Efficient Algorithms for Single-Axis Attitude Estimation[†]

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Abstract

Computationally efficient algorithms are presented for determining single-axis attitude from the measurement of arc lengths and dihedral angles. The dependence of these algorithms on the solution of trigonometric equations has been much reduced. Both single-time and batch estimators are presented along with the covariance analysis of each algorithm.

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I. Introduction

Since nearly every spacecraft is spinning during part of its life--in particular, at the time of orbit injection--spin-axis attitude* estimation is an important segment of almost every mission support operation. Indeed, for spin-stabilized spacecraft there is often no need (or desire) to determine the complete three-axis attitude at every point and, in fact, when accuracy requirements for the spin-axis attitude dictate that many measurements taken at different times be processed simultaneously, the computation of a three-axis attitude may not even be possible.

Very often, three-axis attitude information is definitive data required chiefly by mission scientists and generally processed anytime from several days to several months after the receipt of telemetry. The need for efficient three-axis attitude estimation algorithms in those cases is determined by the definitive data rate. When three-axis attitude information is required in real-time for the purpose of attitude control, this is usually provided on-board by three-axis gyros (e.g. SMM) or on the ground by the spin axis and a third angle, which can be obtained by monitoring some other sensor reading such as IR scanner pitch (e.g. AEM, Magsat).

Spin-axis attitudes by contrast are usually required not only as definitive data but also by the ground support system in near real-time for the purpose of monitoring spacecraft performance and determining large scale attitude maneuvers. Thus, the efficiency of a spin-axis attitude estimation algorithm becomes a factor in the safety and daily operation of the spacecraft.

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* Since the single-axis attitude of interest is invariably the spin-axis attitude these terms will be used almost interchangeably throughout this work.

While a number of highly-efficient algorithms exist for three-axis attitude estimation,[1] the computation of spin-axis attitude[2] is by comparison very clumsy. This is largely because the computation of three-axis attitude uses complete vector measurements in general and can take advantage of the linear properties of Euclidean three-space. The computation of spin-axis attitude, on the other hand, must rely on incomplete vector information (the measurement of arc lengths and dihedral angles) to determine a quantity (the spin-axis) which is restricted to the surface of a sphere. Thus, while three-axis attitude computations need only execute simple matrix operations, the computation of spin-axis attitude is beset with the burden of solving complex relations from spherical trigonometry.

Since spin-axis attitude is usually not computed frequently, the need for efficient algorithms is not immediate, at least not for ground support systems. The determination of the spin-axis attitude from batch measurements of arc lengths and dihedral angles has become highly standardized and reliable[3] and there is no obvious need to replace this software in normal ground support operations.

The need for more efficient algorithms lies in two areas: 1) the eventual implementation of spin-axis attitude computation in onboard microprocessor-based attitude determination systems; and 2) the computation of spin-axis attitude accuracies, which imposes a far greater computational burden than computing just the attitude due to the greater number of terms and because the computation of the attitude covariance involves implicitly the computation of derivatives of the attitude.

The large computational burden imposed by the need to solve spherical trigonometric equations in the computation of spin-axis attitude covariances is evident in the work of Wertz and Chen,[2,4-6] the most complete and careful work to date. The difficulties which are encountered in this approach are of two kinds: 1) the complexity of the trigonometric relations, themselves, and 2) the fact that for certain cases the representation of the quantities being calculated becomes indeterminant while the quantities themselves are well defined. This last difficulty is simply a manifestation of the fact that the representation of rotations by Euler angles is sometimes ambiguous and is overcome in the same way, namely, by changing the representation.

The need for computing spin-axis attitude covariance matrices is two-fold. Firstly, it is necessary to be able to assess the accuracy of a spin-axis attitude computation during the spacecraft mission. Secondly, it is important to be able to predict spin-axis attitude accuracies for mission planning, particularly in the determination of launch windows. For an example of launch window computations using the geometrical approach see Chen.[7]

The purpose of the present work is to develop algorithms for computing spin-axis attitude and the associated covariance matrix without relying as heavily as do current methods on the solution of trigonometric equations. A completely vectorial approach is, of course, not possible owing to the nature of the measurements themselves. However, in large degree many of the trigonometric equations can be abandoned with the result that the spin-axis attitude and, particularly, the covariance matrix can be computed more efficiently.

