boldsymbol{\zeta}_k\Delta\boldsymbol{\zeta}_k^T\} = \sigma_k^2 I_{2\times2}, \tag{26}$$

which is certainly non-singular. $\boldsymbol{\zeta}_k$ is the measurement vector which will be used in the Kalman filter update equations. In addition, we will make a very specific choice for $\hat{\chi}_k$ which simplifies the filter equations.

The Update Equations

Following [5] we write the state vector as

$$\mathbf{x}_k = \begin{bmatrix} \bar{q}_k \\ \mathbf{y}_k \end{bmatrix}, \tag{27}$$

where $\bar{q}_k$ is the attitude quaternion at time t_k and $\mathbf{y}_k$ denotes all other components of the state vector. For the moment, we assume that the unit-vector measurements do not depend explicitly on $\mathbf{y}_k$. Other measurements may depend explicitly on $\mathbf{y}_k$, and for these measurements the Kalman filter update equations will not benefit from the results derived here. It need not be the case, however, that any of the measurements depend on $\mathbf{y}_k$ in order to have a reasonable filter. Consider, in this regard, the case where $\mathbf{y}_k$ denotes the spacecraft angular velocity, which can be updated in the Kalman filter even though the unit-vector measurements do not depend on it explicitly [7].

If $\mathbf{x}_{k|k-1}$ is the predicted state vector, then the state errors at the update times, $\delta\bar{q}_k$ and $\Delta\mathbf{y}_k$ are defined so that

$$\mathbf{x}_k = \begin{bmatrix} \delta\bar{q}_k \otimes \bar{q}_{k|k-1} \\ \mathbf{y}_{k|k-1} + \Delta\mathbf{y}_k \end{bmatrix}, \tag{28}$$

where $\otimes$ denotes quaternion composition and we follow the convention that the earliest quaternion is rightmost. In particular,

$$\bar{q}_{k|k} = \delta\bar{q}_{k|k} \otimes \bar{q}_{k|k-1}, \tag{29a}$$

$$\mathbf{y}_{k|k} = \mathbf{y}_{k|k-1} + \Delta\mathbf{y}_{k|k}. \tag{29b}$$

By definition, these state errors satisfy

$$\delta\bar{q}_{k|k-1} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}, \tag{30a}$$

$$\Delta\mathbf{y}_{k|k-1} = \mathbf{0}. \tag{30b}$$

To linearize the measurement equation, we write, in similar fashion to equations (13) and (14),

$$\delta\bar{q}_k = \begin{bmatrix} \hat{\boldsymbol{\xi}}_k \sin(|\boldsymbol{\xi}_k|/2) \\ \cos(|\boldsymbol{\xi}_k|/2) \end{bmatrix} \simeq \begin{bmatrix} \boldsymbol{\xi}_k/2 \\ 1 \end{bmatrix}, \tag{31}$$

where necessarily,

$$\boldsymbol{\xi}_{k|k-1} = \mathbf{0}. \tag{32}$$

We are led, therefore, to define the state error vector as

$$\delta\mathbf{x}_k = \begin{bmatrix} \boldsymbol{\xi}_k \\ \Delta\mathbf{y}_k \end{bmatrix}. \tag{33}$$

Note that if $\mathbf{y}_k$ has dimension m, then the state error vector has dimension $(m + 3)$, while the state vector has dimension $(m + 4)$, as in [5] (which was specialized to $m = 3$).

The vector $\hat{\chi}_k$ has been undefined except for being close to $A_k \hat{\mathbf{V}}_k$. We may now take advantage of the freedom which choosing this vector provides us. Recalling equation (14), we have to lowest order in ξ_k

$$\begin{aligned}\zeta_k &= U_k(\hat{\chi}_k)(I + [\![\xi_k]\!])\hat{\mathbf{W}}_{k|k-1} + \Delta\zeta_k, \\ &= U_k(\hat{\chi}_k)(\hat{\mathbf{W}}_{k|k-1} - [\![\hat{\mathbf{W}}_{k|k-1}]\!]\xi_k) + \Delta\zeta_k. \qquad (34)\end{aligned}$$

