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A Simple Kalman Filter and Smoother for Spacecraft Attitude

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Abstract

The QUEST solution to the Wahba problem, which previously has been recast as a maximum likelihood estimation problem for the attitude, is now reformulated as a Kalman filter. The resulting "filter QUEST" turns out to have several advantages over the usual Kalman filter mechanization for spacecraft attitude. Firstly, it deals with the whole attitude rather than with incremental corrections. Thus, questionable subtractions need not be made, and convergence of "filter QUEST" is immediate. Secondly, for cases where the effect of process noise contributions can be represented adequately by a limited memory filter, the resulting algorithm is extremely efficient. When the attitude covariance matrix is output with smaller frequency than that of the measurements, "filter QUEST" becomes proportionately more efficient than the corresponding usual implementation of the Kalman filter. Numerical examples are presented.

Introduction

In a previous paper [1], the statistical properties were studied of the least-squares attitude matrix, A^*, which minimizes Wahba's cost function [2],

$$L(A) = \frac{1}{2}\sum_{k=1}^{n} a_k|\hat{\mathbf{W}}_k - A\hat{\mathbf{V}}_k|^2, \tag{1}$$

where $\hat{\mathbf{W}}_k$, $k = 1, \ldots, n$, is a set of unit vector observations in the spacecraft-fixed reference frame, $\hat{\mathbf{V}}_k$, $k = 1, \ldots, n$, are the representations of the same unit vectors with respect to the primary reference frame (the frame to which the attitude is referred), and the a_k are a set of positive weights. It was shown there that the least-squares attitude matrix, A^*, was also the maximum-likelihood estimate of the attitude given the measurements $\hat{\mathbf{W}}_k$, for the measurement model

$$\hat{\mathbf{W}}_k = A\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k, \tag{2}$$

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where the sensor error, $\Delta\hat{\mathbf{W}}_k$, is approximately Gaussian and satisfies to lowest order

$$E\{\Delta\hat{\mathbf{W}}_k\} = \mathbf{0}, \tag{3}$$

$$E\{\Delta\hat{\mathbf{W}}_k\,\Delta\hat{\mathbf{W}}_k^T\} = \sigma_k^2[I - (A\hat{\mathbf{V}}_k)(A\hat{\mathbf{V}}_k)^T]. \tag{4}$$

This approximate but realistic model for the mean and covariance matrix of the attitude sensor error has been used frequently in the past. In fact, it was the basis for the covariance analyses of the QUEST algorithm [3], a very efficient implementation of the solution to the Wahba problem which has been employed in the support of several spacecraft. References to the earlier work on the Wahba problem are contained in [1].

This result has important consequences. Firstly, knowing that the Wahba attitude is a maximum likelihood estimate gives us greater insight into its statistical properties. Secondly, since the Kalman filter is a sequential mechanization of the maximum likelihood estimate for linear Gaussian systems [4], it follows that a sequentialization of the QUEST algorithm must be equivalent to the Kalman filter for the measurement model above. It is the elaboration of a QUEST mechanization of a Kalman filter and smoother for the attitude, which is the subject of this paper.

The second important result in [1] was that the Fisher information matrix, $F_{\theta\theta}$, which asymptotically is equal to the inverse of the estimate-error covariance matrix, $P_{\theta\theta}$, could be calculated directly from the maximum likelihood estimate of the attitude and attitude profile matrix, defined according to

$$B = \sum_{k=1}^{n} a_k \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T. \tag{5}$$

The relation is

$$F_{\theta\theta} \simeq P_{\theta\theta}^{-1} \simeq \mathrm{tr}(A^*B)I - A^*B, \tag{6}$$

where $\mathrm{tr}(\cdot)$ denotes the trace function. This could be solved for B to yield

$$B \simeq \left[\frac{1}{2}\,\mathrm{tr}(P_{\theta\theta}^{-1})I - P_{\theta\theta}^{-1}\right]A^*. \tag{7}$$

In either form B is an *exact* representation of the maximum-likelihood attitude and an approximate representation (good to lowest order in σ_k^2) of the attitude covariance. Thus, the possibility exists of mechanizing a Kalman filter for the attitude in terms of B, which is what we intend to do.

We begin by discussing early attempts to accommodate noisy systems in the Wahba problem within a batch framework. We then develop a filter implementation of the QUEST algorithm for deterministic static systems subject to noisy measurements and extend it to take approximate account of process noise as a fading-memory filter. We then compare the performance of the fading-memory filter, dubbed "filter QUEST" with the correctly modeled Kalman filter. The agreement turns out to be generally good. Finally, the "filter QUEST" algorithm is extended as a Rauch-Tung-Striebel smoother [4].

Early Applications of the Wahba Problem to Dynamical Systems

We are used to thinking of the Kalman filter as a mechanization of the maximum likelihood estimate for measurements which constitute a time series. For the Wahba problem, the measurements are all simultaneous. Hence, a Kalman filter mechanization of the least-square attitude defined above can correspond only to a static system. The Wahba cost function has been applied to dynamical systems, however, most notably the support of the HEAO spacecraft [5]. In this application the spacecraft was equipped with three-axis strap-down gyros, which, if we neglect gyro noise, made possible the computation of the relative attitude Φ_k from time t_k to time t_{k+1},

$$A_{k+1} = \Phi_k A_k . \tag{8}$$

Using these relative attitudes the measurements at time t_k were transformed to the spacecraft body frame at time t_o, the loss function of equation (1) was minimized to find the best estimate of the attitude at time t_o, and the relative attitudes then used again to transform A_o back to A_k. The gyro biases were assumed to be very-slowly varying and estimates of the gyro biases were updated from time to time in a manner independent of the attitude computation, essentially by minimizing the spread of the attitudes which were mapped back to t_o.