The types of measurements studied here are of two kinds:

measurements of arc length, which will always be the angle between the observed direction and the spin axis.

measurements of dihedral angles, i.e., the angle between two planes, where the line of intersection is assumed to be the spin axis.[8]

Dihedral angles, in general, are measured by observing two crossing times in the spacecraft and multiplying by the angular velocity. Arc lengths may be measured in a variety of ways, for example, by direct sighting (as of the Sun or a star) or by measuring the component of a vector along the spin axis (e.g., the magnetic field vector). The measurement of the nadir angle is hybrid in that an arc length (the nadir angle) is determined from the measurement of a dihedral angle (the Earth width). It is the measurement of the nadir angle which is the source of most of the computational complexity.

Estimation algorithms may be classified either as deterministic (usually single-frame, i.e., single-time) algorithms, in which a minimal subset of the available data is chosen to compute the spin-axis attitude, or as optimal (batch) algorithms, in which a larger quantity of data is used from which one computes a "best" result. Three cases are treated in this report

1) A deterministic estimator using two arc-length measurements,

2) A deterministic estimator using the measurements of two arc lengths and the included dihedral angle. (Since in this case the spin-axis attitude is over-determined the question of optimality is also discussed.)

3) An optimal batch estimator utilizing any number of measurements of dihedral angles and arc lengths.

In each case the covariance analysis is presented in detail.

In the appendix the measurement of the nadir angle is presented. It is at this point that trigonometric relations cannot be avoided, at least in so far as measuring instruments (horizon scanners) are presently constructed. The treatment is similar to that of Wertz and his collaborators (Ref. 2) but a method is given for avoiding sign ambiguities.

The treatment of single-axis attitude estimation presented here complements that of Wertz. The advantage of Wertz's treatment is that the variances along two great circles of the celestrial sphere intersecting at the direction of the spin axis and the dihedral angle between these two circles (the correlation angle) is given fairly directly. Much less direct is determining the covariance of the spin-axis vector in inertial space. This part of the calculation falls out simply in the present formalism.

The results presented here are quite simple although they do not seem to be generally known. An important result, which is demonstrated here, is that little accuracy is lost by relaxing the constraint in the optimization that the spin-axis vector be a unit vector and then unitizing post hoc. This is responsible for a great deal of simplification of the methods presented here, especially for batch estimation.

II. Single-Frame Spin-Axis Estimation from the Measurement of Two Arc Lengths

Consider the simplest case in which the measured quantities are β, the Sun angle (the angle between the spin axis and the Sun vector), and η, the nadir angle (the direction between the spin axis and the nadir vector). The case where one of these measurements is replaced by the magnetic field angle is analogous.

Let $\hat{\underset{\sim}{S}}$ denote the Sun unit vector, $\hat{\underset{\sim}{E}}$ the nadir vector, and $\hat{\underset{\sim}{n}}$ the spin axis. Then

$$\hat{\underset{\sim}{S}} \cdot \hat{\underset{\sim}{n}} = \cos \beta \equiv c_S \tag{1a}$$

$$\hat{\underset{\sim}{E}} \cdot \hat{\underset{\sim}{n}} = \cos \eta \equiv c_E \tag{1b}$$

The direction of the spin-axis can then be determined simply by using a method that has been published recently by Grubin,[9] though it has been in use since the beginning of the space program and probably has been known for several hundred years.

If $\hat{\mathbf{S}}$ and $\hat{\mathbf{E}}$ are not parallel, then it is always possible to write

$$\hat{\mathbf{n}} = a_S \hat{\mathbf{S}} + a_E \hat{\mathbf{E}} + a_N \, \hat{\mathbf{S}} \times \hat{\mathbf{E}} \tag{2}$$

The problem is now to determine the coefficients a_S, a_E, a_N.

From Eqs. (1) and the normalization condition we have

$$c_S = \hat{\mathbf{n}} \cdot \hat{\mathbf{S}} = a_S + a_E (\hat{\mathbf{S}} \cdot \hat{\mathbf{E}}) \tag{3a}$$

$$c_E = \hat{\mathbf{n}} \cdot \hat{\mathbf{E}} = a_S (\hat{\mathbf{S}} \cdot \hat{\mathbf{E}}) + a_E \tag{3b}$$

$$1 = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = a_S^2 + a_E^2 + 2 a_S a_E (\mathbf{S} \cdot \mathbf{E}) + a_N^2 |\hat{\mathbf{S}} \times \hat{\mathbf{E}}|^2 \tag{3c}$$

