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# Kalman Filtering for Spacecraft Attitude Estimation

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## I. Introduction

THIS report reviews the methods of Kalman filtering in attitude estimation and their development over the last two decades. This review is not intended to be complete but is limited to algorithms suitable for spacecraft equipped with three-axis gyros as well as attitude sensors. These are the systems to which we feel that Kalman filtering is most applicable.

The Kalman filter uses a dynamical model for the time development of the system and a model of the sensor measurements to obtain the most accurate estimate possible of the system state using a linear estimator based on present and past measurements. It is, thus, ideally suited to both ground-based and on-board attitude determination. However, the applicability of the Kalman filtering technique rests on the availability of an accurate dynamical model.

The dynamic equations for the spacecraft attitude pose many difficulties in the filter modeling. In particular, the external torques and the distribution of momentum internally

due to the use of rotating or rastering instruments lead to significant uncertainties in the modeling. For autonomous spacecraft the use of inertial reference units as a model replacement permits the circumvention of these problems. In this representation the angular velocity of the spacecraft is obtained from the gyro data. The kinematic equations are used to obtain the attitude state and this is augmented by means of additional state-vector components for the gyro biases. Thus, gyro data are not treated as observations and the gyro noise appears as state noise rather than as observation noise.

It is theoretically possible that a spacecraft is three-axis stabilized with such rigidity that the time development of the system can be described accurately without gyro information, or that it is one-axis stabilized so that only a single gyro is needed to provide information on the time history of the system. The modification of the algorithms presented here in order to apply to those cases is slight. However, this is of little practical importance because a control system capable of such

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Eugene J. Lefferts was born in 1925 in New York City and grew up in Philadelphia. He received the B.S. in Education from Westchester State College, Westchester, Pennsylvania in 1951 and the M.A. in Mathematics from Temple University in Philadelphia in 1953. From 1953 until 1967 he was at the Glenn L. Martin Company (now Martin-Marietta Corporation), where he worked on problems of analog simulation, optimal control, filtering theory, and Lyapunov stability theory. From 1969 until 1983 he was a member of the RIAS group at Martin. Since 1967 he has been with NASA Goddard Space Flight Center in Greenbelt, Maryland and since 1972 he has been head of the Attitude Dynamics Section of the Mission and Data Operations Division at GSFC. He has written numerous articles on estimation, control, and stability and has been responsible for attitude mission analysis for numerous NASA missions including recently: Applied Technology Satellite 6 (ATS-6), Solar Maximum Mission, and Landsat-D.

F. Landis Markley was born in 1939 in Philadelphia, Pennsylvania and attended public schools in Cheltenham Township, Pennsylvania. He received a Bachelor of Engineering Physics from Cornell University in 1963 and the Ph. D. in Physics from the University of California at Berkeley in 1967. From 1967 to 1968 he was a National Science Foundation Post Doctoral Fellow at the center for Theoretical Physics at the University of Maryland and from 1968 to 1974 he was assistant Professor of Physics at Williams College, Williamstown, Massachusetts. In 1974 he joined the Systems Sciences Division of the Computer Sciences Corporation in Silver Spring, Maryland, where he was active in mission analysis in software development for several NASA missions, among them, NIMBUS and the Solar Maximum mission, and also in general spacecraft dynamics and simulation studies. Since 1978 Dr. Markley has been a physicist in the Aerospace Systems Division of the Naval Research Laboratory, where he has continued to work extensively in the areas of spacecraft attitude and orbit dynamics, estimation and control. Dr. Markley is the author of numerous journal articles and conference reports in theoretical physics and in Aerospace and Astrodynamics studies. He was also one of the major contributors to the book, "Spacecraft Attitude Determination and Control," edited by J.R. Wertz.

Malcolm D. Shuster was born in 1943 in Boston, Massachusetts and attended public schools in greater Boston. He received the S.B. in Physics from the Massachusetts Institute of Technology, the Ph.D. in Physics from the University of Maryland, and a M.S. in Electrical Engineering from the Johns Hopkins University. Between 1970 and 1977 he pursued an academic career in theoretical physics during which he held positions with the French Atomic Energy Commission, Paris, France (engineer-physicist, 1970-72), the University of Karlsruhe, West Germany (instructor, 1972-73), Tel-Aviv University, Israel (lecturer, 1973-76) and Carnegie-Mellon University, Pittsburgh, Pennsylvania, (visiting assistant professor, 1976-77). During this period he carried out research on the interaction of elementary particles with nuclei and taught and lectured extensively in English, French and German. In 1977 he joined the System Sciences Division of the Computer Sciences Corporation in Silver Spring, Maryland, where he provided mission analysis and software support for a number of NASA missions including: Sonnet, Magast, Applications Explorer Mission, Solar Maximum Mission, Dynamics Explorer Mission, and the Space Telescope. Since 1981 he has been with Business and Technological Systems, Inc., where he works on theoretical and practical problems of estimation, dynamics, and control for space and defense systems. Dr. Shuster has written numerous articles on theoretical physics, estimation, and control and is currently preparing a book on spacecraft attitude estimation.

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accuracy invariably includes gyros in the control loop and these may be sampled to obtain the full angular velocity. Therefore, with little loss of generality, the algorithms here are specialized to spacecraft with three-axis gyros.

Throughout this report the attitude is represented by the quaternion. The development of the Kalman filter for the quaternion representation was motivated by the requirement of real-time autonomous attitude determination for attitude control and the annotation of science data. The quaternion parameterization was chosen for several practical reasons: 1) the prediction equations are treated linearly, 2) the representation is free from singularities (thus the gimbal-lock situation is avoided), and 3) the attitude matrix is algebraic in the quaternion components (thus eliminating the need for transcendental functions).

The use of the quaternion as the attitude state presents some difficulty in the application of the filter equations. This difficulty is due to the lack of independence of the four quaternion components, which are related by the constraint that the quaternion have unit norm. This constraint results in the singularity of the covariance matrix of the quaternion state. The various ways to treat or circumvent this difficulty make up the major part of this report. Each of the filters discussed predicts the quaternion state in the same manner as if each component were independent. Different approaches to the update equations and to the covariance representation and its propagation give rise to the different methods presented.

In principal, one can also estimate other sensor biases by suitably augmenting the state vector. In general, however, these other biases are not estimated simultaneously with the attitude due to problems of observability under normal spacecraft operating conditions. Therefore, the estimation methods for sensor biases other than gyro drift biases will not be treated here.

Section II of this report reviews the relevant literature on Kalman filtering as applied to attitude estimation and on related topics. Since the attitude problem is nonlinear, the vehicle for optimal estimation is the extended Kalman filter, which is reviewed in Sec. III without derivation. Attitude kinematics are reviewed in Sec. IV, with emphasis on the quaternion representation. A discussion of gyros used in the model replacement mode and the modeling of attitude sensors is presented in Sec. V.

The state equation and the equation for attitude prediction are derived in Sec. VI. Various approaches to the filtering equations are discussed in the succeeding five sections. Section XII reviews the advantages of the different methods. Several results of interest have been gathered in the Appendix.

The intent of the present work is to provide a complete account of what now seem to be the relevant filtering algorithms rather than to present a new approach. An attempt has been made to present the current methods in a common framework in order to produce a useful reference for future applications.