This approach has certain disadvantages. Since the gyro biases are changing due to the accumulation of white noise and higher-order Markov processes, the contribution to $A(t_o)$ should not be as great from very late as from very early measurements. A diminution of the effect of later data can be accomplished by down-weighting the data, say by multiplying the weights a_k by a factor

$$f_k = \frac{1}{\sigma_k^2 + \varpi(t_k - t_o)}, \tag{9}$$

where σ_k is the sensor error level and ϖ characterizes the gyro noise, assumed here to be Gaussian and white. Such down-weighting can be expected to be heuristic at best since until recently the Wahba problem was not analyzed in its proper statistical setting. Down-weighting of the data does not make this procedure totally satisfying either, since down-weighting which is appropriate for $A(t_o)$ will not be appropriate for $A(t_k)$ with k different from zero.

A second unpleasantness of this approach is that one must continually augment the size of the batch to obtain estimates for later times than were contained within the original batch or wait until a new batch is accumulated. Thus, this approach does not permit the attitude to be computed on-line with much ease.

Finally, all these enhancements are only heuristic at best. For one to have confidence in a statistical estimate, one must have confidence in the statistical foundations of that estimate. Simply making intuitively satisfying doctorings of an algorithm is not sufficient.

A Filter Solution to the Wahba Problem

As a first step to constructing a filter solution to the Wahba problem for a deterministic dynamical system we consider the sequentialization of the Wahba problem as given by equation (1). Let A_k^* denote the estimate based on the first k measurements. This quantity and its estimate error covariance are determined completely by the attitude profile matrix

$$B_k = \sum_{i=1}^{k} a_i \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T . \tag{10}$$

Earlier implementations of the Wahba problem set

$$\sum_{l=1}^{k} a_l = 1 . \tag{11}$$

It was shown earlier [1], however, that $L(A)$ becomes the data-dependent part of the negative-log-likelihood function for the measurements described by equations (2) through (4) provided we choose instead

$$a_l = \frac{1}{\sigma_l^2} . \tag{12}$$

With this choice B_k satisfies

$$B_k = B_{k-1} + \frac{1}{\sigma_k^2}\hat{\mathbf{W}}_k\hat{\mathbf{V}}_k^T . \tag{13}$$

Since B_k is an exact representation of the maximum-likelihood estimate of the attitude [1], it follows that equation (13) is a mechanization of the Kalman filter for the measurement described by equations (2) through (4) and a static dynamical system, for which A_k is constant. Since B_k is a representation of both the attitude estimate and the estimate error covariance, equation (13) represents both the estimate update equation and the update equation for the covariance simulation of the filter.

This result may be extended to deterministic dynamical systems described by

$$\frac{d}{dt}A(t) = [\![\boldsymbol{\omega}(t)]\!]A(t) , \tag{14}$$

where $[\![\boldsymbol{\omega}]\!]$ denotes the antisymmetric matrix defined as

$$[\![\boldsymbol{\omega}]\!] \equiv \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} . \tag{15}$$

For the system to be deterministic, $\boldsymbol{\omega}(t)$ must be known exactly. In this case, $A_k (\equiv A(t_k))$ satisfies equation (8), with known Φ_k.

If we define now $\hat{\mathbf{W}}_k(t)$ according to

$$\frac{d}{dt}\hat{\mathbf{W}}_k(t) = [\![\boldsymbol{\omega}(t)]\!]\hat{\mathbf{W}}_k(t) , \tag{16}$$

with the boundary condition

$$\hat{\mathbf{W}}_k(t_k) = \hat{\mathbf{W}}_k , \tag{17}$$

where $\hat{\mathbf{W}}_k$ is the measurement which takes place at time t_k, then $\hat{\mathbf{W}}_k(t)$ is simply the representation of the k-th measurement in the body frame at time t. Thus,

$$\hat{\mathbf{W}}_l(t_{k+1}) = \Phi_k\hat{\mathbf{W}}_l(t_k) . \tag{18}$$

The attitude profile matrix at time t_k taking account of all measurements up to and including time t_k is, therefore,

$$B_k = \sum_{l=1}^{k} \frac{1}{\sigma_l^2}\Phi_{k-1}\Phi_{k-2}\ldots\Phi_l\hat{\mathbf{W}}_l\hat{\mathbf{V}}_l^T , \tag{19}$$

which satisfies the recursion relation

$$B_k = \Phi_{k-1} B_{k-1} + \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T . \tag{20}$$

Greater similarity to the usual Kalman filter equations may be obtained by defining

$$\begin{aligned} B_{k|k-1} &\equiv B(t_k) \quad \text{given } \hat{\mathbf{W}}_1, \ldots, \hat{\mathbf{W}}_{k-1} \\ &= \sum_{l=1}^{k-1} \frac{1}{\sigma_l^2} \hat{\mathbf{W}}_l(t_k) \hat{\mathbf{V}}_l^T , \end{aligned} \tag{21}$$

$$\begin{aligned} B_{k|k} &\equiv B(t_k) \quad \text{given } \hat{\mathbf{W}}_1, \ldots, \hat{\mathbf{W}}_k \\ &= \sum_{l=1}^{k} \frac{1}{\sigma_l^2} \hat{\mathbf{W}}_l(t_k) \hat{\mathbf{V}}_l^T . \end{aligned} \tag{22}$$

Then the "filter QUEST" implementation of the maximum likelihood estimator given the measurements described by equations (2) through (4) and the *deterministic* dynamics given by equation (8) is

$$B_{k|k-1} = \Phi_{k-1} B_{k-1|k-1} , \tag{23}$$

$$B_{k|k} = B_{k|k-1} + \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T , \tag{24}$$

which are the prediction and update equations, respectively.