which have the solution

$$a_S = \frac{1}{|\hat{\mathbf{S}} \times \hat{\mathbf{E}}|^2} [c_S - c_E (\hat{\mathbf{S}} \cdot \hat{\mathbf{E}})] \tag{4a}$$

$$a_E = \frac{1}{|\hat{\mathbf{S}} \times \hat{\mathbf{E}}|^2} [c_E - c_S (\hat{\mathbf{S}} \cdot \hat{\mathbf{E}})] \tag{4b}$$

$$a_N = \pm \frac{1}{|\hat{\mathbf{S}} \times \hat{\mathbf{E}}|^2} [|\hat{\mathbf{S}} \times \hat{\mathbf{E}}|^2 - (c_S^2 - 2 c_S c_E (\hat{\mathbf{S}} \cdot \hat{\mathbf{E}}) + c_E^2)]^{1/2} \tag{4c}$$

Note that there are two possible solutions for $\hat{n}$. These are shown geometrically in Figure 1.

It will be convenient to define the following quantities

$$\underset{\sim}{a} \equiv \begin{bmatrix} a_S \\ a_E \end{bmatrix} \qquad \underset{\sim}{c} \equiv \begin{bmatrix} c_S \\ c_E \end{bmatrix} \tag{5}$$

$$\underset{\sim}{U} \equiv \frac{1}{|\hat{\underset{\sim}{S}} \times \hat{\underset{\sim}{E}}|^2} \begin{bmatrix} 1 & -(\hat{\underset{\sim}{S}}\cdot\hat{\underset{\sim}{E}}) \\ -(\hat{\underset{\sim}{S}}\cdot\hat{\underset{\sim}{E}}) & 1 \end{bmatrix} \tag{6}$$

where the tilde below the letter denotes a two-dimensional vector or a 2×2 matrix.

Eqs. (4) can now be written

$$\underset{\sim}{a} = \underset{\sim}{U}\,\underset{\sim}{c} \tag{7a}$$

$$a_N = \pm \frac{1}{|\hat{\underset{\sim}{S}} \times \hat{\underset{\sim}{E}}|^2} [1 - \underset{\sim}{c}^T \underset{\sim}{U}\,\underset{\sim}{c}]^{1/2} \tag{7b}$$

The covariance analysis is now straightforward. Define the three-vector

$$\underset{\approx}{a} \equiv \begin{bmatrix} a_S \\ a_E \\ a_N \end{bmatrix} \tag{8}$$

Then the covariance matrix of the measurements is given by

$$\underset{\sim}{P}_c \equiv \langle \delta \underset{\sim}{c}\ \delta \underset{\sim}{c}^T \rangle \tag{9}$$

where the bracket denotes the expectation value and $\delta \underset{\sim}{c}$ is the error in $\underset{\sim}{c}$. The covariance matrix of the spin-axis direction in the non-orthogonal coordinate system is

$$P_a \equiv \langle \delta \underline{a}\ \delta \underline{a}^T \rangle \tag{10}$$

and in an orthogonal coordinate system

$$P \equiv \langle \delta \hat{\underline{n}}\ \delta \hat{\underline{n}}^T \rangle \tag{11}$$

Substitution of Eqs. (7) in Eq. (10) gives readily

$$P_a = \left[\begin{array}{c|c} \underset{\sim}{M} & \underset{\sim}{V} \\ \hline \underset{\sim}{V}^T & S \end{array} \right] \tag{12}$$

with

$$M \equiv \langle \delta \underset{\sim}{a}\ \delta \underset{\sim}{a}^T \rangle = \underset{\sim}{U}\ \underset{\sim}{P}_c\ \underset{\sim}{U}^T \tag{13a}$$

$$\underset{\sim}{V} = \underset{\sim}{M}\ \underset{\sim}{b} \tag{13b}$$

$$S = \underset{\sim}{b}^T\ \underset{\sim}{M}\ \underset{\sim}{b} \tag{13c}$$

Then the covariance matrix of the measurements is given by

$$P_c \equiv \langle \delta c \; \delta c^T \rangle \tag{9}$$

where the bracket denotes the expectation value and δc is the error in c. The covariance matrix of the spin-axis direction in the non-orthogonal coordinate system is