If we choose now

$$\hat{\chi}_k = \hat{\mathbf{W}}_{k|k-1}, \qquad (35)$$

then the first term in equation (34) vanishes identically because

$$U_k(\hat{\mathbf{W}}_{k|k-1})\hat{\mathbf{W}}_{k|k-1} = \mathbf{0}, \qquad (36)$$

and it follows that

$$\zeta_k \equiv U_{k|k-1}\hat{\mathbf{W}}_k, \qquad (37a)$$

$$= -U_{k|k-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!]\xi_k + \Delta\zeta_k, \qquad (37b)$$

where we have defined

$$U_{k|k-1} \equiv U_k(\hat{\mathbf{W}}_{k|k-1}). \qquad (38)$$

Then, the linearized measurement equation becomes finally

$$\zeta_k = H_k\,\delta\mathbf{x}_k + \Delta\zeta_k, \qquad (39)$$

where the measurement sensitivity matrix, H_k, is given by

$$H_k = [-U_{k|k-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!] \;\vdots\; 0_{2\times m}], \qquad (40)$$

$$\equiv [h_k \;\vdots\; 0_{2\times m}]. \qquad (41)$$

We may write the usual Kalman filter update equations as

$$B_k = H_k P_{k|k-1} H_k^T + R_{\zeta_k}, \qquad (42)$$

$$= h_k (P_{\xi\xi})_{k|k-1} h_k^T + \sigma_k^2 I_{2\times 2}, \qquad (43)$$

$$K_k = P_{k|k-1} H_k^T B_k^{-1}, \qquad (44)$$

$$\begin{aligned}\boldsymbol{\nu}_k &= \zeta_k - H_k\,\delta\mathbf{x}_{k|k-1}, \\ &= \zeta_k, \qquad (45)\end{aligned}$$

$$\begin{aligned}\delta\mathbf{x}_{k|k} &= \delta\mathbf{x}_{k|k-1} + K_k\boldsymbol{\nu}_k, \\ &= K_k\zeta_k, \qquad (46)\end{aligned}$$

$$P_{k|k} = (I - K_k H_k)P_{k|k-1}, \qquad (47)$$

$$= (I - K_k H_k)P_{k|k-1}(I - K_k H_k)^T + K_k R_{\zeta_k} K_k^T. \qquad (48)$$

Here, $\boldsymbol{\nu}_k$ is the innovation, B_k is the innovation covariance, and K_k is the Kalman gain matrix. The submatrix $P_{\xi\xi}$ of the state error covariance matrix is sometimes written $P_{\theta\theta}$ in the references. R_{ζ_k} is given by equation (26). Note that the sensitivity matrix with respect to the attitude error vector is given by

$$h_k \equiv -U_{k|k-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!], \tag{49}$$

$$= [-\hat{\mathbf{b}}_k(\hat{\mathbf{W}}_{k|k-1}) \vdots \hat{\mathbf{a}}_k(\hat{\mathbf{W}}_{k|k-1})]^T. \tag{50}$$

so that the matrix multiplication of equation (49) need never be carried out in practice.

Note that the vectors $\hat{\mathbf{a}}_k(\hat{\mathbf{W}}_{k|k-1})$ and $\hat{\mathbf{b}}_k(\hat{\mathbf{W}}_{k|k-1})$ are arbitrary except for the fact that $\hat{\mathbf{a}}_k(\hat{\mathbf{W}}_{k|k-1})$, $\hat{\mathbf{b}}_k(\hat{\mathbf{W}}_{k|k-1})$ and $\hat{\mathbf{W}}_{k|k-1}$ form a right-hand orthonormal triad. Thus, the only sensor-dependent quantity in these filter equations is the sensor variance, σ_k^2.