If an *a priori* maximum likelihood estimate of the attitude, $A^*_{o|o}$, and corresponding estimate error covariance matrix, $P_{o|o}$, is available, then following equation (7), the attitude profile matrix may be initialized according to

$$B_{o|o} = \left[\frac{1}{2} \operatorname{tr}(P_{o|o}^{-1}) I - P_{o|o}^{-1} \right] A^*_{o|o} , \tag{25}$$

or, if an *a priori* estimate is not available, by

$$B_{o|o} = 0 . \tag{26}$$

Thus, for this deterministic system (subject to noisy measurements) equations (23) through (26) provide a complete mechanization of the Kalman filter in terms of the attitude profile matrix B_k. At any time the estimated attitude matrix and covariance matrix may be extracted using the QUEST algorithm [1,3]. This method has the disadvantage that one must do further work to compute the attitude matrix or quaternion and the covariance matrix from B_k. On the other hand it has certain advantages as well. First, "filter QUEST" deals with the whole attitude rather than with a differential correction and is, therefore, free of dubious subtractions, which usually lead to useless estimates until the filter converges. Secondly, the filter is exact and does not suffer from errors like those introduced into the extended Kalman filter by the linearization. Thirdly, since "filter QUEST" is closer to being an information filter than a covariance filter, it can treat prior-free estimates without the need to concoct not-quite-infinite initial covariance matrices, which add to the initial modeling errors and worsen the filter convergence. Fourthly, if the measurement frequency is much greater than the frequency with which attitude need be computed, then the method is very efficient since the number of steps needed for "filter QUEST" is smaller than that needed for the more usual Kalman filter computation of the attitude [6].

To see the nature of this efficiency in more quantitative terms, we will compare equations (23) through (26) with the equivalent Kalman filter equations. Thus, the corresponding prediction equation is paralleled by the two equations

$$A^*_{k+1|k} = \Phi_k A^*_{k|k}, \qquad P_{k+1|k} = \Phi_k P_{k|k} \Phi_k^T, \tag{27}$$

involving 81 flops* (or only 72 if we use the fact that P is symmetric) as opposed to only 27 flops for equation (23).

The implementation of the update for the equivalent Kalman filter is more complex. To represent the update correction to the attitude, we write

$$A_k = e^{[\![\xi_k]\!]} A^*_{k|k-1}, \tag{28}$$

where ξ_k is expected to be small and

$$\xi^*_{k|k-1} = \mathbf{0}. \tag{29}$$

Then, the measurement equation may be written to first order in ξ_k *as*

$$\hat{\mathbf{W}}_k = A^*_{k|k-1}\hat{\mathbf{V}}_k - [\![A^*_{k|k-1}\hat{\mathbf{V}}_k]\!]\xi_k + \mathbf{v}_k, \tag{30}$$

where $\mathbf{v}_k$ is the measurement noise which, to all practical purposes, is Gaussian, white, and has covariance matrix given by equation (4) with $A\hat{\mathbf{V}}_k$ replaced by $A^*_{k|k-1}\hat{\mathbf{V}}_k$.

Although the measurement is singular, the information matrix associated with the measurement is readily evaluated and is given by

$$R_k^{-1} = \frac{1}{\sigma_k^2}[I - \hat{\mathbf{W}}_{k|k-1}\hat{\mathbf{W}}^T_{k|k-1}]. \tag{31}$$

The notation is not meant to suggest that R_k^{-1} has an inverse. We have defined

$$\hat{\mathbf{W}}_{k|k-1} = A^*_{k|k-1}\hat{\mathbf{V}}_k. \tag{32}$$

Then, in terms of these quantities the Kalman filter equations become

$$P^{-1}_{k|k} = P^{-1}_{k|k-1} + R_k^{-1}, \tag{33}$$

$$\xi^*_{k|k} = \frac{1}{\sigma_k^2} P_{k|k}[\![\hat{\mathbf{W}}_{k|k-1}]\!]\hat{\mathbf{W}}_k. \tag{34}$$

The computation of the filter update has been simplified considerably by recombining some of the matrices and carrying out some of the matrix multiplications explicitly. Also the filter has been cast as an information filter rather than as a covariance filter in order to make the comparison with "filter QUEST" as unbiased as possible.

Given $\xi^*_{k|k}$ one must now update the attitude according to

$$\bar{q}^*_{k|k} = \begin{bmatrix} \xi^*_{k|k}/2 \\ 1 \end{bmatrix} \otimes \bar{q}^*_{k|k-1}, \tag{35}$$

where the indicated operation is quaternion composition, and it is understood that the quaternion must be renormalized after the update. Note that estimates have been indi-

*1 flop (= *floating-point operation*) is the equivalent to one floating-point multiplication or one floating-point division.

cated throughout by an asterisk rather than the more usual caret to avoid confusion with the caret designation of a unit vector.

This is the simplest implementation of the traditional Kalman filter update for the given measurement type. Equations (31) through (35) require 88 flops as opposed to only 12 flops for equation (24). The computation of the quaternion from $B_{k|k}$ using the QUEST algorithm requires an additional 16 flops [7], and the calculation of the covariance matrix from $B_{k|k}$ using the results of [1] requires an additional 45 flops. Thus, "filter QUEST" requires 39 flops if the quaternion and covariance matrix are not to be output and 100 flops otherwise. This should be compared with 169 flops for the more traditional Kalman filter. If the attitude covariance matrix need not be output at every update, the computational savings are substantial.

Inclusion of Process Noise in the Wahba Problem

The Wahba problem, by its formulation, does not allow easily for the inclusion of process noise. Equations (6) and (7) provide the means for computing a covariance matrix from the updated attitude profile matrix, adding a process noise covariance, and computing a new attitude profile matrix. Thus, process noise can be included into the QUEST algorithm exactly. Such a method would be extremely costly, however, and eliminate any computational gains obtained by implementing QUEST in place of the more traditional Kalman filter.