## II. Historical Survey

### Early Applications of Kalman Filtering

The Kalman filter,<sup>1,2</sup> which was originally developed as a tool in linear estimation theory, was soon applied to nonlinear orbital guidance and navigation problems in the Apollo program by Schmidt and his collaborators.<sup>3,5</sup> Almost simultaneously with Kalman's work, Swerling<sup>4,7</sup> developed a recursive theory for satellite navigation which differed from Kalman's in the treatment of the process noise and was not carried out as completely. Standard treatments of Kalman filtering can be found in the review of Sorenson<sup>8</sup> and the textbooks of Jazwinski<sup>9</sup> and Gelb.<sup>10</sup>

An accurate and complete historical survey of Kalman filtering for attitude estimation is not possible since the earliest applications were directed toward national defense and hence could not be published in the open literature. The

earliest published reference is by Farrell,<sup>11,12</sup> who studied the extent to which Kalman filtering of crude attitude measurements from sun sensors and magnetometers could provide attitude accuracy equivalent to that obtained without smoothing from more elaborate instrumentation. Farrell represented the attitude by Euler angles and assumed torque-free motion in the attitude prediction. Cherry and O'Connor,<sup>13</sup> in their design of the lunar excursion module autopilot, considered sequential estimation of the disturbance torques induced by the ascent or descent propulsion system. Potter and Vander Velde<sup>14</sup> used Kalman filtering theory to determine the optimum mixing of gyroscope and star tracker data in an attitude determination system. Generally, as remarked by Sabroff,<sup>15</sup> the application of Kalman filtering to attitude estimation had not shown impressive results up to 1967. Aside from insufficient study, the lack of real success in applying optimal estimation was caused by the inability to model the system dynamics accurately.

Continued effort in this area was evidenced at a symposium on spacecraft attitude determination in 1969, at which six papers were presented on the application of Kalman filtering to this problem. Foudriat<sup>16</sup> and Arneson and Nelson<sup>17</sup> considered spin-stabilized satellites, while Ribarich<sup>18</sup> and Lesinski<sup>19</sup> were interested in the dual-spin case. Pauling, Jackson, and Brown<sup>20</sup> and Toda, Heiss, and Schlee<sup>21</sup> studied the space precision attitude reference system (SPARS), which used gyro measurements in a model replacement mode with periodic star-sensor updates. Pauling et al.<sup>20</sup> used the attitude matrix for state prediction and Euler angles for updates, while Toda et al.<sup>21</sup> used the quaternion for prediction and incremental error angles in the update. Jackson<sup>22</sup> also contributed a paper to the symposium on the application of nonlinear estimation theory to the attitude determination problem, an application that was also studied by Kau, Kumar, and Granley.<sup>23</sup>

In a 1971 review of strapdown navigation by Edwards,<sup>24</sup> Kalman filtering is not treated but the use of the quaternion and error angles for state prediction is discussed. A review article by Schmidtbauer, Samuelsson, and Carlsson<sup>25</sup> extends the history to 1973 and contains a classification and discussion of extended Kalman filter methods for attitude determination, with specific emphasis on algorithms suitable for on-board computation.

### Attitude Representations

The nonlinear aspects of rotational kinematics are discussed in many texts.<sup>26-30</sup> Approaches which treat the spacecraft axes as uncoupled (thus ignoring commutation errors) have been studied by Potter and Vander Velde,<sup>14</sup> Ribarich,<sup>18</sup> Schmidtbauer et al.,<sup>25</sup> and Farrenkopf.<sup>31</sup> This approximation leads to independent linear estimation problems for the three spacecraft axes, and closed-form expressions for the steady-state estimation errors can be obtained in some cases.<sup>14,25,31</sup> Uncoupled-axis Kalman filters employing both gyro data and dynamics modeling have been used in on-board attitude control systems, in the NASA International Ultraviolet Explorer and Solar Maximum Mission<sup>22</sup> spacecraft, for example.

Stuepfnagel<sup>32</sup> and Markley<sup>34</sup> discuss different parameterizations of the attitude that have been used when the decoupled-axis approximation is not applicable. Early strapdown navigation systems used the direction-cosine matrix as the representation of the attitude.<sup>20,24</sup> Due to round-off, quantization, and truncation errors in the attitude propagation, this procedure results in an attitude matrix that is not orthogonal.<sup>33,36</sup> Various orthogonalization schemes for the attitude matrix were developed; but Giardina, Bronson, and Wallen<sup>34</sup> proved that an optimal orthogonalization (one that minimizes the sum of squares of the differences between the elements of the propagated matrix and those of the orthogonalized matrix) requires a computationally expensive

matrix square root. Because of this problem and the redundancy of the nine-parameter direction cosine representation, it has not been widely used recently.

The three-parameter Euler-angle representation<sup>26,28,33,34</sup> was used in several early applications of Kalman filtering to attitude estimation.<sup>11,16,17,19,28,33</sup> However the kinematic equations for Euler angles involve nonlinear and computationally expensive trigonometric functions, and the angles become undefined for some rotations (the gimbal-lock situation), which cause problems in Kalman filtering applications. Despite these difficulties, the Euler angle representation continues to be used for the attitude estimation of spinning spacecraft. Stuefpiegel<sup>33</sup> discusses two other three-parameter representations: the exponential of a  $3 \times 3$  skew-symmetric matrix (rotation-vector representation) and the Cayley parameterization (not to be confused with the Cayley-Klein parameters). The latter is equivalent to the Gibbs-vector parameterization.<sup>34</sup> Neither of these representations has found a Kalman filtering application, to our knowledge. Stuefpiegel proves that no three-parameter representation can be both global and nonsingular.

The global nonsingular four-parameter representation of the attitude in terms of Euler's symmetric parameters, or equivalently the four components of a quaternion, is discussed by many authors.<sup>26,28,33,37</sup> Quaternions were invented by Hamilton<sup>38</sup> in 1843; their use in attitude dynamics simulations was promoted by Robinson<sup>39</sup> and by Mitchell and Rogers.<sup>40</sup> The attitude matrix computed from a quaternion (as a homogeneous quadratic function) is orthogonal if the sum of squares of the quaternion components is unity. If propagation errors result in a violation of this constraint, the quaternion can be renormalized by dividing its components by the (scalar) square root of the sum of their squares; and Giardina et al.<sup>36</sup> showed that the attitude matrix computed from the renormalized quaternion is identical to the one given by their optimal orthogonalization. The application of quaternions to strapdown guidance, with error analyses, was discussed by Wilcox<sup>43</sup> and Mortenson.<sup>41</sup> Quaternion kinematics has been the subject of several recent studies.<sup>42,48</sup> Of particular concern has been the best method for extracting a quaternion from an attitude matrix.<sup>49,52</sup> A recent review of quaternion relations has been given by Friedland.<sup>53</sup>

The advantages of the quaternion parameterization have led to its frequent use in attitude determination systems. One example is an attempt by Lefferts and Markley<sup>54</sup> to model the attitude dynamics of the NIMBUS-6 spacecraft, which indicated that dynamics modeling with elaborate torque models could still not give acceptable attitude determination accuracy. For this reason, most applications of quaternion attitude estimation have used gyros in the dynamic model replacement mode. These include the work by Toda et al.<sup>21</sup> mentioned above, by the group at TRW for the precision attitude determination system (PADS),<sup>55</sup> which was incorporated in the attitude determination system for the High-Energy Astronomy Observatory (HEAO) mission,<sup>56,58</sup> by Yong and Headley<sup>59</sup> for highly maneuverable spacecraft, by Murrell<sup>60</sup> in a design for the NASA Multimission Modular Spacecraft, by Sorenson, Schmidt, and Goka<sup>61</sup> in a square-root filtering application, by Shuster and co-workers<sup>62,64</sup> for a microprocessor-based on-board attitude determination system, and by Markley<sup>65</sup> for an autonomous navigation study.

