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IDENTIFICATION OF VIBRATIONAL MODES FOR A NON-RIGID SPACECRAFT

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ABSTRACT

A methodology is presented for the identification of spacecraft vibrational modes based on maximum-likelihood estimation techniques and realistic models for spacecraft sensor error sources. Special features of the problem, which make possible a partitioned estimation scheme leading to important computational savings, are presented.

INTRODUCTION

The identification of spacecraft attitude and attitude system parameters, even for rigid spacecraft, can present formidable computational burdens. Therefore, in practice, statistically consistent estimation schemes have not been employed, thereby making accuracy assessment difficult. Over the past few years, a consistent and computationally efficient program has gradually been developed based on maximum likelihood estimation techniques and realistic sensor error models. The principal results of that methodology are presented here as they pertain to the identification of the vibrational modes of a non-rigid spacecraft.

The goal of a system identification scheme in the present case is to determine those unknown or poorly-determined parameters which affect the estimation of spacecraft attitude. For the present study, these are the characteristic frequencies of vibrational modes and their equilibria. In practice, it has not been possible to estimate these equilibria adequately from prelaunch calibrations. Once these quantities are determined from inflight data, the estimation of spacecraft attitude, or the orientation in space of any other part of the spacecraft structure, becomes straightforward.

A complete presentation of this work is impossible within the scope of this short note. The present report, therefore, will only outline the main components of the methodology for a somewhat simplified application. The statement and derivations of more general results can be found in works cited in the references.

THE MODEL

Consider a non-rigid spacecraft equipped with three-axis gyros and n line-of-sight sensors, inertially stabilized in almost torque-free space. The non-rigid dynamics of the spacecraft, at a level of detail consistent with the n sensors, may be represented in terms of a quaternion $\bar{q}(t)$ describing the spacecraft attitude and n quaternions, $\bar{q}_i(t), i = 1, \ldots n$, describing the alignment of the line-of-sight sensors with respect to the spacecraft body coordinate axes. In general, the attitude motion of the sensors relative to the spacecraft body axes, is greatly restricted. Therefore, the orientation of these n sensors may be parameterized in terms of a three-vector of small angles [1], $\boldsymbol{\theta}_i(t), i = 1, \ldots n$, measured from the nominal alignment $\bar{q}_i^{(o)}$ of these sensors, which is assumed to be constant in time. Since the equilibrium value of the misalignments is not known *a priori*, it useful to decompose $\boldsymbol{\theta}_i(t)$ into two terms, the equilibrium misalignment $\boldsymbol{\vartheta}_i(t)$, which is, in fact, constant in time, and the dynamical portion of the misalignment $\boldsymbol{\varphi}_i(t)$.

If it is assumed now for simplicity that the spacecraft is inertially stabilized and at sufficiently high altitude that persistent external torques are small compared to random disturbances, then the coupling of the small spacecraft center-of-mass motion to that of the sensor may be neglected and the equations of motion of the sensors can be written as

$$\dot{\boldsymbol{\vartheta}}_i(t) = \mathbf{0}\,, \qquad \dot{\boldsymbol{\varphi}}_i(t) = \boldsymbol{\omega}_i(t)\,, \qquad \dot{\boldsymbol{\omega}}_i(t) = \sum_{j=1}^{n} K_{ij}\,\boldsymbol{\varphi}_j(t) \qquad i = 1, \ldots, n, \tag{1}$$

where the K_{ij} are constant matrices, which depend upon the inertial and elastic properties of the spacecraft. The equations of motion of the spacecraft attitude are given by

$$\dot{\bar{q}}(t) = \frac{1}{2}\Omega(\boldsymbol{\omega}(t))\,\bar{q}(t) \quad , \quad \dot{\boldsymbol{\omega}}(t) = \mathbf{g}(t) + \boldsymbol{\eta}(t), \tag{2}$$

where $\mathbf{g}(t)$ is the output of the three-axis gyros, taken to be quasi-continuous, and $\boldsymbol{\eta}(t)$ is the gyro noise, whose particular random properties will not be important in the present work. In general, gyro noise is smaller than random dynamical disturbances, hence the use of three-axis gyros in a *dynamical replacement mode* [2] eliminates the need to model the attitude dynamics of the spacecraft. Thus, nominally the state will consist of the spacecraft attitude quaternion, n misalignment vectors, and n angular velocities, and whatever additional state variables are necessary to model the dynamics of the gyro random processes.