An alternative to working explicitly with a process noise covariance matrix is to approximate the degradation of the estimate due to process noise by a fading memory approximation [8]. Thus, in predicting the system from time t_k to time t_{k+1}, the Fisher information matrix is multiplied by a factor α where $0 \leq \alpha \leq 1$. Such an approach is approximate, to be sure, but unless one is specifically trying to identify error processes in the system, this approximation leads to an algorithm which is of obvious utility. The fading memory approximation has the advantage also that it makes the filter robust. With this approximation, the filter QUEST formulation takes the form

Initialization

$$B_{o|o} = \left[\frac{1}{2}\,\mathrm{tr}(P_{o|o}^{-1})I - P_{o|o}^{-1}\right]A_{o|o}^{*} \quad \text{or} \quad 0\,. \tag{36}$$

Prediction

$$B_{k+1|k} = \alpha\Phi_k B_{k|k}\,. \tag{37}$$

Update

$$B_{k|k} = B_{k|k-1} + \frac{1}{\sigma_k^2}\hat{\mathbf{W}}_k\hat{\mathbf{V}}_k^T\,. \tag{38}$$

For $\alpha = 1$, we recover the usual QUEST algorithm. For $\alpha = 0$, only the current measurements contribute to the estimate. Note that equation (37) assumes that the measurements are uniformly separated in time. When this is not the case, a more general form of the forgetfulness factor would be

$$\alpha_{k+1,k} = \exp[-\gamma(t_{k+1} - t_k)], \qquad \gamma \geq 0\,, \tag{39}$$

if the measurements are not evenly spaced in time. The traditional Kalman filter equations remain the same except that the covariance prediction becomes

$$P_{k+1|k} = \alpha^{-1}\Phi_k P_{k|k}\Phi_k^T . \tag{40}$$

The problem now becomes the choice of α. Since the maximum likelihood estimate is also the minimum variance estimate for linear Gaussian systems, a logical choice for α is the value which minimizes the steady-state updated covariance of the QUEST filter, i.e., the limit of the updated covariance as the time becomes infinite. This will yield the best possible steady-state estimate. Since spacecraft typically have more than one sensor, this limit is not unique. Thus, we must specify which sensor update is meant. Also, it assumes that as $k \rightarrow \infty$, σ_k for each sensor tends to a constant. In the examples we will study, we will assume that σ_k is independent of time.

Numerical Examples

In the absence of process noise the QUEST algorithm and the Kalman filter must yield identical results, since they both minimize the same function of the data. To test the performance of the filter QUEST algorithm in the presence of process noise the following stochastic model has been chosen

$$\frac{d}{dt}\boldsymbol{\theta}(t) = \boldsymbol{\omega}(t) + \frac{1}{2}[\![\boldsymbol{\omega}(t)]\!]\boldsymbol{\theta}(t) + \mathbf{w}(t) , \tag{41}$$

where $\boldsymbol{\theta}(t)$ is the rotation vector, which is related to the attitude matrix by

$$A = \cos|\boldsymbol{\theta}|I + \frac{(1 - \cos|\boldsymbol{\theta}|)}{|\boldsymbol{\theta}|^2}\boldsymbol{\theta}\boldsymbol{\theta}^T + \frac{\sin|\boldsymbol{\theta}|}{|\boldsymbol{\theta}|}[\![\boldsymbol{\theta}]\!] , \tag{42}$$

and which is assumed to be very small, so that higher order terms in $\boldsymbol{\theta}(t)$ could be dropped from equation (41). The random variable $\mathbf{w}(t)$ is taken to be a continuous white Gaussian process with constant power spectral density Q proportional to the identity matrix. The term $\mathbf{w}(t)$ in this model is the gyro noise, which in the dynamics replacement approximation [6] becomes state process noise. In the actual numerical example, $\boldsymbol{\omega}(t)$ has been chosen to vanish identically since the values of $\boldsymbol{\omega}(t)$ do not affect the performance of either QUEST or the Kalman filter. Thus, the effective discrete process is

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \mathbf{w}_k , \tag{43}$$

where $\mathbf{w}_k$ is a discrete white Gaussian process with process noise covariance Q_k, where for small uniform sample times Δt

$$Q_k \simeq Q\,\Delta t \equiv qI . \tag{44}$$

Two cases were considered for the measurements:

(a) Three simultaneous sightings in each frame of data, which we assume for simplicity to be along each of the three body coordinate axes, each having the same measurement covariance parameter σ.

(b) Two simultaneous sightings in each frame of data along the body x- and y-axes, each having the same measurement covariance parameter σ. This is very close to actual mission configurations except that the angle between the two sightings is seldom 90 degrees.

The "filter QUEST" algorithm and the traditional Kalman filter with fading memory will yield identical results within differences in round-off error. Therefore, we will compare in this section the "filter QUEST" result with the traditional Kalman filter with correct modeling for the prediction step, i.e.,

$$P_{k+1|k} = \Phi_k P_{k|k} \Phi_k^T + Q_k , \tag{45}$$

As a consequence of this, it might be argued that the numerical example is a test of the traditional Kalman filter with fading memory rather than of any specifically QUEST-based algorithm. This, of course, does not diminish the value of knowing how well a fading-memory filter operates within the QUEST context. We must now choose the optimal value of α.

We may regard $\hat{\mathbf{W}}_{k-l}$ for $l \geq 0$ as a measurement of $\boldsymbol{\theta}_k$. To show this using the fact that $\boldsymbol{\theta}_k$ is very small we note

$$\begin{aligned} \hat{\mathbf{W}}_{k-l} &\simeq (I + [\![\boldsymbol{\theta}_{k-l}]\!])\hat{\mathbf{V}}_{k-l} + \Delta\hat{\mathbf{W}}_{k-l} \\ &= \hat{\mathbf{V}}_{k-l} - [\![\hat{\mathbf{V}}_{k-l}]\!]\boldsymbol{\theta}_{k-l} + \Delta\hat{\mathbf{W}}_{k-l} \\ &= \hat{\mathbf{V}}_{k-l} - [\![\hat{\mathbf{V}}_{k-l}]\!](\boldsymbol{\theta}_k - \mathbf{w}_{k-1} - \mathbf{w}_{k-2} - \dots - \mathbf{w}_{k-l}) + \Delta\hat{\mathbf{W}}_{k-l} \\ &= \hat{\mathbf{V}}_{k-l} - [\![\hat{\mathbf{V}}_{k-l}]\!]\boldsymbol{\theta}_k + \Delta\hat{\mathbf{W}}_{k,k-l} , \end{aligned} \tag{46}$$

where

$$\Delta\hat{\mathbf{W}}_{k,k-l} \equiv \Delta\hat{\mathbf{W}}_{k-l} - [\![\hat{\mathbf{V}}_{k-l}]\!]\sum_{m=1}^{l} \mathbf{w}_{k-m} . \tag{47}$$