#### Gyro Noise Models

The first papers describing statistical models of gyro drifts were by Newton<sup>66</sup> and Hammon<sup>67</sup> in 1960. Newton considered additive white noise in the gyro drifts, while Hammon assumed that the gyro drift rates were an exponentially correlated random process. Dushman<sup>68</sup> considered a drift-rate model obtained by adding a random walk component to Hammon's autocorrelation function. Early implementations

of gyro noise models in Kalman filters were generally incomplete. Thus, Potter and Vander Velde<sup>14</sup> include only the random walk term in the drift rate, while both Pauling et al.<sup>20</sup> and Toda et al.<sup>21</sup> have only the white noise term. Farrenkopf<sup>31,69</sup> considered a gyro model including all the terms discussed above. This model was used in the subsequent HEAO system development.<sup>36,57</sup>

### III. The Kalman Filter

We review in this section the principal equations for the extended Kalman filter<sup>9,10</sup> in order to introduce the necessary notation for the sections which follow.

The state equation may be written as

$$\frac{d}{dt}x(t) = f(x(t), t) + g(x(t), t)w(t) \quad (1)$$

where  $x(t)$  is the state vector and  $w(t)$ , the process noise, is a Gaussian white-noise process whose mean and covariance function are given by

$$E\{w(t)\} = 0 \quad (2)$$

$$E\{w(t)w^T(t')\} = Q(t)\delta(t-t') \quad (3)$$

$E$  denotes the expectation and  $T$  the matrix transpose. The initial mean and covariance of the state vector are given by

$$E\{x(t_0)\} = \bar{x}(t_0) = x_0 \quad (4)$$

$$E\{[x(t_0) - x_0][x(t_0) - x_0]^T\} = P(t_0) = P_0 \quad (5)$$

#### Prediction

Given the initial conditions on the state vector and the state covariance matrix, the minimum variance estimate of the state vector at a future time  $t$  is given in the absence of measurements by the conditional expectation

$$\hat{x}(t) = E\{x(t) | \hat{x}(t_0) = x_0\} \quad (6)$$

This predicted estimate satisfies the differential equation

$$\frac{d}{dt}\hat{x}(t) = E\{f(x(t), t)\} = f(\hat{x}(t), t) \quad (7)$$

which we write approximately as

$$\frac{d}{dt}\hat{x}(t) = f(\hat{x}(t), t) \quad (8)$$

Equation (8) may be integrated formally to give

$$\hat{x}(t) = \phi(t, \hat{x}(t_0), t_0) \quad (9)$$

The state error vector and covariance matrix are defined by

$$\Delta x(t) = x(t) - \hat{x}(t) \quad (10)$$

$$P(t) = E\{\Delta x(t)\Delta x^T(t)\} \quad (11)$$

Neglecting terms which are higher than first order in the state error vector and the process noise, the state error vector satisfies the differential equation

$$\frac{d}{dt}\Delta x(t) = F(t)\Delta x(t) + G(t)w(t) \quad (12)$$

where

$$F(t) = \left. \frac{\partial}{\partial x} f(x, t) \right|_{\hat{x}(t)} \quad (13)$$

$$G(t) = g(\hat{x}(t), t) \quad (14)$$

Equation (12) may be integrated formally to give

$$\Delta x(t) = \Phi(t, t_0) \Delta x(t_0) + \int_{t_0}^t \Phi(t, t') G(t') w(t') dt' \quad (15)$$

where  $\Phi(t, t_0)$  is the transition matrix, which satisfies

$$\frac{\partial}{\partial t} \Phi(t, t_0) = F(t) \Phi(t, t_0) \quad (16)$$

$$\Phi(t_0, t_0) = I \quad (17)$$

Note that for nonlinear systems  $\Phi(t, t_0)$  also depends implicitly on  $\hat{x}(t_0)$ , which for notational convenience will be suppressed.

The predicted covariance matrix satisfies the Riccati equation

$$\frac{d}{dt} P(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) \quad (18)$$

which may be integrated to give

$$P(t) = \Phi(t, t_0) P(t_0) \Phi^T(t, t_0) + \int_{t_0}^t \Phi(t, t') G(t') Q(t') G^T(t') \Phi^T(t, t') dt' \quad (19)$$

In more compact notation,  $\hat{x}_k(-)$  and  $P_k(-)$  denote the predicted values of the state vector and state covariance matrix at time  $t_k$ , and  $\hat{x}_k(+)$  and  $P_k(+)$  denote the same quantities immediately following a measurement at time  $t_k$ . Thus, in obvious notation

$$\hat{x}_{k+1}(-) = \Phi(t_{k+1}, \hat{x}_k(+), t_k) \quad (20)$$

$$P_{k+1}(-) = \Phi_k P_k(+)\Phi_k^T + N_k \quad (21)$$

#### Filtering

The measurement vector at time  $t_k$  is related to the state vector by

$$z_k = h(x_k) + v_k \quad (22)$$

where  $v_k$ , the measurement noise, is a discrete Gaussian white-noise process

$$E[v_k] = 0 \quad (23)$$

$$E[v_k v_l^T] = R_k \delta_{kl} \quad (24)$$

The minimum variance estimate of  $x_k$  immediately following the measurement is given by

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k [z_k - h(\hat{x}_k(-))] \quad (25)$$

where the Kalman gain matrix is given by

$$K_k = P_k(-) H_k^T [H_k P_k(-) H_k^T + R_k]^{-1} \quad (26)$$

and the measurement sensitivity matrix is given by

$$H_k = \left. \frac{\partial h(x)}{\partial x} \right|_{x_k(-)} \quad (27)$$

The covariance matrix immediately following the measurement is given by

$$P_k(+) = (I - K_k H_k) P_k(-) \quad (28)$$

$$= (I - K_k H_k) P_k(-) (I - K_k H_k)^T + K_k R_k K_k^T \quad (29)$$

Equation (29) is the starting point for many efficient and numerically stable factored forms of the Kalman filter.<sup>70</sup>

In the sections which follow, the implementation of the above equations is examined for three different representations of the attitude estimation problem. In each case, explicit expressions are developed for the transition matrix  $\Phi$  and the sensitivity matrix  $H$ .

#### IV. Attitude Kinematics

In the systems investigated in the present study the attitude is represented by the quaternion defined as

$$\hat{q} = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} \quad (30)$$

where

$$q_0 = \hat{n} \sin(\theta/2) \quad q_s = \cos(\theta/2) \quad (31)$$

The unit vector  $\hat{n}$  is the axis of rotation and  $\theta$  the angle of rotation. Quaternions will always be denoted by an overbar. The quaternion possesses three degrees of freedom and satisfies the constraint

$$\hat{q}^T \hat{q} = 1 \quad (32)$$

The attitude matrix is obtained from the quaternion according to the relation

$$A(\hat{q}) = (|q_s|^2 - |\mathbf{q}|^2) I_{3 \times 3} + 2\mathbf{q}\mathbf{q}^T + 2q_s[\mathbf{q}] \quad (33)$$

where

$$[\mathbf{q}] = \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} \quad (34)$$

This notation will be used generally for any  $3 \times 3$  skew-symmetric matrix generated from a three-vector. The convention will be followed that  $A$  is the matrix which transforms representations of vectors in the reference (usually geocentric inertial) coordinate system to representations in the body-fixed coordinate system.