Further Simplification

Let us note that

$$h_k^T B_k^{-1} h_k = h_k^T\{h_k(P_{\xi\xi})_{k|k-1}h_k^T + \sigma_k^2 I_{2\times 2}\}^{-1}h_k,$$

$$= [\![\hat{\mathbf{W}}_{k|k-1}]\!]^T\{[\![\hat{\mathbf{W}}_{k|k-1}]\!](P_{\xi\xi})_{k|k-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!]^T + \sigma_k^2 I_{3\times 3}\}^{-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!], \tag{51}$$

which may be obtained either by expanding B_k^{-1} in a Taylor series, collecting terms, and finally recombining the terms of the Taylor series as a new inverse, or by repeated application of the Sherman-Morrison-Woodbury formula [13]. Similarly,

$$h_k^T B_k^{-1} U_{k|k-1} = -[\![\hat{\mathbf{W}}_{k|k-1}]\!]^T\{[\![\hat{\mathbf{W}}_{k|k-1}]\!](P_{\xi\xi})_{k|k-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!]^T + \sigma_k^2 I_{3\times 3}\}^{-1}. \tag{52}$$

If we now define

$$\mathcal{H}_k \equiv [-[\![\hat{\mathbf{W}}_{k|k-1}]\!] \vdots 0_{3\times m}], \tag{53}$$

$$\mathcal{B}_k \equiv \mathcal{H}_k(P_{\xi\xi})_{k|k-1}\mathcal{H}_k^T + \sigma_k^2 I_{3\times 3}, \tag{54}$$

$$= [\![\hat{\mathbf{W}}_{k|k-1}]\!](P_{\xi\xi})_{k|k-1}[\![\hat{\mathbf{W}}_{k|k-1}]\!]^T + \sigma_k^2 I_{3\times 3}, \tag{55}$$

then it follows that

$$K_k H_k = \mathcal{K}_k \mathcal{H}_k, \tag{56}$$

and

$$K_k \boldsymbol{\nu}_k = \mathcal{K}_k \hat{\mathbf{W}}_k, \tag{57}$$

where

$$\mathcal{K}_k \equiv (P_{\xi\xi})_{k|k-1}\mathcal{H}_k^T \mathcal{B}_k^{-1}. \tag{58}$$

These relations permit us to rewrite the Kalman filter equations as

$$\mathcal{B}_k \equiv \mathcal{H}_k^T(P_{\xi\xi})_{k|k-1}\mathcal{H}_k + \mathcal{R}_k, \tag{59}$$

$$\mathcal{K}_k \equiv (P_{\xi\xi})_{k|k-1}\mathcal{H}_k^T\mathcal{B}_k^{-1}, \tag{60}$$

$$\boldsymbol{\nu}_k \equiv \hat{\mathbf{W}}_k - \mathcal{H}_k\,\delta\mathbf{x}_{k|k-1}, \tag{61}$$

$$= \hat{\mathbf{W}}_k, \tag{62}$$

$$\delta\mathbf{x}_{k|k} = \delta\mathbf{x}_{k|k-1} + \mathcal{K}_k\boldsymbol{\nu}_k, \tag{63}$$

$$= \mathcal{K}_k\hat{\mathbf{W}}_k. \tag{64}$$

$$P_{k|k} = (I - \mathcal{K}_k \mathcal{H}_k) P_{k|k-1}, \tag{65}$$

$$= (I - \mathcal{K}_k \mathcal{H}_k) P_{k|k-1} (I - \mathcal{K}_k \mathcal{H}_k)^T + \mathcal{K}_k \mathcal{R}_k \mathcal{K}_k^T, \tag{66}$$

and

$$\mathcal{R}_k \equiv \sigma_k^2 I_{3\times 3}. \tag{67}$$

Thus, the projection matrix, $U_{k|k-1}$, no longer appears. In its place we must invert a 3×3 matrix rather than a 2×2.

The physical interpretation of the new Kalman filter equations is that we have replaced a singular three-dimensional measurement by a statistically equivalent[3] non-singular three-dimensional measurement of the form

$$\mathbf{W}_k' = A_k \hat{\mathbf{V}}_k + \Delta \mathbf{W}_k', \tag{68}$$

with $\mathbf{W}_k'$ identical in value to $\mathbf{W}_k$. The new equivalent measurement noise satisfies

$$E\{\Delta \mathbf{W}_k'\} = \mathbf{0}, \tag{69}$$

$$E\{\Delta \mathbf{W}_k' \, \Delta \mathbf{W}_k'^T\} = \sigma_k^2 I_{3\times 3} = \mathcal{R}_k. \tag{70}$$

To equation (34) there now corresponds

$$\mathbf{W}_k' = \hat{\mathbf{W}}_{k|k-1} - [\![\hat{\mathbf{W}}_{k|k-1}]\!] \boldsymbol{\xi}_k + \Delta \mathbf{W}_k'. \tag{71}$$

The first term of equation (71) no longer vanishes by itself: it does not contribute in the new Kalman filter equations, however, because

$$\mathcal{H}_k^T \hat{\mathbf{W}}_{k|k-1} = \mathbf{0}. \tag{72}$$

Equation (2) in our filter computations is thus effectively replaced by equation (68) but with $\mathbf{W}_k'$ a unit vector by accident, not by constraint.