As a result of the process noise all the measurements will be correlated with one another. If the indices a and b denote the measurement axes (1, 2, and 3), k the temporal index of the state vector (here, $\boldsymbol{\theta}_k$), and $k - l$ and $k - l'$ the temporal indices of the measurements (so that each measurement is labeled by an axis index and a temporal index), then the covariance matrix of the measurements is

$$\mathrm{Cov}\{\Delta\hat{\mathbf{W}}_{k,a,k-l}, \Delta\hat{\mathbf{W}}_{k,b,k-l'}\} = (\sigma^2 \delta_{ab}\delta_{ll'} + q \min(l, l'))[\![\hat{\mathbf{V}}_{a,k-l}]\!][\![\hat{\mathbf{V}}_{b,k-l'}]\!]^T , \tag{48}$$

where $\min(i, j)$ denotes the minimum of i and j. The triple index should be clear. The indices a and l label the measurement (l is the number of time intervals prior to k at which the measurement occurs) and k is the temporal label of the current frame (for which $A^*_{k|k}$ is being estimated).

To determine the best value for the forgetfulness factor α, the QUEST attitude matrix is computed according to the prescriptions of equations (36) through (38), the attitude error, $\Delta\boldsymbol{\theta}$, is computed, and the covariance of this attitude error is computed using the covariance matrix of equation (48) to give the *true* covariance of the QUEST attitude. The quantity α is then chosen to minimize the true covariance of the QUEST attitude. The covariance matrix which QUEST would calculate from equation (6) need not be the same as the true covariance since equation (6) assumes that the measurements are uncorrelated.

Example 1: Three Measurements

The information matrix for a single frame of data in this case is

$$R_k^{-1} = \frac{1}{\sigma^2}(I - \hat{\mathbf{u}}_1\hat{\mathbf{u}}_1^T) + \frac{1}{\sigma^2}(I - \hat{\mathbf{u}}_2\hat{\mathbf{u}}_2^T) + \frac{1}{\sigma^2}(I - \hat{\mathbf{u}}_3\hat{\mathbf{u}}_3^T) = \frac{2}{\sigma^2} I , \tag{49}$$

or

$$R_k = \frac{\sigma^2}{2} I \equiv R \, . \tag{50}$$

The steady-state Kalman filter covariances are the solutions of

$$P(+)^{-1} = P(-)^{-1} + R^{-1}, \qquad P(-) = P(+) + Q \, , \tag{51}$$

where

$$P(+) \equiv \lim_{k\to\infty} P_{k|k} \, , \qquad P(-) \equiv \lim_{k\to\infty} P_{k|k-1} \, . \tag{52}$$

Since R and Q are proportional to the identity matrix it follows that

$$P(\pm) = p(\pm) I \, , \tag{53}$$

which are readily calculable to give

$$p^{\mathrm{KF}}(\pm) = \frac{\sigma^2}{2} \frac{\mp 1 + \sqrt{1 + 2x}}{x} \, , \tag{54}$$

where

$$x \equiv \sigma^2 / q \, . \tag{55}$$

For the special case considered in the present analysis, the QUEST attitude is calculated most simply in terms of $\boldsymbol{\theta}_k$ directly. From equation (46) we have that $\boldsymbol{\theta}^{*\mathrm{QUEST}}_{k|k}$ must minimize

$$L(\boldsymbol{\theta}_k) = \frac{1}{2} \sum_{l=0}^{k-1} \frac{\alpha_{k,k-l}}{\sigma_l^2} |\hat{\mathbf{W}}_{k-l} - \hat{\mathbf{V}}_{k-l} + [\![\hat{\mathbf{V}}_{k-l}]\!] \boldsymbol{\theta}_k|^2 , \tag{56}$$

which in our example becomes

$$L(\boldsymbol{\theta}_k) = \frac{1}{2} \sum_{l=0}^{k-1} \sum_{a=1}^{3} \frac{\alpha^l}{\sigma^2} |\hat{\mathbf{W}}_{a,k-l} - \hat{\mathbf{V}}_{a,k-l} + [\![\hat{\mathbf{V}}_{a,k-l}]\!] \boldsymbol{\theta}_k|^2 . \tag{57}$$

With this QUEST loss function $\boldsymbol{\theta}^{*\mathrm{QUEST}}_{k|k}$ is given by

$$\boldsymbol{\theta}^{*\mathrm{QUEST}}_{k|k} = P^{\mathrm{QUEST}}_{k|k} \mathcal{V}^{\mathrm{QUEST}}_{k|k} , \tag{58}$$

where

$$\mathcal{V}^{\mathrm{QUEST}}_{k|k} = \sum_{l=0}^{k-1} \sum_{a=1}^{3} \frac{\alpha^l}{\sigma^2} [\![\hat{\mathbf{V}}_{a,k-l}]\!] (\hat{\mathbf{W}}_{a,k-l} - \hat{\mathbf{V}}_{a,k-l}) \, , \tag{59}$$

$$P^{\mathrm{QUEST}}_{k|k} = \left(\sum_{l=0}^{k-1} \sum_{a=1}^{3} \frac{\alpha^l}{\sigma^2} [I - \hat{\mathbf{V}}_{a,k-l} \hat{\mathbf{V}}^T_{a,k-l}] \right)^{-1} \tag{60}$$

$$= \frac{1 - \alpha}{1 - \alpha^k} \frac{\sigma^2}{2} I \, . \tag{61}$$

$P^{\mathrm{QUEST}}_{k|k}$ as given by equations (60) and (61) is the covariance matrix that QUEST would calculate from its own internal assumptions which do not take account of the correlations between the measurements. To evaluate the true covariance of the