In addition to gyro readings, the spacecraft attitude determination system performs measurements at intervals Δt, of the lines of sight of n celestial objects. In general these are the directions of stars, the Sun, the Earth, or the geomagnetic field. For the present model these measurements are taken to be the measured direction expressed as a unit vector $\mathbf{U}_i(t_k)$ in the sensor frame and related to the known direction $\mathbf{V}_i(t_k)$ in the inertial frame according to

$$\mathbf{U}_i(t_k) = A^T(\bar{q}_i(t_k))\,A(\bar{q}(t_k))\,\mathbf{V}_i(t_k) + \Delta\mathbf{U}_i(t_k) \quad i = 1, \ldots n, \tag{3}$$

where $A(\cdot)$ gives the dependence of the spacecraft attitude matrix on the attitude quaternion or the equivalent dependence of the sensor alignment matrix on the sensor alignment quaternion. The random measurement noise, $\Delta\mathbf{U}_i(t_k)$ is assumed to be zero mean and have a covariance matrix given by [3]

$$E\{\Delta\mathbf{U}_i(t_k)\,\Delta\mathbf{U}_j^T(t_k)\} = \sigma_i^2\,\delta_{ij}\,[I_{3\times3} - \mathbf{U}_i(t_k)\,\mathbf{U}_j^T(t_k)]. \tag{4}$$

The line-of-sight measurements depend explicitly on the spacecraft attitude.

PARTITIONING INTO CENTER-OF-MASS AND RELATIVE MOTION

The dependence of the measurements on both the spacecraft attitude and the alignments of the sensors relative to the spacecraft leads to a coupling of these dynamical variables. In addition, the gyro error sources are correlated in time, which further increases the computational burden, requiring a Kalman filter of very high dimension. A solution to this problem is to represent the sensor data in terms of pseudo-measurements which are independent of the spacecraft attitude. If there are n line-of-sight sensors, then these provide $2n$ equivalent scalar measurements. Three linear combinations of these measurements, effectively, are used to determine the attitude leaving $2n-3$ equivalent measurements which can depend on the sensor misalignments alone. The pseudo-measurement proposed here is

$$z_{ij}(t_k) \equiv \mathbf{W}_i^{(o)}(t_k) \cdot \mathbf{W}_j^{(o)}(t_k) - \mathbf{V}_i(t_k) \cdot \mathbf{V}_j(t_k), \tag{5}$$

where $\mathbf{W}_i^{(o)}(t_k) = A(\bar{q}_i^{(o)})\,\mathbf{U}_i(t_k)$ is the estimated observation in spacecraft body coordinates. It is a simple matter to show that

$$z_{ij}(t_k) = (\mathbf{W}_i^{(o)}(t_k) \times \mathbf{W}_j^{(o)}(t_k)) \cdot (\boldsymbol{\theta}_i(t_k) - \boldsymbol{\theta}_j(t_k)) + \eta_{ij}(t_k), \tag{6}$$

where $\eta_{ij}(t_k)$ is zero mean and has nonvanishing covariances given by

$$E\{\eta_{ij}^2(t_k)\} = (\sigma_i^2 + \sigma_j^2)\,|\mathbf{W}_i^{(o)}(t_k) \times \mathbf{W}_j^{(o)}(t_k)|^2\,, \tag{7}$$

$$E\{\eta_{ij}(t_k)\,\eta_{i\ell}(t_k)\} = \sigma_i^2\,(\mathbf{W}_i^{(o)}(t_k) \times \mathbf{W}_j^{(o)}(t_k)) \cdot (\mathbf{W}_i^{(o)}(t_k) \times \mathbf{W}_\ell^{(o)}(t_k))\,, \quad j \neq \ell. \tag{8}$$