$$P_a \equiv \langle \delta a \; \delta a^T \rangle \tag{10}$$

and in an orthogonal coordinate system

$$P \equiv \langle \delta \hat{n} \; \delta \hat{n}^T \rangle \tag{11}$$

Substitution of Eqs. (7) in Eq. (10) gives readily

$$P_a = \left[\begin{array}{c|c} M & V \\ \hline V^T & S \end{array} \right] \tag{12}$$

with

$$M \equiv \langle \delta a \; \delta a^T \rangle = U \, P_c \, U^T \tag{13a}$$

$$V = M \, b \tag{13b}$$

$$S = b^T \, M \, b \tag{13c}$$

where

$$B = \left[\begin{array}{c|c} \underset{\sim}{I} & \underset{\sim}{b} \\ \hline \underset{\sim}{b}^T & 0 \end{array}\right] \tag{18}$$

Equations (17) and (14) may now be combined to give

$$P = \sum_{i=1}^{2} \sum_{j=1}^{2} M_{ij} \underset{\sim}{X}_i \underset{\sim}{X}_j^T \tag{19}$$

where

$$\underset{\sim}{X}_1 = \hat{\underset{\sim}{S}} + b_S(\hat{\underset{\sim}{S}} \times \hat{\underset{\sim}{E}}) \tag{20a}$$

$$\underset{\sim}{X}_2 = \hat{\underset{\sim}{E}} + b_E(\hat{\underset{\sim}{S}} \times \hat{\underset{\sim}{E}}) \tag{20b}$$

Eq. (16) is again satisfied since

$$\underset{\sim}{X}_i \cdot \hat{\underset{\sim}{n}} = 0 \qquad i=1,2 \tag{21}$$

III. Single-Frame Spin-Axis Estimation from the Measurement of Two Arc Lengths and the Included Dihedral Angle

The ambiguity in determining the spin-axis observed in the previous section is removed if the included dihedral angle is also measured. The dihedral angle ψ is defined as the angle between the $(\hat{\underset{\sim}{S}},\hat{\underset{\sim}{n}})$ and $(\hat{\underset{\sim}{E}},\hat{\underset{\sim}{n}})$ planes and is easily shown to be given by

$$\sin\psi = \frac{\hat{\underset{\sim}{n}}\cdot(\hat{\underset{\sim}{S}} \times \hat{\underset{\sim}{E}})}{\sqrt{(1-(\hat{\underset{\sim}{S}}\cdot\hat{\underset{\sim}{n}})^2)(1-(\hat{\underset{\sim}{E}}\cdot\hat{\underset{\sim}{n}})^2)}} \tag{22a}$$

$$\cos\psi = \frac{(\hat{\underline{S}}\cdot\hat{\underline{E}}) - (\hat{\underline{S}}\cdot\hat{\underline{n}})(\hat{\underline{E}}\cdot\hat{\underline{n}})}{\sqrt{(1-(\hat{S}\cdot\hat{n})^2)(1-(\hat{E}\cdot\hat{n})^2)}} \tag{22b}$$

$$\tan\psi = \frac{\hat{\underline{n}}\cdot(\hat{\underline{S}}\times\hat{\underline{E}})}{(\hat{\underline{S}}\cdot\hat{\underline{E}}) - (\hat{\underline{S}}\cdot\hat{\underline{n}})(\hat{\underline{E}}\cdot\hat{\underline{n}})} \tag{22c}$$

The geometry is depicted in Figure 2.

To determine the spin axis attitude it will be convenient to define

$$c_N = \sqrt{(1-c_S^2)(1-c_E^2)}\ \sin\psi \tag{23}$$

and

$$\underset{\sim}{c} = \begin{bmatrix} c_S \\ c_E \\ c_N \end{bmatrix} \tag{24}$$

The vector $\underset{\sim}{a}$ is now determined by four equations

$$c_S = a_S + a_E(\hat{\underline{S}}\cdot\hat{\underline{E}}) \tag{25a}$$

$$c_E = a_S(\hat{\underline{S}}\cdot\hat{\underline{E}}) + a_E \tag{25b}$$

$$c_N = |\hat{\underline{S}}\times\hat{\underline{E}}|^2\ a_N \tag{25c}$$

$$1 = a_S^2 + a_E^2 + 2\,a_S a_E(\hat{\underline{S}}\cdot\hat{\underline{E}}) + a_N^2\ |\hat{\underline{S}}\times\hat{\underline{E}}|^2 \tag{25d}$$