In contrast to the usual convention for quaternion composition established by Hamilton<sup>34</sup> the product of two quaternions will be written in the same order as the corresponding rotation matrices. Thus,

$$A(\hat{q}') A(\hat{q}) = A(\hat{q}' \otimes \hat{q}) \quad (35)$$

The composition of quaternions is bilinear, i.e.,

$$\hat{q}' \otimes \hat{q} = [\hat{q}'] \hat{q} \quad (36)$$

with

$$[\hat{q}'] = \begin{bmatrix} q'_0 & q'_1 & -q'_2 & q'_3 \\ -q'_1 & q'_0 & q'_3 & q'_2 \\ q'_2 & -q'_3 & q'_0 & q'_1 \\ -q'_3 & -q'_2 & -q'_1 & q'_0 \end{bmatrix} \quad (37)$$

or

$$\hat{q}' \otimes \hat{q} = \{\hat{q}\} \hat{q}' \quad (38)$$

with

$$\dot{q} = \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \quad (39)$$

The rate of change of the attitude matrix with time defines the angular velocity vector  $\omega$

$$\frac{d}{dt}A(t) = [\omega(t)]A(t) \quad (40)$$

The corresponding rate of change of the quaternion is given by

$$\frac{d}{dt}\dot{q}(t) = \frac{1}{2}\Omega(\omega(t))\dot{q}(t) \quad (41)$$

with

$$\Omega(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \quad (42)$$

It will be convenient in later sections to define the four-component quantity

$$\dot{\omega} = \begin{bmatrix} \omega \\ 0 \end{bmatrix} \quad (43)$$

whence

$$\frac{d}{dt}\dot{q}(t) = \frac{1}{2}\dot{\omega}(t) \otimes \dot{q}(t) \quad (44)$$

If the direction of  $\omega$  is constant over the time interval of interest or if the "rotation vector" defined by

$$\Delta\theta = \int_t^{t+\Delta t} \omega(t') dt' \quad (45)$$

is small, then the solution of Eq. (41) is

$$\dot{q}(t+\Delta t) = M(\Delta\theta)\dot{q}(t) \quad (46)$$

where

$$M(\Delta\theta) = \cos(|\Delta\theta|/2)I_{4 \times 4} + \frac{\sin(|\Delta\theta|/2)}{|\Delta\theta|} \Omega(\Delta\theta) \quad (47)$$

**V. Sensor Models**

The spacecraft being studied is assumed to be equipped with three-axis gyros as well as (nongyroscopic) attitude sensors.

**Gyro Models**

We use a simple but realistic model for gyro operation developed by Farrenkopf<sup>31</sup> and applied to the HEAO mission by Hoffman and McElroy.<sup>37</sup> In this model the spacecraft angular velocity is related to the gyro output vector  $u$  according to

$$\omega = u - b - \eta_1 \quad (48)$$

The vector  $b$  is the drift-rate bias and  $\eta_1$  is the drift-rate noise.  $\eta_1$  is assumed to be a Gaussian white-noise process

$$E[\eta_1(t)] = 0 \quad (49)$$

$$E[\eta_1(t)\eta_1^T(t')] = Q_1(t)\delta(t-t') \quad (50)$$

The drift-rate bias is itself not a static quantity but is driven by a second Gaussian white-noise process, the gyro drift-rate ramp noise

$$\frac{d}{dt}b = \eta_2 \quad (51)$$

with

$$E[\eta_2(t)] = 0 \quad (52)$$

$$E[\eta_2(t)\eta_2^T(t')] = Q_2(t)\delta(t-t') \quad (53)$$

The two noise processes are assumed to be uncorrelated

$$E[\eta_1(t)\eta_2^T(t')] = 0 \quad (54)$$

In general,  $u$ ,  $b$ ,  $\eta_1$ , and  $\eta_2$  will be linear combinations of the outputs of three or more gyros, which need not be aligned along the spacecraft axes.

An alternative equation for the drift-rate bias

$$\frac{d}{dt}b = -b/\tau + \eta_2 \quad (55)$$

gives rise to an exponentially correlated noise term as in the model considered by Hammon.<sup>47</sup> In order to reproduce realistic gyro data, a superposition of several drift terms, with different values of the time constant may be needed.<sup>48</sup> Since relatively persistent drifts are observed in actual gyro operation, at least one time constant must be very large. Letting  $\tau$  be infinite, which leads to Eq. (51), is adequate for most applications.

In the propagation equations to be used in the Kalman filter, Eq. (48) is assumed to be integrated continuously. The model thus assumes that the gyros are used in a rate mode. In practice, however, rate-integrating gyros are used, which sense the spacecraft angular rates continuously but are sampled at discrete intervals. The spacecraft attitude is also propagated at discrete intervals, equal to either the gyro sampling interval or some multiple thereof. If the attitude update interval is much shorter than the Kalman filter update interval, as it always is in practice, the approximation of continuous gyro updates will be good.

**Attitude Sensors**

The attitude sensors considered here may be any sensor for which the measured quantity depends solely on the direction of some object in the sensor coordinate system. Thus, we treat the most general case possible, although in most practical applications of Kalman filters for attitude estimation, these sensors will be vector sun sensors and star trackers, which are the most accurate.

The direction of a body in the sensor coordinate system  $p_S$  is related to the direction in the reference coordinate system  $p_R$  according to

$$p_S = TA(\dot{q})p_R \quad (56)$$

where  $A(\dot{q})$  is the spacecraft attitude and  $T$  the sensor alignment matrix. Note that the measurement depends explicitly on the attitude but not on parameters such as gyro biases.

It will be assumed throughout this report that the sensor measurements are scalar and uncorrelated. In general, the

correlation between the measurements is small. When this is not the case, a set of uncorrelated measurements can always be achieved by choosing a representation in which the measurement covariance matrix is diagonal.

### VI. The State Equation

The spacecraft attitude state is given by the attitude quaternion and the gyro drift-rate bias vector

$$x(t) = \begin{bmatrix} \hat{q}(t) \\ b(t) \end{bmatrix} \quad (57)$$

and thus has dimension seven. The quaternion and the bias vector have been shown to satisfy the coupled differential equations

$$\frac{d}{dt} \hat{q}(t) = \frac{1}{2} \Omega(u(t) - b(t) - \eta_1(t)) \hat{q}(t) \quad (58)$$

$$\frac{d}{dt} b(t) = \eta_2(t) \quad (59)$$

Noting that the matrix function  $\Omega$  is linear and homogeneous in its argument and defining the  $4 \times 3$  function  $\Xi(\hat{q})$  by

$$\Omega(b) \hat{q} = \Xi(\hat{q}) b \quad (60)$$

Eq. (58) may be rewritten as

$$\frac{d}{dt} \hat{q}(t) = \frac{1}{2} \Omega(u(t) - b(t)) \hat{q}(t) - \frac{1}{2} \Xi(\hat{q}(t)) \eta_1(t) \quad (61)$$

Equations (59) and (61) are now in the same form as Eq. (1).

The matrix  $\Xi(\hat{q})$  has the explicit form

$$\Xi(\hat{q}) = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \quad (62)$$

The properties of the matrix  $\Xi(\hat{q})$  are discussed in the Appendix.

#### Prediction

The predicted state vector is defined as in Sec. II. Taking the expectation of Eqs. (59) and (61) leads, within the approximation of Eq. (8), to

$$\frac{d}{dt} \hat{q} = \frac{1}{2} \Omega(\hat{\omega}) \hat{q} \quad (63)$$

$$\frac{d}{dt} \hat{b} = 0 \quad (64)$$

where

$$\hat{\omega} = u - \hat{b} \quad (65)$$

is the estimated angular velocity.

From Eq. (64) we see immediately that  $\hat{b}$  is constant over the prediction interval. Thus,  $\hat{\omega}$  depends only on  $u(t)$  and the initial value of the state vector. Therefore, Eq. (63) can be integrated directly to give

$$\hat{q}(t) = \Theta(t, t_0) \hat{q}(t_0) \quad (66)$$

with

$$\frac{\partial}{\partial t} \Theta(t, t_0) = \frac{1}{2} \Omega(\hat{\omega}(t)) \Theta(t, t_0) \quad (67)$$

$$\Theta(t_0, t_0) = I_{7 \times 7} \quad (68)$$

When the direction of  $\hat{\omega}(t)$  is constant throughout the time interval or the angular displacements of the axes is small, then  $\Theta(t, t_0)$  can be approximated by Eq. (47).