The Information Filter

The equations for the information filter may be written directly now that the measurement sensitivity matrix has been determined [14, 15]. Thus, in information form

$$P_{k|k}^{-1} = P_{k|k-1}^{-1} + H_k^T R_k^{-1} H_k, \tag{73}$$

which leads directly to

$$P_{k|k}^{-1} = P_{k|k-1}^{-1} + \begin{bmatrix} \sigma_k^{-2}(I - \hat{\mathbf{W}}_{k|k-1}\hat{\mathbf{W}}_{k|k-1}^T) & 0_{3\times m} \\ 0_{m\times 3} & 0_{m\times m} \end{bmatrix}, \tag{74}$$

and the Kalman filter gain is given by

$$K_k = P_{k|k} H_k^T R_k^{-1}, \tag{75}$$

or

$$K_k = P_{k|k} \begin{bmatrix} \sigma_k^{-2} [\![\hat{\mathbf{W}}_{k|k-1}]\!] U_{k|k-1}^T \\ 0_{m\times 2} \end{bmatrix}. \tag{76}$$

[3] Equivalent in the sense that it leads to the same estimator.

Again,

$$\delta \mathbf{x}_{k|k} = K_k \zeta_k . \tag{77}$$

Similar simplifications can also be obtained for the formulation in terms of the information vector [11]. Using the results for the covariance just derived[4], we have immediately for the effective three-dimensional measurements that

$$P_{k|k}^{-1} = P_{k|k-1}^{-1} + \mathcal{H}_k^T \mathcal{R}_k^{-1} \mathcal{H}_k , \tag{78}$$

which, when evaluated, leads again to equation (74). The Kalman filter gain for the three-dimensional measurement in the information filter form is

$$\mathcal{K}_k = P_{k|k} \mathcal{H}_k^T \mathcal{R}_k^{-1} , \tag{79}$$

which becomes

$$\mathcal{K}_k = P_{k|k} \begin{bmatrix} \sigma_k^{-2} [\![\hat{\mathbf{W}}_{k|k-1}]\!] \\ 0_{m \times 3} \end{bmatrix} . \tag{80}$$

and again,

$$\delta \mathbf{x}_{k|k} = \mathcal{K}_k \hat{\mathbf{W}}_k . \tag{81}$$

This result has already been used in [4] for the special case that the state vector consists of the attitude alone.

Alignment Estimation

The above results rely fundamentally on the assumption that the measured directions depend explicitly only on the attitude and not on other components of the state vector. For many parameters which are estimated in a typical mission (for example, gyro biases) this is true. Sensor alignments, unfortunately, do not fall into this category. The extension of these results to include the estimation of sensor alignments, however, is not difficult.

Assume that the spacecraft is equipped with n vector sensors for which we wish to estimate alignments in the filter and let $\hat{\mathbf{U}}_{i,k}$ denote the direction measured by sensor i at time t_k in the sensor frame. It will be convenient to define the temporal index so that there is only one sensor measurement at each time t_k. Simultaneous measurements are handled by letting several consecutive t_k have the same value. The $\hat{\mathbf{U}}_{i,k}$ is related to the measured direction in spacecraft body coordinates by

$$\hat{\mathbf{W}}_{i,k} = S_i \hat{\mathbf{U}}_{i,k}, \qquad i = 1, \ldots, n , \tag{82}$$

which defines S_i, the alignment matrix, which is proper orthogonal. Thus, we may write equivalently in sensor coordinates

$$\hat{\mathbf{U}}_{i,k} = S_i^T A_k \hat{\mathbf{V}}_{i,k} + \Delta \hat{\mathbf{U}}_{i,k}, \qquad i = 1, \ldots, n , \tag{83}$$

with

$$E\{\Delta \hat{\mathbf{U}}_{i,k}\} = \mathbf{0} . \tag{84a}$$

[4]The same results follow from a rigorous treatment.

