TECHNICAL NOTE

A Comment on Fast Three-Axis Attitude Determination

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Abstract

The solutions to the Wahba problem for three-axis attitude determination, first posed in 1965, are reviewed. A recent solution proposed by Tietze is examined in detail.

The Wahba Problem and its Solutions

Recently, Tietze [1] has proposed a new method for computing least-squares three-axis attitude. This algorithm offers a supposedly faster method for computing the solution to an optimal least-squares attitude estimation problem first proposed by Wahba [2], namely, to find the special orthogonal matrix R which minimizes the loss function

$$L(R) = \frac{1}{2}\sum_{i=1}^{n} a_i|\hat{\mathbf{W}}_i - R\hat{\mathbf{V}}_i|^2 \tag{1}$$

where $\hat{\mathbf{W}}_i$ $(i = 1, \ldots, n)$ and $\hat{\mathbf{V}}_i$ $(i = 1, \ldots, n)$ are sets of observation and reference unit vectors, respectively, and a_i $(i = 1, \ldots, n)$ are a set of positive weights, which may be normalized to have unit sum.

The solution of this problem has a long (and continuing) history. The earliest published solutions [3–8] solve this problem directly for the rotation matrix (taking the six constraints dictated by orthogonality into explicit account) with the result

$$R = B\Lambda^{-1} \tag{2}$$

where

$$B = \sum_{i=1}^{n} a_i\hat{\mathbf{W}}_i\hat{\mathbf{V}}_i^T \tag{3}$$

and Λ is the symmetric positive definite solution to the equation

$$\Lambda^2 = B^TB \tag{4}$$

A different approach was followed by Davenport [9], who avoided the need to compute the matrix square root by solving instead for the Gibbs vector [10], which was uncon-

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strained. This approach led to an iterative solution for the Gibbs vector which could be shown to converge provided that a unique solution existed for the attitude. A variation of this algorithm was used in support of the Orbiting Astronomical Observatory [11]. Further refinements were developed by Fraiture [12] and by Davenport [13].

The most important advance was made by Davenport (as reported by Keat [14]), who showed that the quaternion representation of the optimal attitude was the eigenvector with maximum eigenvalue of the 4 × 4 matrix

$$K = \left[\begin{array}{c|c} S - \sigma I & \mathbf{Z} \\ \hline \mathbf{Z}^T & \sigma \end{array}\right] \tag{5}$$

where

$$S = B + B^T \tag{6}$$

$$\sigma = \mathrm{Tr}\, B \tag{7}$$

and

$$\mathbf{Z} = (B_{23} - B_{32}, B_{31} - B_{13}, B_{12} - B_{21})^T \tag{8}$$

The optimal quaternion could now be found by solving the eigenvalue equation

$$K\bar{q}_{opt} = \lambda_{max}\bar{q}_{opt} \tag{9}$$

The "q-algorithm" was used in support of the High Energy Astromony Observatory (HEAO) [15, 16] and was also part of the ground support software for the Solar Maximum Mission [17].

The solution of the eigenvalue problem was much less burdensome computationally than the evaluation of the matrix square root or the iterative algorithms developed earlier but still somewhat costly. A large share of the remaining computational burden was eliminated by Shuster [18, 19], who showed that, without loss of usable accuracy, the eigenvalue could be set equal to

$$\lambda_{max0} = \sum_{i=1}^{n} a_i = 1 \tag{10}$$

For the Magsat mission, in fact, for which the sensor accuracies were on the order of 20 arc sec, this approximation was correct to eight significant figures. This substitution converted the eigenvalue problem into a simple algebraic problem, and the Gibbs vector ($\mathbf{Y} = \mathbf{q}/q_4$) for the optimal rotation was given by

$$\mathbf{Y}_0 = [(\lambda_{max0} + \sigma)I - S]^{-1}\mathbf{Z} \tag{11}$$

to this same number of significant figures.

Equation (11) is nothing more than the solution of the first three rows of equation (9) with λ_{max} replaced by λ_{max0}. As is pointed out in [19, 20], this scheme must fail if the angle of rotation is close to π (because the Gibbs vector becomes infinite at that value). When this situation occurs, one simply solves three different rows of equation (9). An alternate procedure, which sidesteps the possibility of a singularity by avoiding the computation of the Gibbs vector as an intermediate quantity, was also developed in

[19]. This nonsingular implementation still becomes inaccurate when the angle of rotation is close to π due to large cancellations which occur then. For that case, a very simple transformation of the matrix B (the signs of two of the rows are changed) leads to a numerically well-behaved problem which yields the identical quaternion components without loss of significance. This transformation is not needed very often, however. For the Magsat mission, for example, if one were willing to accept a numerical accuracy of 10^{-4} arc sec, it was estimated that this transformation would have to be applied once in 250 million attitude computations.

The nonsingular implementation has the added advantage that the quantity $1 - \lambda_{max}$ is obtained as an intermediate result. This quantity has proved to be extremely useful as a measure of the consistency of the data for fault isolation and also for system identification. Because $1 - \lambda_{max}$ is calculated to lowest nonvanishing order, the computation of the quaternion is more accurate as well.

Reference [20] repeated the derivations of [19] and extended this work by providing a complete covariance analysis of the algorithm, as well as showing its connection to the deterministic triad algorithm. Reference [21] used this algorithm further as the basis for developing efficient algorithms for the inflight estimation of attitude-sensor accuracies and misalignments. The algorithm of [19] and [20] was incorporated in the ground support software for the Magsat spacecraft [22]. In addition to extensive prelaunch testing, the algorithm was exercised more than 80 million times during the course of that mission. The algorithm has also been selected for implementation in the ground support systems of the Earth Radiation Budget Satellite (ERBS) [23] and the Space Telescope [24].