The three components of $\underset{\sim}{a}$ are now overdetermined. The most convenient solution is obtained by solving the first three equations, which are linear, leading to

$$\underset{\sim}{a} = U \underset{\sim}{c} \tag{26}$$

where

$$U = \frac{1}{|\hat{\underset{\sim}{S}} \times \hat{\underset{\sim}{E}}|^2} \begin{bmatrix} 1 & -(\hat{\underset{\sim}{S}} \cdot \hat{\underset{\sim}{E}}) & 0 \\ -(\hat{\underset{\sim}{S}} \cdot \hat{\underset{\sim}{E}}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

The spin-axis $\underset{\sim}{n}$ given by this $\underset{\sim}{a}$, however, is not properly normalized since the measurements are not exact. A properly normalized spin-axis vector is then obtained by simply normalizing the solution

$$\hat{n} = \underset{\sim}{n}/|\underset{\sim}{n}| \tag{28}$$

The covariance matrix of $\underset{\sim}{a}$ is given simply by

$$P_a = U \, P_c \, U^T \tag{29}$$

and the covariance matrix for the <u>unnormalized</u> spin-axis is given by

$$P_n = T \, P_a \, T^T \tag{30}$$

similarly to Eq. (14). The covariance matrix of the properly normalized spin-axis vector is recovered simply as

$$P = \frac{1}{|\underset{\sim}{n}|^2} Q \, P_n \, Q \tag{31}$$

where

$$Q = I - \hat{\underset{\sim}{n}}\,\hat{\underset{\sim}{n}}^T \tag{32}$$

It is well to ask how good is the approximation of ignoring the normalization condition and then normalizing the solution post hoc. Instead of this seemingly brutal approach one can find the best solution to Eqs. (25abc) subject to the constraint of Eq. (25d), i.e., one seeks to minimize the loss function

$$L(\underset{\sim}{a}) = (\underset{\sim}{c} - A\underset{\sim}{a})^T P_c^{-1} (\underset{\sim}{c} - A\underset{\sim}{a}) \tag{33}$$

subject to the constaint

$$\underset{\sim}{a}^T A \underset{\sim}{a} = 1 \tag{34}$$

where

$$A = U^{-1} = \begin{bmatrix} 1 & (\hat{\underset{\sim}{S}}\cdot\hat{\underset{\sim}{E}}) & 0 \\ (\hat{\underset{\sim}{S}}\cdot\hat{\underset{\sim}{E}}) & 1 & 0 \\ 0 & 0 & |\underset{\sim}{S} \times \underset{\sim}{E}|^2 \end{bmatrix} \tag{35}$$

The solution is straightforward and yields

$$\underset{\sim}{a}_{opt} = (A - \lambda P_c)^{-1} \underset{\sim}{c} \tag{36}$$

where λ is the Lagrange multiplier for the constraint and from Eq. (34) is the root of the equation

$$\underset{\sim}{c}^T \frac{1}{A - \lambda P_c} A \frac{1}{A - \lambda P_c} \underset{\sim}{c} = 1 \tag{37}$$

which yields the smallest value of the loss function.

Equation (36) may be rewritten

$$\underline{a}_{opt} = (I-\lambda P_c U)^{-1}\underline{a} \tag{38}$$

where $\underline{a}$ is given by Eq. (26). Since $\underline{a}_{opt}$ is expected to be close to $\underline{a}$, it follows that $\lambda P_c U$ must be small. An approximate solution for $\underline{a}_{opt}$ can be obtained by expanding Eqs. (37) and (38) in λP_{cc} and solving. This yields

$$\underline{a}_{opt} - \underline{a} \approx \frac{1}{2}\frac{(1-\underline{a}^T A\underline{a})}{a^T A P_c A^{-1} a} P_c U \underline{a} \tag{39}$$

Now

$$\langle 1-\underline{a}^T A\underline{a}\rangle = Tr(P_c U) \tag{40a}$$

$$\langle (1-\underline{a}^T A\underline{a})^2\rangle = 4\, \underline{a}^T P_c \underline{a} \tag{40b}$$

so that the additional root mean square (rms) error in $\underline{a}$ when optimality is not taken into account is of the same order of magnitude as the rms error in the cosine measurements. However, the source of this additional error, as shown by Eqs. (40) is the error in the normalization. Hence this error will be greatly reduced when the unit vector is normalized.