### VII. The Seven-Dimensional Covariance Representation

The simplest approach to express the state-error vector and the covariance matrix is to define these in terms of the complete state vector as in Eqs. (10) and (11).

#### Prediction

The seven-dimensional state error vector satisfies the differential equation

$$\frac{d}{dt} \Delta x(t) = F(t) \Delta x(t) + G(t) w(t) \quad (69)$$

with

$$F(t) = \begin{bmatrix} \frac{1}{2} \Omega(\hat{\omega}) & -\frac{1}{2} \Xi(\hat{q}) \\ 0_{3 \times 4} & 0_{3 \times 3} \end{bmatrix} \quad (70)$$

$$G(t) = \begin{bmatrix} -\frac{1}{2} \Xi(\hat{q}) & 0_{4 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (71)$$

$$w(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \quad (72)$$

and, therefore,

$$Q(t) = \begin{bmatrix} Q_1(t) & 0_{3 \times 3} \\ 0_{3 \times 3} & Q_2(t) \end{bmatrix} \quad (73)$$

$F(t)$  and  $G(t)$  are determined straightforwardly from Eqs. (13), (14), (59), and (61).

The transition matrix has the general form

$$\Phi(t, t_0) = \begin{bmatrix} \Theta(t, t_0) & \Psi(t, t_0) \\ 0_{3 \times 4} & I_{3 \times 3} \end{bmatrix} \quad (74)$$

where  $\Theta(t, t_0)$  is given by Eqs. (67-68) and

$$\frac{\partial}{\partial t} \Psi(t, t_0) = \frac{1}{2} \Omega(\hat{\omega}(t, t_0)) \Psi(t, t_0) - \frac{1}{2} \Xi(\hat{q}(t)) \quad (75)$$

subject to

$$\Psi(t_0, t_0) = 0_{4 \times 3} \quad (76)$$

Equation (75) may be integrated directly to yield

$$\Psi(t, t_0) = -\frac{1}{2} \int_{t_0}^t \Theta(t, t') \Xi(\hat{q}(t')) dt' \quad (77)$$

#### Filtering

Since the measurements are assumed to be scalar and to depend only on the attitude and not the bias vector, as

discussed in Sec. V, it follows that the sensitivity matrix is a seven-dimensional row vector of the form

$$H = [h, 0^T] \tag{78}$$

where

$$h = \left( \frac{\partial h}{\partial p_s} \frac{\partial p_s}{\partial \hat{q}} \right) \Big|_{\hat{q}(-)} \tag{79}$$

Define the vector  $r$  by

$$r = \left[ \frac{\partial h}{\partial p_s} T \right]^T \tag{80}$$

then it can be shown that

$$h = \left[ \frac{\partial}{\partial \hat{q}} r \cdot A(\hat{q}) p_R \right] \Big|_{\hat{q}(-)} = 2(r \times \hat{p}_B)^T \Xi^T(\hat{q}(-)) \tag{81}$$

where

$$\hat{p}_B = A(\hat{q}(-)) p_R \tag{82}$$

**VIII. Singularity of the Covariance Matrix**

The covariance matrix for the seven-dimensional state vector is singular. This follows immediately from the constraint on the quaternion norm so that

$$\Delta \hat{q}^T \hat{q} = 0 \tag{83}$$

and hence

$$\begin{bmatrix} \hat{q}(t) \\ 0 \end{bmatrix}$$

is a null vector of  $P(t)$ .

This singularity is difficult to maintain numerically due to the accumulation of round-off error. If fact,  $P(t)$  may even develop a negative eigenvalue. The simplest way to maintain the singularity is to represent  $P(t)$  by a matrix of smaller dimension. There are three possible approaches to accomplishing this, two of which lead to the same result. The examination of these three approaches forms the subject matter for the remainder of this report.

**IX. Reduced Representation of the Covariance Matrix**

A  $6 \times 6$  representation of the covariance matrix is induced by the form of the transition matrix prescribed above.

**Prediction**

It can be shown (see the Appendix) that

$$\Theta(t, t') \Xi(\hat{q}(t')) = \Xi(\hat{q}(t)) \Lambda(t, t') \tag{84}$$

where

$$\Lambda(t, t') = A(\hat{q}(t)) A^T(\hat{q}(t')) \tag{85}$$

is the  $3 \times 3$  rotation matrix which transforms the estimated attitude matrix from time  $t'$  to time  $t$ . Substituting this into Eq. (77) leads to

$$\Psi(t, t_0) = \Xi(\hat{q}(t)) K(t, t_0) \tag{86}$$

with

$$K(t, t_0) = -\frac{1}{2} \int_{t_0}^t \Lambda(t, t') dt' \tag{87}$$

It is also shown in the Appendix that

$$\Theta(t, t_0) = \Xi(\hat{q}(t)) \Lambda(t, t_0) \Xi^T(\hat{q}(t_0)) + \hat{q}(t) \hat{q}^T(t_0) \tag{88}$$

Substituting these expressions into the expression for the transition matrix gives

$$\Phi(t, t_0) = S(\hat{q}(t)) \tilde{\Phi}(t, t_0) S^T(\hat{q}(t_0)) + \begin{bmatrix} \hat{q}(t) \hat{q}^T(t_0) & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \tag{89}$$

where

$$S(\hat{q}(t)) = \begin{bmatrix} \Xi(\hat{q}(t)) & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \tag{90}$$

$$\tilde{\Phi}(t, t_0) = \begin{bmatrix} \Lambda(t, t_0) & K(t, t_0) \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \tag{91}$$

The second term of the right member of Eq. (89) annihilates  $P(t_0)$  on the left and  $P(t)$  on the right. Thus, if the  $6 \times 6$  covariance matrix  $\tilde{P}(t)$  is defined by

$$\tilde{P}(t) = S^T(\hat{q}(t)) P(t) S(\hat{q}(t)) \tag{92}$$

this satisfies the integrated Riccati equation of the form

$$\begin{aligned} \dot{\tilde{P}}(t) &= \tilde{\Phi}(t, t_0) \tilde{P}(t_0) \tilde{\Phi}^T(t, t_0) \\ &+ \int_{t_0}^t \tilde{\Phi}(t, t') \tilde{G}(t') Q(t') \tilde{G}^T(t') \tilde{\Phi}^T(t, t') dt' \end{aligned} \tag{93}$$

where

$$\tilde{G}(t) = S^T(\hat{q}(t)) G(t) = \begin{bmatrix} -\frac{1}{2} I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \tag{94}$$

In obtaining Eq. (93) repeated use has been made of the equation

$$\Xi^T(\hat{q}) \hat{q} = 0 \tag{95}$$

At any time the  $7 \times 7$  covariance matrix  $P(t)$  can be reconstructed via

$$P(t) = S(\hat{q}(t)) \tilde{P}(t) S^T(\hat{q}(t)) \tag{96}$$

**Filtering**

The Kalman filter may also be mechanized using  $\tilde{P}(t)$ . Define the  $1 \times 6$  matrix  $\tilde{H}_k$  and the  $6 \times 1$  matrix  $\tilde{K}_k$  according to

$$\tilde{H}_k = H_k S(\hat{q}_k(-)) \tag{97}$$

$$\tilde{K}_k = \tilde{P}_k(-) \tilde{H}_k^T [\tilde{H}_k \tilde{P}_k(-) \tilde{H}_k^T + R_k]^{-1} \tag{98}$$

where  $H_k$  is as in Sec. VIII. Then it is easy to show that

$$\tilde{P}_k(+ ) = (I_{6 \times 6} - \tilde{K}_k \tilde{H}_k) \tilde{P}_k(-) \tag{99}$$

$$K_k = S(\hat{q}_k(-)) \tilde{K}_k \tag{100}$$

In implementing this method the  $7 \times 7$  covariance matrix need never be computed, the  $6 \times 6$  covariance is propagated and updated using Eqs. (93) and (99), respectively. The ex-



pression for the  $1 \times 6$  sensitivity matrix  $\hat{H}_y$  can be simplified; but this expression will be derived in Sec. XI, since it falls out more directly in that approach. That section contains an independent derivation of the prediction and filtering algorithms which turn out to be identical to those presented here.