A recent solution to the Wahba problem has also been offered by Bar-Itzhack and Reiner [25], who compute the attitude matrix directly in an extended Kalman filter, without the orthogonality constraint. The orthogonality is then restored post hoc using an efficient iterative process [26, 27]. (There exists also an optimal algorithm [28] for orthogonalizing an approximate attitude matrix, which rests to some degree on the solution of the Wahba problem.) The computational burden of calculating the attitude matrix in this manner is necessarily large but still workable if the attitude data rates can be accommodated. Bar-Itzhack and Oshman [29] implement the extended Kalman filter to estimate the quaternion directly with greater efficiency than is possible for the direct estimation of the attitude matrix.

Tietze's Solution

The development of Tietze's algorithm [1] follows extremely closely the development of [19] and [20]. In fact, of the twenty-five equations in [1], the first twenty-two simply repeat virtually identical equations in [20] (not cited by Tietze), with only slightly different notation and conventions. The sole innovation of [1] is the suggestion that the optimal quaternion be computed from equation (9) using the inverse-iteration method [30]. This procedure, Tietze contrasts with standard procedures for solving the complete eigenvector-eigenvalue problem for a symmetric system, namely "to transform the matrix to tridiagonal form, and then to find the eigenvalues and eigenvectors. Finally, the resulting eigenvectors are back-transformed to the original coordinates." The implication of this statement is that this is the method followed in the earlier work to which Tietze refers.

This, however, is not the case. Reference [19] (the report cited by Tietze) showed that it was not necessary to solve an eigenvalue problem at all, no matter how stringent the accuracy requirements of the mission. If the measurements have a typical error, ε, then equations (10) and (11) will be correct to within terms of order ε^2. There is clearly no need to construct an algorithm which is capable of solving this problem with greater accuracy. The implementation now in use in NASA mission software, however, because it also computes $1 - \lambda_{max}$ to lowest nonvanishing order, is correct to within terms of order ε^4. It is difficult to imagine a situation for which this level of accuracy is not adequate.

Tietze would replace this simple one-step algorithm, which requires no more than one inversion of a well-behaved 3×3 matrix, with an iterative procedure which requires at each iteration the solution of four ill-conditioned equations. This he carries out by Gaussian elimination. Apart from the final renormalization of the quaternion, his procedure requires 16 floating-point multiplications (or divisions) to set up the Gaussian elimination, 13 floating-point multiplications for the first iteration, and 16 for each succeeding iteration. Thus, the three iterations of Tietze's examples require 61 floating-point multiplications. This should be compared with only 51 floating-point multiplications for the supposedly slower nonsingular method of [19] and [20]. The computational burden of Tietze's approach is seen to be even greater if it is noted that this accounting does not include the computation of the eigenvalue, which requires an additional four floating-point multiplications, bringing the total number of floating-point multiplications for Tietze's implementation to 65. If only a single iteration is necessary and the value of $1 - \lambda_{max}$ is not computed, then Tietze's implementation requires 29 multiplications compared to only 16 multiplications for the method of [19] and [20] if equation (11) is evaluated by Gaussian elimination. Thus, Tietze's supposedly fast algorithm requires roughly from 30 to 80 percent more floating-point multiplications than the method currently in use.

A word should be said about the problem of singular cases. It has been pointed out that the algorithm of [19] and [20] behaves poorly when the angle of rotation is close to π (or, equivalently, when the fourth component of the quaternion is very small). A similar situation also occurs in Tietze's implementation when the starting quarternion is nearly orthogonal to the desired solution (or, equivalently, when the component of the quarternion along the starting value is small). The two situations occur with approximately equal frequency in the two respective approaches (and with a very low frequency at that). The necessary repair in each case results in a doubling of the number of computations. Thus, Tietze's implementation does *not* avoid the problem of bad cases indicated in [19] and [20], nor does it treat them more efficiently.

Examining [1] in detail, we remark that equation (14) of that work gives the vector **Z**, with the wrong sign. That equation is identical to an equation appearing in [20], but Tietze has adopted a different convention for one of the symbols. However, that expression does not seem to have been used in his calculations, since the values for **Z** presented in his tables have the correct sign.

It may be remarked that, for the case where there are three mutually orthogonal measurements with equal weights and equal accuracies (the case in Tietze's examples), the loss function has a distribution which is given approximately by

$$L(R_{opt}) = \frac{1}{2}\sigma^2\chi_3^2 \tag{12}$$

where σ^2 is the variance of the measurement error per axis and χ_3^2 is a chi-square variable with three degrees of freedom. For Gaussian measurement errors the statement is exact. Equation (12) is consistent with the expression for the expected value of λ_{max} given in [21], except that Tietze's convention for the a_i has been followed. From [1]

$$\sigma^2 = (0.017)^2/3 \tag{13}$$

$$L(R_{opt}) = 0.162 \times 10^{-3} \tag{14}$$

from which it follows that

$$\chi_3^2 = 3.37 \tag{15}$$

The expected value of this variable is 3 and the variance is 6. Thus, Tietze's numerical example is statistically very typical. Since the convergence properties of the algorithm are determined by $L(R_{opt})$, this means that the number of iterations required by Tietze is also typical.

Acknowledgement

The author is grateful to Professor I. Y. Bar-Itzhack for making him aware of some earlier work on this subject and to P. B. Davenport, E. J. Lefferts, and F. L. Markley for many interesting discussions on this topic over the years.

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