IV. Batch Estimation

The value of avoiding trigonometric expressions becomes more obvious in dealing with batch estimation. The computational advantage of the present approach over the geometrical approach[3] is substantial.

For batch estimation the non-orthogonal basis cannot be used since only the Sun vector is constant (and then only for relatively short data spans). The present treatment focuses on the case where the measurements consist of two arc lengths and the included dihedral angle. The extension to other cases is straightforward.

Let $c_S(i)$, $c_E(i)$, $c_N(i)$ be a series of measurements of the Sun projection, the nadir projection, and the Sun-nadir dihedral angle, respectively. Then the best solution for the spin-axis is obtained by minimizing

$$L(\hat{\underline{n}}) = \sum_{i=1}^{N} \Big\{ \frac{1}{\sigma_S^2} \left| c_S - \hat{\underline{n}}\cdot\hat{\underline{S}}_i \right|^2$$

$$+ \frac{1}{\sigma_E^2} \left| c_E - \hat{\underline{n}}\cdot\hat{\underline{E}}_i \right|^2$$

$$+ \frac{1}{\sigma_N^2} \left| c_N - \hat{n}\cdot(\hat{\underline{S}} \times \underline{E})_i \right|^2 \Big\} \qquad (41)$$

subject to the constraint

$$\hat{\underline{n}}\cdot\hat{\underline{n}} = 1 \qquad (42)$$

In order to decrease the number of subscripts in the expressions it has been assumed that each data type is available at each time and that each measurement type has a single characteristic error. Except for a proliferation of subscripts the expressions which follow are not changed when this assumption is removed.

The minimization of Eq. (41) subject to the constraint is straightforward and leads to

$$\hat{\underset{\sim}{n}} = (M-\lambda I)^{-1} \underset{\sim}{V} \tag{43}$$

where

$$M = \sum_{i=1}^{N} \{\frac{1}{\sigma_S^2} \hat{\underline{S}}_i \hat{\underline{S}}_i^T + \frac{1}{\sigma_E^2} \hat{\underline{E}}_i \hat{\underline{E}}_i^T + \frac{1}{\sigma_N^2} (\hat{\underline{S}} \times \hat{\underline{E}})_i (\hat{\underline{S}} \times \hat{\underline{E}})_i^T\} \tag{44a}$$

$$\underset{\sim}{V} = \sum_{i=1}^{N} \{\frac{1}{\sigma_S^2} \hat{c}_S(i) \hat{\underline{S}}_i + \frac{1}{\sigma_E^2} c_E(i) \hat{\underline{E}}_i + \frac{1}{\sigma_N^2} c_N(i) (\hat{\underline{S}} \times \hat{\underline{E}})_i\} \tag{44b}$$

and λ is the root of

$$\underset{\sim}{V}^T \frac{1}{(M-\lambda I)^2} \underset{\sim}{V} = 1 \tag{45}$$

which leads to the smallest value of Eq. (41).

As in the previous section it can be expected that the constraint can be ignored ($\lambda \approx 0$) and the solution be approximated by

$$\hat{\underline{n}} = \underline{n}/|\underline{n}| \tag{46}$$

where

$$\underset{\sim}{n} = M^{-1} \underset{\sim}{V} \tag{47}$$

This approximation has been tested for one spacecraft[10] and been observed to be quite good. The covariance of $\underline{n}$ is given by

$$P_n = M^{-1} \tag{48}$$

and the covariance of the normalized solution is given again by

$$P = \frac{1}{|\underline{n}|^2} Q P_n Q \tag{49}$$

V. Measurement Errors

The computation of the spin-axis covariance matrix requires as input a model for the covariance matrix of the cosine measurements. Expressions are derived here for computing these for the case of Sun and Nadir measurements. The treatment when one of these measured quantities is the magnetic field is treated in the same way.

Sun Measurements

The quantity measured is usually the Sun angle, β. Hence,

$$\delta c_S = -\sin\beta\,\delta\beta \tag{50}$$

Nadir Measurements

If the spacecraft has angular velocity ω, then the Earth width is given by

$$\Omega = \omega(t_0 - t_I) \tag{51}$$

where t_I and t_0 are the in- and out-triggering times, respectively, of the Earth scan (for a momentum-wheel mounted scanner, ω will be the angular velocity of the momentum wheel).