For n sensors there are $n(n+1)/2$ pseudo-measurements z_{ij}, of which no more than $2n-3$ are statistically independent. Thus, the pseudo-measurements are redundant for $n > 3$. However, if the n unit vectors $\mathbf{W}_i^{(o)}(t_k)$ at t_k are not pairwise parallel, then one can show that the set of pseudo-measurements z_{1j}, $j = 2, \ldots, n$ and z_{2j}, $j = 3, \ldots, n$ form a maximal statistically independent set. Otherwise, a singular value decomposition or a square-root information filter [4] must be employed to obtain a statistically meaningful set of pseudo-measurements.

ESTIMATION OF VIBRATIONAL PARAMETERS

Having now delineated a set of pseudo-measurements which are statistically independent and decoupled from the center of mass motion, it is now possible to apply maximum likelihood methods [5] to estimate the parameters. Let $\mathbf{Z}_k$ denote the $(2n-3)$-dimensional stacked vector of effective measurements at time t_k, with measurement error vector denoted by $\mathbf{v}_k$. The measurement noise vector, $\mathbf{v}_k$, is a discrete zero-mean white Gaussian process whose covariance is formed from the covariances of equs. (7) and (8). These now become imputs to a Kalman filter [6], whose state vector may be written as

$$\mathbf{x}(t) \equiv [\boldsymbol{\vartheta}_1^T(t), \ldots, \boldsymbol{\vartheta}_n^T(t), \boldsymbol{\varphi}_1^T(t), \ldots, \boldsymbol{\varphi}_n^T(t), \boldsymbol{\omega}_1^T(t), \ldots, \boldsymbol{\omega}_n^T(t)]^T \tag{9}$$

and whose state equations are given by equs. (1). The measurement equation becomes

$$\mathbf{Z}_k = H_k\,\mathbf{x}_k + \mathbf{v}_k, \tag{10}$$

where H_k is the sensitivity matrix of $\mathbf{Z}_k$ to $\mathbf{x}_k$, which may be constructed from equ. (6).

Let $\boldsymbol{\alpha}$ be the parameter vector, consisting of the values of $\boldsymbol{\vartheta}(0)$ and the time constants which characterize the K_{ij}. Then both $\mathbf{Z}_k$ and $\mathbf{x}_k$ in equ. (1) above depend on $\boldsymbol{\alpha}$, as does the transition matrix Φ_k (through K_{ij})

$$\mathbf{x}_{k+1}(\boldsymbol{\alpha}) = \Phi_k(\boldsymbol{\alpha})\,\mathbf{x}_k(\boldsymbol{\alpha}). \tag{11}$$

Equations (10) and (11) are the basis for a Kalman filter, as part of whose mechanization we calculate the innovation $\boldsymbol{\nu}_k(\boldsymbol{\alpha})$ and innovation covariance $B_k(\boldsymbol{\alpha})$ [6], which depend on α through the filter equations. The negative-log-likelihood function [5] J, is readily computable in terms of these two quantities as

$$J = \frac{1}{2}\sum_{k=1}^{N}\left\{\boldsymbol{\nu}_k(\boldsymbol{\alpha})^T B_k(\boldsymbol{\alpha})^{-1}\boldsymbol{\nu}_k(\boldsymbol{\alpha}) + \log\det B_k(\boldsymbol{\alpha}) + (2n-3)\log 2\pi\right\}. \tag{12}$$