QUEST solution we note that the true value of $\boldsymbol{\theta}_k$ is $\mathbf{0}$, and, hence, equations (58) through (61) yield

$$\Delta\boldsymbol{\theta}_{k|k}^{*\mathrm{QUEST}} = \frac{1}{2}\frac{1-\alpha}{1-\alpha^k}\sum_{l=0}^{k-1}\sum_{a=1}^{3}\alpha^l[\![\hat{\mathbf{V}}_{a,k-l}]\!]\,\Delta\hat{\mathbf{W}}_{k,a,k-l}. \tag{62}$$

Thus,

$$P_{true\text{-}k|k}^{\mathrm{QUEST}} = \left(\frac{1}{2}\frac{1-\alpha}{1-\alpha^k}\right)^2\sum_{l=0}^{k-1}\sum_{l'=0}^{k-1}\sum_{a=1}^{3}\sum_{b=1}^{3}\alpha^{l+l'}[\![\hat{\mathbf{V}}_{a,k-l}]\!] \times \mathrm{Cov}\{\Delta\hat{\mathbf{W}}_{k,a,k-l},\Delta\hat{\mathbf{W}}_{k,b,k-l'}\}[\![\hat{\mathbf{V}}_{a,k-l}]\!]^T. \tag{63}$$

The asymptotic covariance matrix is computed by letting $k \to \infty$, which yields

$$P_{true}^{\mathrm{QUEST}}(+) = p_{true}^{\mathrm{QUEST}}(+)I, \tag{64}$$

with

$$p_{true}^{\mathrm{QUEST}}(+) = \frac{\sigma^2}{2}\left[\frac{1-\alpha}{1+\alpha} + \frac{2}{x}\frac{\alpha^2}{1-\alpha^2}\right], \tag{65}$$

which is a minimum for

$$\alpha_{opt} = \frac{x+1-\sqrt{1+2x}}{x}. \tag{66}$$

Substituting (66) into (65) leads to

$$p_{true\text{-}min}^{\mathrm{QUEST}}(+) = \frac{\sigma^2}{2}\frac{-1+\sqrt{1+2x}}{x}, \tag{67}$$

so that the true covariance of the QUEST attitude in steady-state is the same as that for the Kalman filter. This is not surprising since in the present case the covariance prediction in the Kalman filter is tantamount to multiplication by a forgetfulness factor. Since the Kalman filter solution lies within the domain of possible QUEST solutions, it is clear that the QUEST solution with minimum variance must be the Kalman filter solution. The asymptotic covariance matrix which QUEST would calculate based on its internal statistical assumptions is given by equation (61) in the limit that $k \to \infty$, which leads to

$$P^{\mathrm{QUEST}}(+) = (1-\alpha)\frac{\sigma^2}{2}I. \tag{68}$$

Substituting α_{opt} in this expression leads, surprisingly, to the same result for $p_{min}^{\mathrm{QUEST}}(+)$ as that given in equations (54) and (67).

Example 2: Two Measurements

In this case, the covariance matrix is not proportional to the identity matrix. The steady-state covariance of the Kalman filter is

$$P^{\mathrm{KF}}(\pm) = \begin{bmatrix} p_1^{\mathrm{KF}}(\pm) & 0 & 0 \\ 0 & p_1^{\mathrm{KF}}(\pm) & 0 \\ 0 & 0 & p_2^{\mathrm{KF}}(\pm)\end{bmatrix}. \tag{69}$$

with

$$p_1^{KF}(\pm) = \frac{\sigma^2}{2}\frac{\mp 1 + \sqrt{1 + 4x}}{x}, \tag{70a}$$

$$p_2^{KF}(\pm) = \frac{\sigma^2}{2}\frac{\mp 1 + \sqrt{1 + 2x}}{x}. \tag{70b}$$

The equivalent result for the QUEST attitude relative to the true statistics is

$$P_{true}^{QUEST}(+) = \frac{\sigma^2}{2}\left(\frac{1 - \alpha}{1 + \alpha}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{x}\frac{\alpha^2}{1 - \alpha^2}I\right). \tag{71}$$

Since the covariance matrix is not a multiple of the identity matrix, we choose α_{opt} to be the value which minimizes the trace of the true covariance of the QUEST attitude. The trace of the Kalman filter covariance is

$$\mathrm{tr}^{KF} \equiv \mathrm{tr}\, P^{KF}(+) = \frac{3}{2}\sigma^2\left[\frac{-1 + (2/3)\sqrt{1 + 4x} + (1/3)\sqrt{1 + 2x}}{x}\right], \tag{72}$$

while that for QUEST relative to true statistics is

$$\mathrm{tr}_{true}^{QUEST} \equiv \mathrm{tr}\, P_{true}^{QUEST}(+) = \frac{3}{2}\sigma^2\left[\frac{5}{3}\frac{1 - \alpha}{1 + \alpha} + \frac{2}{x}\frac{\alpha^2}{1 - \alpha^2}\right]. \tag{73}$$

Minimizing $\mathrm{tr}_{true}^{QUEST}$ over α leads to

$$\alpha_{opt} = \frac{y + 1 - \sqrt{1 + 2y}}{y}, \qquad \text{where } y = 5x/3, \tag{74}$$

and

$$\mathrm{tr}_{true\text{-}min}^{QUEST} = \frac{3}{2}\sigma^2\frac{-1 + \sqrt{1 + 2y}}{x}. \tag{75}$$

The trace computed by QUEST based on its own internal statistical assumptions has the same value as equation (75).

How different are the results of equations (72) and (75)? For $x \ll 1$ the errors in the predicted steady-state attitudes will be dominated by the process noise so that for both filter QUEST and the Kalman filter the estimates should be the same and simply the estimate for one frame of data. The specific result on taking the limits of the two expressions above is indeed

$$\mathrm{tr}^{KF},\ \mathrm{tr}_{true\text{-}min}^{QUEST} \to \frac{5}{2}\sigma^2 \quad \text{as } x \to 0. \tag{76}$$

As $x \to \infty$ both the Kalman filter and QUEST will have very long memories and it is expected that in this case the greatest divergence between the two cases will occur. The limiting behavior in the two cases is

$$\mathrm{tr}^{KF} \to 2.71\sigma^2/\sqrt{x} \quad \text{as } x \to \infty, \tag{77}$$

$$\mathrm{tr}_{true\text{-}min}^{QUEST} \to 2.74\sigma^2/\sqrt{x} \quad \text{as } x \to \infty. \tag{78}$$

Thus, the two agree to within approximately one per cent. For finite x the comparison is given in Table 1.