$$E\{\Delta\hat{\mathbf{U}}_{i,k}\,\Delta\hat{\mathbf{U}}_{i,k}^T\} = \sigma_{i,k}^2(I - (S_i^T A_k \hat{\mathbf{V}}_{i,k})(S_i^T A_k \hat{\mathbf{V}}_{i,k})^T). \tag{84b}$$

If we wish to estimate sensor alignments as well as the attitude in the filter, we must linearize the dependence of the measurements on the sensor alignments also. Thus, analogously to equation (13) we define the misalignment matrix, $M_{i,k}$, by

$$S_{i,k} = M_{i,k} S_{i,k|k-1}, \tag{85}$$

where $S_{i,k|k-1}$ is the predicted estimate of the alignment at t_k based on all the data previous to t_k. Note that although the alignment matrices are constant,

$$S_{i,k} = S_{i,k-1} = S_i, \tag{86}$$

(leading to a prediction step, $S_{i,k|k-1} = S_{i,k-1|k-1}$) the estimates of the alignment matrices are not. In similar fashion to the definition of the attitude error, the misalignment vectors, $\boldsymbol{\theta}_{i,k}, i = 1, \ldots, n$, are defined by

$$M_{i,k} \equiv e^{[\![\boldsymbol{\theta}_{i,k}]\!]}, \qquad i = 1, \ldots, n, \tag{87}$$

and to lowest non-vanishing order in $\boldsymbol{\theta}_{i,k}$

$$S_{i,k} = (I + [\![\boldsymbol{\theta}_{i,k}]\!])S_{i,k|k-1}. \tag{88}$$

Combining equations (83), (85), (87), and (13) leads to

$$\hat{\mathbf{U}}_{i,k} = S_{i,k|k-1}^T e^{-[\![\boldsymbol{\theta}_{i,k}]\!]} e^{[\![\boldsymbol{\xi}_k]\!]} A_{k|k-1}^* \hat{\mathbf{V}}_{i,k} + \Delta\hat{\mathbf{U}}_{i,k}, \tag{89}$$

We now define the "misaligned" measurement by

$$\hat{\mathbf{W}}_{i,k}^o \equiv S_{i,k|k-1}\hat{\mathbf{W}}_{i,k}, \tag{90}$$

from which it follows immediately that

$$\hat{\mathbf{W}}_{i,k}^o = e^{-[\![\boldsymbol{\theta}_{i,k}]\!]} e^{[\![\boldsymbol{\xi}_k]\!]} \hat{\mathbf{W}}_{i,k|k-1} + \Delta\hat{\mathbf{W}}_{i,k}^o, \tag{91}$$

with $\hat{\mathbf{W}}_{i,k|k-1}$ defined analogously to equation (18),

$$\hat{\mathbf{W}}_{i,k|k-1} \equiv A_{k|k-1}^* \hat{\mathbf{V}}_{i,k}. \tag{92}$$

$\Delta\hat{\mathbf{W}}_{i,k}^o$ is, therefore, a discrete white Gaussian process satisfying

$$E\{\Delta\hat{\mathbf{W}}_{i,k}^o\} = \mathbf{0}, \tag{93a}$$

$$E\{\Delta\hat{\mathbf{W}}_{i,k}^o\,\Delta\hat{\mathbf{W}}_{i,k}^{oT}\} = R_{\hat{\mathbf{W}}_{i,k}^o}, \tag{93b}$$

and

$$R_{\hat{\mathbf{W}}_{i,k}^o} = S_{i,k|k-1} R_{\hat{\mathbf{U}}_{i,k}} S_{i,k|k-1}^T. \tag{94}$$

Equation (91) is very similar in form to equation (13). Linearizing equation (91) leads directly to

$$\hat{\mathbf{W}}_{i,k}^o = \hat{\mathbf{W}}_{i,k|k-1} - [\![\hat{\mathbf{W}}_{i,k|k-1}]\!](\boldsymbol{\xi}_k - \boldsymbol{\theta}_{i,k}) + \Delta\hat{\mathbf{W}}_{i,k}^o. \tag{95}$$