### X. The Truncated Covariance Representation

The most obvious approach to reducing the dimension of the covariance matrix is to delete one of the quaternion components from the state error vector. We may let this be the fourth component, although in principle any component could be deleted.

The truncated state error vector is then

$$\Delta y = \begin{bmatrix} \Delta q \\ \Delta b \end{bmatrix} \quad (101)$$

This state error vector cannot be treated as simply as the seven-dimensional state error vector of Sec. VIII. In Sec. VIII, the four components of the quaternion were treated as independent variables with the normalization constraint being maintained by the form of the equation of motion. In the present case the fourth component of the quaternion error must be determined from the constraint according to

$$\Delta q_4 = -\frac{1}{\hat{q}_4} \hat{q} \cdot \Delta q \quad (102)$$

Thus, partial differentiation must be treated differently in the two cases.

#### Prediction

The truncated covariance matrix is defined as

$$P_y(t) = E[\Delta y(t) \Delta y^T(t)] \quad (103)$$

which satisfies

$$P_y(t) = \Phi_y(t, t_0) P_y(t_0) \Phi_y^T(t, t_0) + \int_{t_0}^t \Phi_y(t, t') G_y(t') Q(t') G_y^T(t') \Phi_y^T(t, t') dt' \quad (104)$$

where

$$\Phi_y(t, t_0) = \begin{bmatrix} \Theta_y(t, t_0) & \Psi_y(t, t_0) \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (105)$$

is the  $6 \times 6$  transition matrix.  $G_y(t)$  is defined below.

The submatrices of  $\Phi_y(t, t_0)$  are easily determined from the corresponding submatrices of  $\Phi(t, t_0)$ . This is simpler than attempting to determine  $\Phi_y(t, t_0)$  by direct integration of a six-dimensional state error equation.

Note that

$$[\Theta_y(t, t_0)]_y = \left( \frac{\partial q_i(t)}{\partial q_j(t_0)} \right), \quad (106)$$

$$[\Psi_y(t, t_0)]_y = \left( \frac{\partial q_i(t)}{\partial b_j(t_0)} \right), \quad (107)$$

where the subscript  $y$  indicates that the partial derivatives are to be computed subject to the constraint. No similar constraint exists in the corresponding expression for  $\Theta(t, t_0)$  and  $\Psi(t, t_0)$ , the constraint being satisfied by the state equation. Noting this fact leads immediately to

$$[\Theta_y(t, t_0)]_y = [\Theta(t, t_0)]_y - \left( \frac{\hat{q}_4(t_0)}{\hat{q}_4(t_0)} \right) [\Theta(t, t_0)]_4 \quad (108)$$

$$[\Psi_y(t, t_0)]_y = [\Psi(t, t_0)]_y \quad (109)$$

and also

$$[G_y(t)]_y = [G(t)]_y \quad (110)$$

The  $7 \times 7$  covariance matrix can be recovered from  $P_y(t)$  by constructing the missing elements according to

$$E[\Delta q_4(t) \Delta y_y(t)] = -\frac{1}{\hat{q}_4(t)} \sum_{i=1}^3 \hat{q}_i(t) [P_y(t)]_4 \quad (111)$$

$$E[\Delta q_4(t) \Delta q_4(t)] = \frac{1}{|\hat{q}_4(t)|^2} \sum_{i=1}^3 \sum_{j=1}^3 \hat{q}_i(t) [P_y(t)]_{ij} \hat{q}_j(t) \quad (112)$$

#### Filtering

Analogously to Eqs. (78) and (79) in Sec. VII the sensitivity matrix is given by

$$H_y = [\ell_y, 0^T] \quad (113)$$

where the three-dimensional row vector  $\ell_y$  is given by

$$\ell_y = \left( \frac{\partial h}{\partial p_s} \frac{\partial p_s}{\partial q} \right) \Big|_{\hat{q}_4}, \quad (114)$$

which reduces to

$$(\ell_y)_i = \ell_i - (\hat{q}_i / \hat{q}_4) \ell_4 \quad (115)$$

and  $\ell$  is given by Eq. (81).

In implementing the Kalman filter in this case also the  $7 \times 7$  covariance matrix need never be computed. The state vector is updated by computing

$$\Delta \hat{x}(+) = K_y [z - h(\hat{q}(-))] = \begin{bmatrix} \Delta \hat{q}(+) \\ \Delta \hat{b}(+) \end{bmatrix} \quad (116)$$

where  $K_y$  is computed from  $P_y(-)$  and  $H_y$  using Eq. (26).  $\Delta \hat{q}_4(+)$  is obtained from Eq. (102) and the updated state vector is given by

$$\hat{x}(+) = \hat{x}(-) + \Delta \hat{x}(+) = \hat{x}(-) + \begin{bmatrix} \Delta \hat{q}(+) \\ \Delta \hat{b}(+) \end{bmatrix} \quad (117)$$

Note that the symbol  $\Delta \hat{x}(+)$  in Eq. (117) denotes simply the quantity  $\hat{x}(+) - \hat{x}(-)$  and should not be confused with  $\Delta x(t)$ , which denotes the state error. Since the state error never appears with a caret (its expectation vanishes by definition), confusion is unlikely. Note also from Eq. (102) that truncation can lead to large errors when  $\hat{q}_4$  is small. This can be avoided by always deleting the quaternion component which is largest in magnitude.

### XI. The Body-Fixed Covariance Representation

In this section we develop an approximate body-referenced representation of the state vector and covariance matrix. The quaternion error in this representation is expressed not as the arithmetic difference between the true and the estimated quaternion but as the quaternion which must be composed with the estimated quaternion in order to obtain the true quaternion. Since this incremental quaternion corresponds almost certainly to a small rotation, the fourth component will be close to unity (to second order in the vector components) and, hence, all the attitude information of interest is

contained in the three vector components. Therefore, the six-component object defined by the vector components of the incremental quaternion and the drift-bias vector will provide a nonredundant representation of the state error. This representation will turn out to be identical to that developed in Sec. IX.