Since the agreement between tr^{KF} and $\mathrm{tr}^{QUEST}_{true\text{-}min}$ has been optimized, it is of interest to ask how well the individual elements of the covariance matrices agree. $P^{QUEST}_{true\text{-}min}(+)$ (and $P^{QUEST}(+)$) have the same structure as $P^{KF}(+)$ as given by equation (69). From equation (71) we have

$$(p_1)^{QUEST}_{true\text{-}min}(+) = \sigma^2(c(x) + d(x)), \tag{79a}$$

$$(p_2)^{QUEST}_{true\text{-}min}(+) = \sigma^2\left(\frac{1}{2}c(x) + d(x)\right), \tag{79b}$$

with

$$c(x) \equiv \frac{1 - \alpha_{opt}}{1 + \alpha_{opt}}, \qquad d(x) \equiv \frac{1}{x}\frac{\alpha^2_{opt}}{1 - \alpha^2_{opt}}. \tag{80}$$

As $x \rightarrow 0$

$$p_1^{KF}(+), \qquad (p_1)^{QUEST}_{true\text{-}min}(+) \rightarrow \sigma^2, \tag{81a}$$

$$p_2^{KF}(+), \qquad (p_2)^{QUEST}_{true\text{-}min}(+) \rightarrow \frac{1}{2}\sigma^2, \tag{81b}$$

and as $x \rightarrow \infty$

$$p_1^{KF}(+) \rightarrow \sigma^2/\sqrt{x}, \qquad p_2^{KF}(+) \rightarrow .71\sigma^2/\sqrt{x} \tag{82a}$$

$$(p_1)^{QUEST}_{true\text{-}min}(+) \rightarrow 1.00\sigma^2/\sqrt{x}, \qquad (p_2)^{QUEST}_{true\text{-}min}(+) \rightarrow .73\sigma^2/\sqrt{x}. \tag{82b}$$

As a typical intermediate value consider $x = 1$. For this value

$$p_1^{KF}(+) = .62\sigma^2, \qquad p_2^{KF}(+) = .38\sigma^2 \tag{83a}$$

$$(p_1)^{QUEST}_{true\text{-}min}(+) = .62\sigma^2, \qquad (p_2)^{QUEST}_{true\text{-}min}(+) = .37\sigma^2, \tag{83b}$$

demonstrating the overall good level of agreement.

How well does the filter QUEST algorithm compare to the Kalman filter for data samples of finite duration? Examine the first example as a function of the time index. Thus, we need only examine scalar quantities. The updated Kalman filter covariance with correctly modeled process noise obeys

$$(p^{KF}_{k+1|k+1})^{-1} = (p^{KF}_{k|k} + q)^{-1} + (\sigma^2/2)^{-1}. \tag{84}$$

TABLE 1. Comparison of QUEST and Kalman Filter Steady-State Covariances

x	$\mathrm{tr}^{KF}/\sigma^2$	$\mathrm{tr}^{QUEST}_{true\text{-}min}/\sigma^2$
1	1.60	1.62
10	.72	.73
100	.26	.26
1000	.084	.085

For QUEST the true finite sample covariance is

$$p^{\text{QUEST}}_{true\text{-}k|k} = \frac{\sigma^2}{2}\left[\frac{1-\alpha}{1+\alpha}\mathcal{A}_k + \frac{2}{x}\frac{\alpha^2}{1-\alpha^2}\mathcal{B}_k\right], \tag{85}$$

with

$$\mathcal{A}_k = \frac{1+\alpha^k}{1-\alpha^k}, \tag{86a}$$

$$\mathcal{B}_k = (1-\alpha^k)^{-2}[(1-\alpha^k)(1-\alpha^k-2\alpha^{k-1}) + k\alpha^{2k-2}(1-\alpha^2)]. \tag{86b}$$

Since filter QUEST is an information filter, it is correctly initialized at t_o. It is customary among many researchers (and users!) of the Kalman filter to set the initial covariance to some very large value multiple of the identity matrix and allow the filter to converge to correct values even though the filtered state is well defined after the first frame of measurements. We will not follow that practice but instead correctly initialize $p_{1|1}$ for both filter QUEST and the Kalman filter to $\sigma^2/2$. The comparison of the Kalman filter variances and those of QUEST at the optimal value of α is shown in Table 2. σ has been chosen to make $p_{o|o} = 100$ and q has been chosen to make $x = 100$, so that the time scale over which the Kalman Filter and QUEST converge will be large.

Thus, filter QUEST and the Kalman Filter stay close even for finite data samples.

A Quest Smoother

Having developed a filter-like implementation of the QUEST algorithm, which in the absence of process noise is identical to the Kalman filter, it is natural to ask what form a smoother implementation of the QUEST algorithm would take. The "smoother QUEST," in fact, can be constructed from the "filter QUEST" by inspection.

TABLE 2. Comparison of QUEST and Kalman Filter Covariances for Finite Data Samples

k	Kalman Filter	QUEST
1	100.	100.
2	50.5	50.7
3	34.4	34.7
4	26.7	27.0
5	22.3	22.7
6	19.5	19.9
7	17.7	18.1
8	16.5	16.8
9	15.6	15.9
10	15.0	15.2
	...	...
20	13.27	13.34
	...	...
50	13.17746	13.17752
	...	...
100	13.17745	13.17745

Suppose we are given measurements $\hat{\mathbf{W}}_k, k = 1, \ldots, N$. Then the smoothed attitude profile matrix $B_{k|N}$ at time $t_k, k = 0, \ldots, N$, is given by

$$\begin{aligned} B_{k|N} = {} & \alpha^k \left(\prod_{i=0}^{k-1} \Phi_i \right) B_{o|o} \\ & + \sum_{i=1}^{k-1} \alpha^{|k-i|} \Phi_{k-1} \ldots \Phi_i \frac{1}{\sigma_i^2} \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T \\ & + \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \\ & + \sum_{i=k+1}^{N} \alpha^{|k-i|} \Phi_k^{-1} \ldots \Phi_{i-1}^{-1} \frac{1}{\sigma_i^2} \hat{\mathbf{W}}_i \hat{\mathbf{V}}_i^T , \end{aligned} \tag{87}$$

where the last three lines of equation (87) correspond to past, present, and future measurement data. The interpretation of the various terms is straightforward. The factors of the transition matrices transform the measurements to the body frame at time t_k and the factors of $\alpha^{|k-i|}$ downgrade the data to reflect the ravages of process noise. $B_{o|o}$ is given by equation (36).