The value of $\boldsymbol{\alpha}$ which minimizes J is by definition the maximum-likelihood estimate $\hat{\boldsymbol{\alpha}}_{ML}$. Provided that J has an isolated minimum, then $\hat{\boldsymbol{\alpha}}_{ML}$ can be found by solving the equations [5]

$$\begin{aligned}\frac{\partial J}{\partial \alpha_m} &= \sum_{k=1}^{N}\left\{\frac{\partial \boldsymbol{\nu}_k(\boldsymbol{\alpha})}{\partial \alpha_m}^T B_k(\boldsymbol{\alpha})\,\boldsymbol{\nu}_k(\boldsymbol{\alpha}) - \frac{1}{2}\,\boldsymbol{\nu}_k(\boldsymbol{\alpha})^T B_k^{-1}(\boldsymbol{\alpha})\,\frac{\partial B_k(\boldsymbol{\alpha})}{\partial \alpha_m}\,B_k^{-1}(\boldsymbol{\alpha})\,\boldsymbol{\nu}_k(\boldsymbol{\alpha}) + \frac{1}{2}\,\mathrm{tr}\left[B_k^{-1}(\boldsymbol{\alpha})\,\frac{\partial B_k(\boldsymbol{\alpha})}{\partial \alpha_m}\right]\right\}\\ &= 0,\end{aligned} \tag{13}$$

which may be solved for $\boldsymbol{\alpha}$ using the Newton-Raphson method. The derivatives of $\boldsymbol{\nu}_k(\boldsymbol{\alpha})$ and $B_k(\boldsymbol{\alpha})$ are computed by differentiating the Kalman filter equations with respect to $\boldsymbol{\alpha}$. Asymptotically, the Kalman filter furnishes the estimate error covariance for $\boldsymbol{\alpha}$ as the inverse of the Fisher information matrix, which is given approximately by [7,8]

$$F_{\ell m} = \sum_{k=1}^{N} \left\{ \frac{1}{2} \operatorname{tr} \left[B_k^{-1}(\boldsymbol{\alpha}) \, \frac{\partial B_k(\boldsymbol{\alpha})}{\partial \alpha_\ell} \, B_k^{-1}(\boldsymbol{\alpha}) \, \frac{\partial B_k(\boldsymbol{\alpha})}{\partial \alpha_m} \right] + \left[\frac{\partial \boldsymbol{\nu}_k(\boldsymbol{\alpha})}{\partial \alpha_\ell}^T \, B_k^{-1}(\boldsymbol{\alpha}) \, \frac{\partial \boldsymbol{\nu}_k(\boldsymbol{\alpha})}{\partial \alpha_m} \right] \right\} . \qquad (14)$$

DISCUSSION AND CONCLUSIONS

We have presented a formalism for computing the characteristic frequencies and equilibria for the vibrational modes of a non-rigid spacecraft. This work is a generalization of the estimation of static misalignments described in Refs. [1, 4]. A number of generalizations of this methodology are possible and have been applied to other complex systems with good results. In particular, the state equations (1) may be augmented to include random error sources. The computational burden entailed by this additional complication is much greater than that of the example treated here, but further partitioning of both the state and the parameter vector reduces this considerably.

A matter which we have gracefully overlooked is the problem of observability. Since at any measurement time there are only $2n$ measurements but at least $3n + 3$ components of the state vector, it is clear that the system may not necessarily be easily observable. The observability problem cannot necessarily be solved simply by collecting more data, because even in the rigid-body case [1] it can be shown that a common misalignment of all the sensors cannot be distinguished from a change in the spacecraft attitude. Thus, the estimation of the misalignments, static or dynamic, *and* the attitude requires that prelaunch calibration provide a prior distribution. The usability of that prelaunch calibration may be questioned since it must be propagated across launch, which entails further unknown error sources. The results of Ref. [1], however, indicate that both the error levels in the prelaunch alignments and the destructive powers of launch shock may both have been needlessly exaggerated.

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8. Note that Ref. [5] contains an error in its result for equ. (14).