Then, using the results from the appendix

$$\delta c_E = \delta \cos\eta$$

$$= \frac{\partial \cos\eta}{\partial \cos\frac{\Omega}{2}} \delta\cos\frac{\Omega}{2}$$

$$= -\frac{\sin\eta}{\cot\gamma - \cot\eta} \delta\cos\frac{\Omega}{2}$$

$$= \frac{\omega}{2}\frac{\sin\eta}{\cot\gamma - \cot\eta} (\sin\frac{\Omega}{2}) (\delta t_0 - \delta t_I) \quad (52)$$

where γ is the scan-cone half angle.

Dihedral Angle Measurements

The dihedral angle ψ is determined from the time interval from the Sun crossing to the mid-point of the horizon scan

$$\psi = \omega[t_S - \tfrac{1}{2}(t_0 + t_I)] \quad (53)$$

Thus, (β, Ω, ψ) or (β, η, ψ) is a set of statistically independent variables. The "dihedral cosine" c_N, however, is given by

$$c_N = \sin\beta \sin\eta \sin\psi \quad (54)$$

hence

$$\delta c_N = c_N[\cot\beta\ \delta\beta + \cot\eta\ \delta\eta + \cot\psi\ \delta\psi] \quad (55)$$

From Eqs. (50-55) the covariance matrix P_c can easily be calculated.

To a large degree, much of the trigonometric complexity which has been removed from the attitude solution has simply been shifted to the computation of a derived measurement covariance matrix. There is, however, a substantial gain because the covariance matrix need not be computed to the same degree of accuracy as the spin-axis attitude itself. Hence, a great deal of computational approximation is possible, such as approximation of the trigonometric functions by simple rational functions.

Appendix - Measurement of the Nadir Angle

Because the Earth is an extended body the nadir vector is not measured directly but determined from measurements of the Earth width. Earth widths are measured by a horizon scanner, which effectively is a sensor mounted on a rotating cone (of half-cone angle γ) about the spacecraft spin axis, which detects the crossings of the Earth horizon on the scan cone. The Earth has an effective angular radius of ρ, which is a function of altitude and (for a non-spherical Earth) latitude. The Earth width is the dihedral angle between the in- and out-crossings (H_I and H_0) the horizon by the scanner and is denoted by Ω. These quantities are related by the spherical law of cosines[2]

$$\cos\rho = \cos\gamma \cos\eta + \sin\gamma \sin\eta \cos(\Omega/2) \qquad \text{(A-1)}$$

The geometry is depicted in Figure 3.

Eq. (A-1) may be solved to give

$$\cos\eta = \frac{\cos\rho \cos\gamma \pm \sin\rho \cos(\Omega/2) \sqrt{A-\cos^2\rho}}{A} \qquad \text{(A-2)}$$

where

$$A = \cos^2\rho + \sin^2\gamma \cos^2(\Omega/2) \tag{A-3}$$

The sign ambiguity may be eliminated if another measurement is present, say that of the Sun angle, β, and the Sun-Earth dihedral angle, γ. Let ξ be the arc length from the Sun direction to the mid scan point

$$\cos\xi = \cos\beta \cos\gamma + \sin\beta \sin\gamma \cos\psi \tag{A-4}$$

Then it is possible to show that the underdetermined sign in Eq. (A-2) must be the same as that of

$$(\cos\beta - \cos\gamma)\ (\hat{E}\cdot\hat{S} - \cos\xi)$$

Alternatively, one may consider simultaneously Sun and horizon measurements. This leads to three simultaneous equations

$$\cos\beta \cos\eta + \sin\beta \sin\eta \cos\psi = \hat{E}\cdot\hat{S} \tag{A-5a}$$

$$\cos\gamma \cos\eta + \sin\gamma \sin\eta \cos(\Omega/2) = \cos\rho \tag{A-5b}$$

$$\cos^2\eta + \sin^2\eta = 1 \tag{A-5c}$$

Equation (A-2) was obtained by solving Eqs. (A-5b) and (A-5c) simultaneously. One could just as easily solve Eqs. (A-5a) and (A-5b) for $\cos\eta$ and $\sin\eta$. The result will not necessarily satisfy Eq. (A-5c) but the two equations have the advantage of being linear. The solutions can then be renormalized to satisfy Eq. (A-5c).

This approach of ignoring the proper normalization for the trigonometric functions has another advantage in that a simultaneous solution to Eqs. (A-5b) and (A-5c) may not exist in certain extreme cases because the measurements are not exact. By solving Eqs. (A-5a) and (A-5b) a solution will always exist.

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