If the infinitesimal attitude error angles are defined as twice the vector components of the incremental quaternion, then this treatment becomes very similar to that employed by several other authors.<sup>21,29-33</sup>

Define the error quaternion as

$$\delta q = \hat{q} \otimes \hat{q}^{-1} \quad (118)$$

and the six-dimension body-referenced state vector as

$$\bar{x} = \begin{bmatrix} \delta q \\ b \end{bmatrix} \quad (119)$$

From Eqs. (30) and (62) we have that

$$\dot{\hat{q}} = (\hat{q}) \delta \dot{\hat{q}} = [\Xi(\hat{q}) : \hat{q}] \delta \dot{\hat{q}} \quad (120)$$

whence

$$\delta \dot{q} = \Xi^T(\hat{q}) \dot{\hat{q}} \quad (121)$$

$$\delta \dot{q}_e = \hat{q}^T \dot{\hat{q}} \quad (122)$$

and

$$\bar{x} = \begin{bmatrix} \Xi^T(\hat{q}) & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \begin{bmatrix} \hat{q} \\ b \end{bmatrix} \quad (123)$$

$$= S^T(\hat{q}) x \quad (124)$$

From Eq. (95) it then follows that

$$\dot{\bar{x}} = S^T(\hat{q}) \dot{x} = \begin{bmatrix} 0 \\ \delta \end{bmatrix} \quad (125)$$

$$\Delta \bar{x} = S^T(\hat{q}) \Delta x = \begin{bmatrix} \delta q \\ \Delta b \end{bmatrix} \quad (126)$$

Note, however, that although

$$\Delta x = S(\hat{q}) \Delta \bar{x} \quad (127)$$

it happens that

$$\dot{\bar{x}} = S(\hat{q}) \dot{x} \quad (128)$$

The results of Sec. IX may now be derived in short order.

**Prediction**

From

$$\frac{d}{dt} \hat{q} = \frac{1}{2} \omega \otimes \hat{q} \quad (129)$$

$$\frac{d}{dt} \hat{q} = \frac{1}{2} \hat{\omega} \otimes \hat{q} \quad (130)$$

it follows immediately that

$$\begin{aligned} \frac{d}{dt} \delta \hat{q} &= \frac{1}{2} [\hat{\omega} \otimes \delta \hat{q} - \delta \hat{q} \otimes \hat{\omega}] \\ &= \frac{1}{2} [\hat{\omega} \otimes \delta \hat{q} - \delta \hat{q} \otimes \hat{\omega}] + \frac{1}{2} \delta \omega \otimes \delta \hat{q} \end{aligned} \quad (131)$$

where

$$\delta \omega = \begin{bmatrix} \delta \omega \\ 0 \end{bmatrix} = \begin{bmatrix} \omega - \hat{\omega} \\ 0 \end{bmatrix} = \begin{bmatrix} -\Delta b - \eta_1 \\ 0 \end{bmatrix} \quad (132)$$

Now

$$\frac{1}{2} [\hat{\omega} \otimes \delta \hat{q} - \delta \hat{q} \otimes \hat{\omega}] = \begin{bmatrix} -\hat{\omega} \times \delta q \\ 0 \end{bmatrix} \quad (133)$$

$$\delta \omega \otimes \delta \hat{q} = \delta \omega + O(|\delta \omega| |\delta q|) \quad (134)$$

Thus, neglecting the second-order terms

$$\frac{d}{dt} \delta q = -\hat{\omega} \times \delta q - \frac{1}{2} (\Delta b + \eta_1) \quad (135)$$

$$\frac{d}{dt} \delta q_e = 0 \quad (136)$$

from which it follows that the state error equation may be written

$$\frac{d}{dt} \Delta \bar{x}(t) = \hat{F}(t) \Delta \bar{x} + \hat{G}(t) w(t) \quad (137)$$

where

$$\hat{F}(t) = \begin{bmatrix} [\hat{\omega}(t)] & -\frac{1}{2} I_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \quad (138)$$

$$\hat{G}(t) = \begin{bmatrix} -\frac{1}{2} I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (139)$$

$\hat{G}$  is identical to Eq. (94).

The transition matrix may be written as

$$\hat{\Phi} = \begin{bmatrix} \hat{\Theta} & \hat{\Psi} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (140)$$

with

$$\frac{\partial}{\partial t} \hat{\Theta}(t, t_0) = [\hat{\omega}(t)] \hat{\Theta}(t, t_0) \quad (141)$$

$$\frac{\partial}{\partial t} \hat{\Psi}(t, t_0) = [\hat{\omega}(t)] \hat{\Psi}(t, t_0) - \frac{1}{2} I_{3 \times 3} \quad (142)$$

subject to

$$\hat{\Theta}(t_0, t_0) = I_{3 \times 3}, \quad \hat{\Psi}(t_0, t_0) = 0_{3 \times 3} \quad (143)$$

It now follows immediately that

$$\hat{\Theta}(t, t_0) = \Lambda(t, t_0) \quad (144)$$

$$\Psi(t, t_0) = -\frac{1}{2} \int_{t_0}^t \Lambda(t, t') dt' = K(t, t_0) \quad (145)$$

so that

$$\Phi(t, t_0) = \begin{bmatrix} \Lambda(t, t_0) & K(t, t_0) \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (146)$$

as in Eq. (91).

The covariance matrix is defined as

$$\tilde{P}(t) = E[\Delta \tilde{x}(t) \Delta \tilde{x}^T(t)] \quad (147)$$

and from Eq. (127) it follows that

$$\tilde{P}(t) = S^T(\hat{q}(t)) P(t) S(\hat{q}(t)) \quad (148)$$

$$P(t) = S(\hat{q}(t)) \tilde{P}(t) S^T(\hat{q}(t)) \quad (149)$$

as in Sec. IX, and  $\tilde{P}(t)$  satisfies the Riccati equation

$$\frac{d}{dt} \tilde{P}(t) = \tilde{F}(t) \tilde{P}(t) + \tilde{P}(t) \tilde{F}^T(t) + \tilde{G}(t) Q(t) \tilde{G}^T(t) \quad (150)$$

whose integral form is simply Eq. (93).

Filtering

In analogy with the previous sections

$$\tilde{H} = [\tilde{L}, \tilde{O}^T] \quad (151)$$

where the three-dimensional row vector  $\tilde{L}$  is given by

$$\tilde{L} = \left( \frac{\partial h}{\partial p_S} \frac{\partial p_S}{\partial (\delta q)} \right) \Big|_{\hat{q}(-)} \quad (152)$$

From

$$A(\hat{q}) = A(\delta \hat{q}) A(\hat{q}(-)) \quad (153)$$

it follows that

$$\tilde{L} = r \cdot \left( \frac{\partial}{\partial (\delta q)} A(\delta \hat{q}) \hat{p}_B \right) \Big|_{\hat{q}(-)} \quad (154)$$

where

$$\hat{p}_B = A(\hat{q}(-)) p_B \quad (155)$$

Noting that

$$A(\delta \hat{q}) = I_{3 \times 3} + 2[\delta q] \quad (156)$$

the differentiation may be carried out immediately to give

$$\tilde{L} = 2(r \times \hat{p}_B)^T \quad (157)$$

The implementation of these algorithms is identical to those of Sec. IX. An apparent but not actual difference is given by the state vector update where

$$\Delta \tilde{x}(+) = \begin{bmatrix} \delta \hat{q}(+) \\ \Delta \hat{b}(+) \end{bmatrix} = K[z - h(\hat{q}(-))] \quad (158)$$

from which we have

$$\hat{q}(+) = \delta \hat{q}(+) \otimes \hat{q}(-) \quad (159)$$

$$\hat{b}(+) = \hat{b}(-) + \Delta \hat{b}(+) \quad (160)$$

with

$$\delta \hat{q}(+) = \begin{bmatrix} \delta \hat{q}(+) \\ 1 \end{bmatrix} \quad (161)$$

Noting from Eqs. (37) and (60) that

$$\delta \hat{q} \otimes \hat{q} = \hat{q} + \Omega(\delta \hat{q}) \hat{q} \quad (162)$$

$$= \hat{q} + \Xi(\hat{q}) \delta \hat{q} \quad (163)$$

we see that this algorithm is identical to implementing Eq. (100).

We note again that the quantities  $\Delta x(+)$ ,  $\delta \hat{q}(+)$ , and  $\Delta \hat{b}(+)$ , which appear in Eqs. (158-163), do not denote state errors but rather filter corrections to the corresponding predicted quantities, namely,  $\hat{x}(+) - \hat{x}(-)$ ,  $\hat{q}(+) \otimes \hat{q}(-)^{-1}$ , and  $\hat{b}(+) - \hat{b}(-)$ , respectively. Again, the presence of the caret helps us avoid confusion.