The two-dimensional measurements are defined analogously to equation (37).

$$\boldsymbol{\zeta}_{i,k} \equiv [\zeta_{1,i,k}, \zeta_{2,i,k}]^T = U_k(\hat{\mathbf{W}}_{i,k|k-1})\hat{\mathbf{W}}_{i,k}^o, \tag{96a}$$

$$\equiv U_{i,k|k-1}\hat{\mathbf{W}}_{i,k}^o. \tag{96b}$$

The steps which led from equation (37) to equations (59) through (67) will not be repeated here in the context of combined attitude and alignment estimation. The results will simply be sketched. The reader should have little difficulty in completing the derivations.

The state vector in the context of combined attitude and alignment estimation is now

$$\mathbf{x}_k = [\bar{q}_k^T, \bar{q}_{1,k}^T, \ldots, \bar{q}_{n,k}^T, \mathbf{y}_k^T]^T, \tag{97}$$

where $\bar{q}_{i,k}, i = 1, \ldots, n$, are the alignment quaternions, which have the same relation to the respective alignment matrices as the attitude quaternion has to the attitude matrix. $\mathbf{y}_k$ now represents all the remaining components of the state vector which again are assumed to not depend explicitly on the attitude. Similarly to the earlier development, the state error vector is defined as

$$\delta\mathbf{x}_k = [\xi_k^T, \theta_{1,k}^T, \ldots, \theta_{n,k}^T, \Delta\mathbf{y}_k^T]^T, \tag{98}$$

so that $\mathbf{x}_k$ has dimension $[4(n + 1) + m]$, while $\delta\mathbf{x}_k$ has dimension $[3(n + 1) + m]$. We have now[5] (always with an additional subscript to label the active sensor)

$$\zeta_{i,k} = h_{i,k}\xi_k - h_{i,k}\theta_{i,k} + \Delta\zeta_{i,k}, \tag{99}$$

$$\equiv H_{i,k}\,\delta\mathbf{x}_k + \Delta\zeta_{i,k} \tag{100}$$

with

$$H_{i,k} = [h_{i,k} \vdots 0_{2\times3} \vdots \cdots \vdots -h_{i,k} \vdots 0_{2\times3} \vdots \cdots \vdots 0_{2\times m}] \tag{101}$$

and

$$h_{i,k} \equiv -U_{i,k|k-1}[\![\hat{\mathbf{W}}_{i,k|k-1}]\!]. \tag{102}$$

In equation (101) the matrix $-h_{i,k}$ is located in the submatrix of $H_{i,k}$ which multiplies $\theta_{i,k}$. With these new definitions, equations (42) and (44) through (48) remain unchanged in the context of alignment estimation except for the expansion of the subscript to include the designation of the active sensor. Equation (43) corresponds in the current context to

$$B_{i,k} = h_{i,k}[(P_{\xi\xi})_{k|k-1} - (P_{\xi\theta_i})_{k|k-1} - (P_{\theta_i\xi})_{k|k-1} + (P_{\theta_i\theta_i})_{k|k-1}]h_{i,k}^T + R_{\zeta_{i,k}}. \tag{103}$$

Equations (51) through (67) have analogous results, *mutatis mutandis*, in the context of alignment estimation. The only changes are that the new equations which correspond to equations (51) and (52) contain replacements similar to those in equation (103), $\mathcal{H}_{i,k}$ has an additional non-vanishing submatrix in similar fashion to $H_{i,k}$ in equation (101), and all quantities now have an additional subscript to designate the active sensor. (Note that since only one value of the active sensor index corresponds to each temporal index k, the labeling of all quantities by both i and k is redundant (except, of course, for $\theta_{i,k}$ and the like). However, the double index makes the notation clearer in most cases.)

[5]Note that $H_{i,k}$ defined here is not the same sensitivity matrix as appears in equations (17a) and (17b).

In actual practice, sensor misalignments generally have not been estimated in a Kalman filter but rather with sub-optimal but attitude-independent batch estimators [16, 17].