Equation (87) may be rewritten as

$$B_{k|N} = B_{k|k} + D_k, \qquad k = 0, \ldots, N, \tag{88}$$

where $B_{k|k}$ is the "filtered" attitude profile matrix, which satisfies the previous "filter" (forward) recursion relations, equations, and is given by the first three lines of equation (87). D_k is the contribution of the future measurements, which is given by the last line of equation (82). By inspection, we see that D_k satisfies a backward recursion relation,

$$D_N = 0 , \tag{89}$$

$$D_{k-1} = \alpha \Phi_{k-1}^{-1} \left[D_k + \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{V}}_k^T \right] , \tag{90}$$

in complete analogy to the usual Rauch-Tung-Striebel Kalman filter/smoother [4]. The information matrix again is given by equation (6). Since Φ_k in the present application is orthogonal, the inverse is given by the transpose. Thus, the set of smoothed attitude estimates for an interval of data is obtained with only twice the computational burden of the calculation of the filtered estimates.

Discussion and Conclusions

The QUEST algorithm is seen to provide a useful estimator as a batch estimator, as a filter, and as a smoother. For two cases it has been shown to perform well even in the presence of process noise. The use of the QUEST algorithm is rather limited, however, since it computes only the attitude. Once other quantities are estimated as well, the computational advantages of using QUEST largely disappear. However, in the ground support of many near-Earth spacecraft, it is often only the attitude which is estimated continuously. Other quantities such as sensor biases and misalignments are often estimated "off-line" and only a few times during the mission. Thus, the batch version of QUEST has been very useful despite its limitations, and the "filter QUEST" presented here should also prove of value.

It may be noted that in batch mode the QUEST algorithm has proven itself to be very robust having been exercised more than 80,000,000 times with real mission data. While the algorithm has not been exercised specifically as a sequential algorithm, the manner in which the attitude profile matrix is accumulated in batch form is identical to the way it would be calculated as a sequential processor. The good performance of such a batch algorithm with non-static systems has already been demonstrated in the field [5].

Note that we have taken our implementation of QUEST in equation (57) to be one in which the estimator is totally unaware of process noise, not only in its effect on correlation but also of its effect on the "true" variances of the measurements. Thus, σ_i^2 is given simply by σ^2. We could, of course, have computed an improved σ_i^2 using equation (46). This would have led to a more realistic estimator but also one that is extremely complicated to evaluate and offers no computational advantages. As it is, the simplest choice performed quite well.

The method of choosing the optimum value for α, the forgetfulness factor, cannot, in general, be carried out in closed form, as was the case in our two numerical examples. Thus, some good starting value is needed to calculate α iteratively, say by finding the value which minimizes the true covariance of the "filter QUEST" algorithm by a Newton-Raphson iteration. For this a good starting value is needed. An obvious choice is

$$\alpha_o = \frac{\mathrm{tr}(P^{KF}(+))}{\mathrm{tr}(P^{KF}(+) + Q)}, \tag{91}$$

since this is the approximate scale of the fractional decrease in the true filter information in the limit that $k \rightarrow \infty$. For the first example, where all covariances which appear in the filter are multiples of the identity matrix, we expect equation (91) to give the same result as the explicit optimization, as, in fact, it does. Interestingly enough, the agreement is exact in the second example also. This exact agreement is not expected to hold in general.

In general, equation (91) is easier to compute than α_{opt}. Therefore, it is well to ask what is the effect of choosing a non-optimal value for α. To compute this in terms of the true covariance of "filter QUEST" can be obtained in the case of the numerical examples considered above by calculating the sensitivities of the expressions to α. However, we note that asymptotically the "hypothetical" covariance matrix computed by the QUEST algorithm is proportional to $(1 - \alpha)$, so that near the optimal value we expect small errors in α_{opt} to lead to similar fractional errors in the attitude-estimate-error covariance.

The level of agreement between "filter QUEST" and the traditional Kalman filter is quite high, better than to a few percent over the entire range of variables. One is tempted to ask if a less simple system model in which the measurement-noise and process-noise covariance matrices were very far from commuting would lead to the same level of agreement. The answer to this question is probably in the negative, although the performance of "filter QUEST" would probably be adequate in all but the most pathological cases. To this must be added the remark that, as a practical matter, models of process-noise covariance are usually very simple, essentially, the model we have used in the examples above. Moreover, except in the journal and conference literature there has been a tendency in spacecraft attitude applications to avoid the use of Kalman filters altogether in favor of simpler (and usually better behaved) batch algorithms. The present offering has the very modest goal of making a very popular batch algorithm more useful. We do not wish to break new ground here but only to make the old ground a bit easier to work.

The benefits to be derived from studies of the Wahba problem seem to be far from exhausted. Variations of the QUEST algorithm have been used to estimate spacecraft sensor misalignments [9]. Recently, Markley has extended his singular-value decomposition solution of the Wahba problem [10] to the estimation of other parameters, in particular, gyro biases [11]. A similar extension exists also for the QUEST implementation [12]. The QUEST algorithm also provides a useful data-compressor for the traditional Kalman filter [13]. The Wahba problem, posed so inconspicuously more than two decades ago, would appear to have a good deal of life to it yet and will probably yield additional insights in the future.

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