If instead of the incremental quaternion the incremental error angles given by

$$\delta q = \frac{1}{2} \delta \theta \quad (164)$$

are used, the expressions simplify somewhat in that the factors of  $\frac{1}{2}$ ,  $\frac{1}{4}$ , and 2 in the expressions for  $\tilde{F}(t)$ ,  $\tilde{G}(t)$ ,  $K(t, t_0)$ , and  $\tilde{L}$  disappear. The implementation is otherwise unchanged.

## XII. Conclusions

In all the implementations of the Kalman filter presented above the propagation of the state vector is performed identically using Eqs. (63) and (64). In the propagation the four components of the quaternion are treated as independent variables. Within numerical errors the constraint on the quaternion normalization is maintained by the structure of Eq. (63).

This normalization can be destroyed by the accumulation of round-off error and the linearization approximation inherent in the update equation. This need not cause concern in the propagation of the quaternion since this operation is linear. Care must be taken, however, when the attitude matrix is to be calculated or when an update is to be performed. The renormalization of the quaternion can then be carried out explicitly.

It is in the propagation of the state covariance matrix and the update of the state vector and covariance matrix that differences in method appear. The three approaches presented above: the use of the full  $7 \times 7$  covariance matrix, the truncated  $6 \times 6$  representation, and the  $6 \times 6$  body representation will now be examined.

The  $7 \times 7$  representation presented in Sec. VII is the most direct but also the most burdensome computationally, since the number of elements in the covariance matrix is largest. A more significant problem, however, is the need to maintain the singularity of the covariance matrix, which is of rank 6. As pointed out in Sec. VIII, round-off errors can lead to terms which raise the rank of the matrix and even lead to negative eigenvalues.

The truncated  $6 \times 6$  representation of the covariance matrix maintains the proper rank but does not result in a computational saving. This is true because the full  $7 \times 7$  transition matrix must be computed at each time and the related  $6 \times 6$  transition matrix  $\Phi$ , computed only at the time of an update. The burden of reducing the dimension of the transition matrix cancels the savings resulting from the implementation of the update equations in a space of smaller dimension.

The 6x6 body representation of the covariance matrix presented in Secs. IX and XI preserves the proper rank of the covariance matrix while simplifying the computation considerably. The transition matrix and the covariance matrix are computed as 6x6 matrices throughout. Of particular value in this representation is the fact that the elements of the covariance matrix have a simple interpretation in terms of gyro bias errors and angular errors in the body frame. In addition, the matrices  $\hat{G}$  and  $\hat{H}$  have a particularly simple form.

In any Kalman filter implementation the largest computational burden is imposed in general by the computation of the transition matrix and the contribution of the process noise to the state covariance matrix. These quantities are not needed at the same level of accuracy as the state vector and hence may be computed at much larger intervals than is required by the state vector propagation. For this same reason, approximate forms of these quantities can also be implemented.

**Appendix: The Matrix  $\Xi(\dot{q})$**

The matrix  $\Xi(\dot{q})$  defined in Eq. (62) has some very useful properties which greatly simplify the computation of the covariance matrix and the transition matrices. In this Appendix the following results will be proved:

$$\frac{d}{dt} \Xi(\dot{q}(t)) = \frac{1}{2} \Omega(\omega) \Xi(\dot{q}(t)) - \Xi(\dot{q}(t)) [\underline{\omega}] \quad (A1)$$

$$\Theta(t, t_0) \Xi(\dot{q}(t_0)) = \Xi(\dot{q}(t)) \Lambda(t, t_0) \quad (A2)$$

$$\Lambda(t, t_0) = \Xi^T(\dot{q}(t)) \Theta(t, t_0) \Xi(\dot{q}(t_0)) \quad (A3)$$

$$\Theta(t, t_0) = \Xi(\dot{q}(t)) \Lambda(t, t_0) \Xi^T(\dot{q}(t_0)) + \dot{q}(t) \dot{q}^T(t_0) \quad (A4)$$

In the above equations the quaternion is assumed to be deterministic, that is,

$$\frac{d}{dt} \dot{q}(t) = \frac{1}{2} \Omega(\omega) \dot{q}(t) \quad (A5)$$

$$\dot{q}(t) = \Theta(t, t_0) \dot{q}(t_0) \quad (A6)$$

and, hence, truly corresponds to  $\dot{q}(t)$  elsewhere in this report. However, to simplify the notation carets will not be written over  $\dot{q}$  and  $\omega$ . Also, although this is not written explicitly,  $\omega$  may depend on the time.

To prove Eq. (A1) note that by definition for any three-vector  $c$

$$\Xi(\dot{q})c = \Omega(c)\dot{q} \quad (A7)$$

Differentiating Eq. (A7) leads to

$$\dot{\Xi}(\dot{q})c + \Xi(\dot{q})\dot{c} = \Omega(\dot{c})\dot{q} + \Omega(c)\dot{\dot{q}} \quad (A8)$$

Let the value of  $c$  at time  $t$  to be arbitrary and let the time development of  $c$  be given by

$$\dot{c} = [\underline{\omega}]c = -\omega \times c \quad (A9)$$

Substituting Eqs. (A5) and (A9) into Eq. (A8) leads to

$$\dot{\Xi}(\dot{q})c + \Xi(\dot{q})[\underline{\omega}]c = -\Omega(\omega \times c)\dot{q} + \frac{1}{2}\Omega(c)\Omega(\omega)\dot{q} \quad (A10)$$

It is a simple matter to prove that

$$\Omega(\omega \times c) = -\frac{1}{2}[\Omega(\omega)\Omega(c) - \Omega(c)\Omega(\omega)] \quad (A11)$$

Substituting this expression into Eq. (A10) above, recalling Eq. (A7), and noting that  $c(t)$  is arbitrary leads directly to Eq. (A1).

To prove Eq. (A2) examine the quantity

$$C(t, t_0) = \Theta(t, t_0) \Xi(\dot{q}(t_0)) - \Xi(\dot{q}(t)) \Lambda(t, t_0) \quad (A12)$$

Differentiating this expression and using Eq. (A1) leads to

$$\frac{\partial}{\partial t} C(t, t_0) = \frac{1}{2} \Omega(\omega) C(t, t_0) \quad (A13)$$

which may be integrated directly to yield

$$C(t, t_0) = \Theta(t, t_0) C(t_0, t_0) \quad (A14)$$

But from Eq. (A12)

$$C(t_0, t_0) = 0_{6 \times 6} \quad (A15)$$

and Eq. (A2) follows.

Equation (A3) follows directly by applying the relation

$$\Xi^T(\dot{q}) \Xi(\dot{q}) = I_{3 \times 3} \quad (A16)$$

to Eq. (A2).

To obtain Eq. (A4) note that the converse relation

$$\Xi(\dot{q}) \Xi^T(\dot{q}) = I_{6 \times 6} - \dot{q} \dot{q}^T \quad (A17)$$

applied to Eq. (A2) leads to

$$\Theta(t, t_0) (I_{6 \times 6} - \dot{q}(t_0) \dot{q}^T(t_0)) = \Xi(\dot{q}(t)) \Lambda(t, t_0) \Xi^T(\dot{q}(t_0)) \quad (A18)$$

Noting

$$\Theta(t, t_0) \dot{q}(t_0) \dot{q}^T(t_0) = \dot{q}(t) \dot{q}^T(t_0) \quad (A19)$$

Equation (A4) follows immediately.

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