The QUEST Algorithm as Data Compressor

Because the Kalman filter can be regarded as the mechanization of a maximum likelihood estimate [18], and the QUEST attitude is the maximum likelihood estimate of the attitude given a set of simultaneous vector measurements [2], it follows that the QUEST attitude provides a sufficient statistic [11] for the attitude and may be used in the filter as an effective measurement in place of the vector measurements $\hat{\mathbf{W}}_k$. This substitution is certainly exact if the state vector consists only of the attitude. For the more general case, where other quantities are also being estimated in the filter, this substitution is only approximate, holding to lowest order in σ^2. However, since σ is almost always a very small quantity, this approximation is generally more than satisfactory. This attitude estimation scheme has been implemented recently in the on-board attitude determination system for the Star Tracker mission [7] and also in studies of Kalman-filter-based attitude systems for a gravity-gradient stabilized spacecraft [19] operating at more modest accuracy levels.

To see how this may be accomplished in practice, let C_k^* denote the QUEST attitude matrix[9] solution for a frame of data (i.e., a set of simultaneous vector measurements) at time t_k. Provided that the frame at t_k contains at least two non-collinear vectors, the "per-frame" maximum likelihood estimate of the attitude will be unambiguously determined. Thus, the QUEST solution at time t_k may be written

$$C_k^* = e^{[\mathbf{v}_k]} A_k , \tag{104}$$

where $\mathbf{v}_k$ is the attitude error from QUEST, which, given the vector measurement model of equation (2), satisfies [1]

$$E\{\mathbf{v}_k\} = \mathbf{0} , \tag{105a}$$

$$E\{\mathbf{v}_k \mathbf{v}_k^T\} = R_k^{\text{QUEST}} . \tag{105b}$$

with

$$(R_k^{\text{QUEST}})^{-1} = \sum_{i=1}^{n_k} \frac{1}{\sigma_{i,k}^2} [I - (A_k \hat{\mathbf{V}}_{i,k})(A_k \hat{\mathbf{V}}_{i,k})^T] . \tag{106}$$

Substituting equation (13) into equation (104) leads to

$$C_k^* = e^{[\mathbf{v}_k]} e^{[\boldsymbol{\xi}_k]} A_{k|k-1}^* . \tag{107}$$

Again, $A_{k|k-1}^*$ is the predicted attitude at time t_k based on all previous frames of data up to and including time t_{k-1}, and C_k^* is the QUEST estimate of the attitude matrix based only on the data at time t_k. It is this equation which we must linearize, which is simple for the attitude matrix and only slightly more complicated for the quaternion [7].

[9]QUEST actually computes a quaternion. However, the present discussion will be somewhat simpler in terms of the attitude matrix. Reference [7] presents a complete mechanization of the filter in terms of the QUEST quaternion.

To accomplish this linearization we define $\mathbf{z}_k$ by

$$e^{[\mathbf{z}_k]} \equiv C_k^* A_{k:k-1}^{*T} . \tag{108}$$

Then from equations (107),

$$e^{[\mathbf{z}_k]} = e^{[\mathbf{v}_k]} e^{[\boldsymbol{\xi}_k]} . \tag{109}$$

Since all the quantities in the matrix exponents are expected to be much smaller than unity, we may write immediately to lowest order in $\mathbf{v}_k$

$$\mathbf{z}_k = \boldsymbol{\xi}_k + \mathbf{v}_k , \tag{110}$$

whence,

$$\mathbf{z}_k = H_k \, \delta\mathbf{x}_k + \mathbf{v}_k , \tag{111}$$

with

$$H_k = [I_{3\times3} \vdots 0_{3\times m}] , \tag{112}$$

and

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k^{\mathrm{QUEST}}) . \tag{113}$$

The construction of the Kalman filter update is now straightforward.

Discussion and Conclusions

General equations have been presented for the processing of direction measurements within the Kalman filter using the QUEST measurement model of the sensor errors and assuming that the observed directions are a function of the attitude alone. Because the QUEST measurement covariance matrix is singular owing to the presence of redundant components in the measurements, the constraint must first be projected out of the measurement before the measurement can be incorporated in the Kalman filter. Algorithms are presented for doing this which take advantage of the special structure of the QUEST model and the Kalman filter. It has been shown that the projection may be dispensed with if the QUEST measurement model is replaced with a non-singular measurement model, which essentially regards the unit normalization of the measurements as accidental rather than constrained. The computational burden of the projection is now replaced by the computational burden of inverting a 3×3 rather than a 2×2 matrix. The extension of the methodology to include alignment estimation was also presented. Lastly, when two or more simultaneous direction measurements are available in a given frame, the QUEST algorithm provides a useful sufficient statistic for compressing this simultaneous data.

The greatest benefit of the QUEST model in the filter implementation, perhaps, is that it leads to a Kalman filter in which only the geometry of the measured unit vectors is important and not the construction of the sensor. Thus, both equation (37) and, even more manifestly, equation (71) depend only on the measured direction and not on the direction of any sensor coordinate axes. The only sensor dependent quantity in this model is the sensor variance σ_k^2 (or $\sigma_{i,k}^2$). This makes the QUEST model particularly attractive as a simulation tool. The

QUEST measurement model is surely the simplest measurement model for line-of-sight sensors which correctly reflects the geometrical properties of those measurements. The realism of these results rests, of course, on the faithfulness of the QUEST model as a representation of the sensor errors. For spacecraft with very limited fields of view, the QUEST model provides in general a very realistic representation of sensor errors. The adequacy of the QUEST model in actual mission support has been demonstrated for numerous spacecraft including the three HEAO spacecraft [6], Magsat [2], and the Solar Maximum Mission, to name only a few.

The alignment estimation methodology presented here is of interest beyond the context of the QUEST model. Provided that the alignment degrees of freedom are represented in the filter according to equations (85) and (87) and the attitude degrees of freedom according to equation (13), then the form of the sensitivity matrix as given by equation (101) (i.e., the sensitivity of the measurement to the misalignments will differ only in sign from the sensitivity to the attitude error) will be obtained no matter what the measurement model.

What is the advantage of being able to discard the normalization constraint in the three components of the line-of-sight measurements? Since the Kalman filter is usually most efficient when vector measurements are processed as a simultaneous sequence of scalar measurements, this means that the addition of a redundant component to the line-of-sight measurements increases the processing of these measurements in the update step by fifty per cent. This increment is offset somewhat by the elimination of the projection operation. Thus, the two approaches are roughly equivalent in computational burden, especially when one considers that the update from line-of-sight measurements represents only a part of the filter computational burden.

What is the physical origin of the result that we can simply ignore the unity constraint on the line-of-sight measurements and treat them as if they had three statistically independent degrees of freedom? The reason for this is fairly obvious, though we would be wary of immediately constructing the filter equations (59) through (66) on the basis of these arguments. The argument is as follows:

Consider equation (1), which is the equation for a vector measurement of a sensor which provides vectors with three statistically independent components (such as a vector magnetometer). If we multiply $\mathbf{W}_k$, $\mathbf{V}_k$, and $\Delta\mathbf{W}_k$, by a common factor, the equation is satisfied, clearly, by the same value of the attitude matrix as before. Thus, the magnitude of $\mathbf{W}_k$ is insensitive to the attitude. Hence, if we linearize the magnitude of $\mathbf{W}_k$ as a function of ξ_k in the manner of equation (39), we must find that the sensitivity matrix corresponding to the magnitude of $\mathbf{W}_k$ vanishes. Thus, no matter what the variance of the magnitude of $\mathbf{W}_k$, this magnitude will not contribute at all to the estimate and we may replace the variance of the magnitude of $\mathbf{W}_k$ by any value. What the rigorous derivation of this result (equations (51) through (67)) tells us is that this statement is true even if the variance of the magnitude of $\mathbf{W}_k$ vanishes! In an equivalent systems theory problem we would say that we have a pole-zero cancellation. Also, it is clear that with minor alterations, this result holds for any measurement model for line-of-sight measurements.

Acknowledgments

The author is grateful to T. E. Strikwerda and H. L. Fisher for many interesting discussions. As usual, begrudging thanks are extended to F. Landis Markley, the author's most exacting and constant critic.

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