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The Generalized Wahba Problem

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To generalize is to be an idiot.

William Blake (1757–1827)

People who like quotations love meaningless generalizations.

Graham Greene (1904–1991)
A Burnt-Out Case (1960)

Abstract

The Generalized Wahba Problem, which can accept as input both measured directions and measured attitudes, is defined and examined in terms of both the attitude profile matrix B and the Davenport matrix K. The possibility of extending the generalization to scalar measurements is also examined. We obtain a number of new results relating these two matrices to the attitude estimate and to the attitude-error covariance matrix. We compare the generalized Wahba problem also with a less restrictive approach to attitude estimation.

Introduction: The Wahba Problem

The Generalized Wahba Problem is an extension of the original Wahba problem [1] which extends the inputs to include not only direction measurements but also measurements of whole attitudes. Its advantage over the original Wahba problem is that it permits the inclusion of an initial condition as well as star-camera attitude estimates as inputs.

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The Wahba problem, first proposed by Grace Wahba in 1965 [1], seeks the *proper orthogonal* matrix (attitude matrix, direction-cosine matrix) A which minimizes the cost function[2]

$$J(A) = \frac{1}{2} \sum_{k=1}^{N} a_k |\hat{\mathsf{W}}_k - A\hat{\mathsf{V}}_k|^2 \tag{1}$$

where $\hat{\mathsf{W}}_k$, $k = 1, \ldots, N$, is a set of N measured directions observed in the spacecraft body frame, $\hat{\mathsf{V}}_k$, $k = 1, \ldots, N$, the corresponding reference directions, and a_k, $k = 1, \ldots, N$, a set of positive weights.

Several solutions [3] were offered almost immediately for solving this problem, none of which was really suitable at the time for practical spacecraft mission support. However, they all shared one common property, which has persisted also in the fast solutions: they took advantage of the fact that Wahba's cost function could be written as

$$J(A) = \lambda_o - g(A) = \lambda_o - \mathrm{tr}[B^{\mathrm{T}} A] \tag{2}$$

where

$$\lambda_o = \sum_{k=1}^{N} a_k \quad \text{and} \quad B = \sum_{k=1}^{N} a_k \hat{\mathsf{W}}_k \hat{\mathsf{V}}_k^{\mathrm{T}} \tag{3ab}$$

and tr denotes the trace operation. This writer generally refers to B as the *attitude profile matrix.* The great power of the Wahba problem arises from the fact that the gain function $g(A)$ is linear in the attitude matrix A.

The next significant development of the Wahba problem was made more than a decade later by Paul Davenport,[3] who showed that as a function of the quaternion the gain function could be written as[4]

$$g(\bar{q}) = \bar{q}^{\mathrm{T}} K \bar{q} \tag{4}$$

where K is the 4×4 symmetric matrix

$$K = \begin{bmatrix} S - s\, I_{3\times3} & Z \\ Z^{\mathrm{T}} & s \end{bmatrix} \tag{5}$$

with

$$s \equiv \mathrm{tr}\, B\,, \qquad S \equiv B + B^{\mathrm{T}}\,, \tag{6ab}$$

$$Z \equiv [B_{23} - B_{32}, \quad B_{31} - B_{13}, \quad B_{12} - B_{21}]^{\mathrm{T}} \tag{6c}$$

The optimal quaternion estimate, $\bar{q}^{*\prime}$, Davenport showed, was the characteristic vector of K' with the largest characteristic value. Thus,[5]

[2] $\hat{\mathsf{W}}_k$ and $\hat{\mathsf{V}}_k$ and all other vectors in this work are column vectors and not (abstract) physical vectors and, therefore, following the conventions of reference [2], are denoted by bold unslanted sans-serif characters.

[3] Unpublished by Davenport but presented in references [3] and [4].

[4] To be more rigorous one should write $g_A(A)$ and $g_{\bar{q}}(\bar{q}) \equiv g_A(A(\bar{q}))$, and similarly for $J(A)$, $J(\bar{q})$, and $J(\boldsymbol{\epsilon})$ below, but the reader should have no problem interpreting our expressions. Likewise, we have been somewhat casual about distinguishing random variables from their sampled values, when a result is true for both. We have not written g', K', B', etc., in equations (4) and (5) because the equations hold true both for the random variables and the sampled values. We have written equations (1) through (6) in terms of random vector measurements and related quantities to avoid a multitude of subscripts or superscripts.

[5] The quantity $\lambda'_{\max}$ in equation (7) is the sampled value of a random variable variable, $\lambda_{\max}$, i.e., the value for the sampled values of the measurements, and λ_o, is the true value of $\lambda_{\max}$, i.e., the value for the true value of

$$K'\bar{q}^{*\prime} = \lambda'_{\max}\bar{q}^{*\prime} \tag{7}$$

Davenport computed $\bar{q}^{*\prime}$ using Householder's method [5]. In this manner, Davenport's q-method was used to support the HEAO missions [6]. Davenport's q-method is the immediate ancestor of the QUEST algorithm [3, 4], of Mortari's many ESOQ algorithms [3, 7, 8], and a new unchristened algorithm of Bruccoleri, Lee, and Mortari in terms of the modified Rodrigues parameters [9]. Markley's SVD algorithm [3, 10] and FOAM algorithm [3, 11], on the other hand, are direct descendants of the Wahba problem.

To obtain a prescription for the weights, a_k, $k = 1, \ldots, N$, appearing in equations (1) and (3), the QUEST measurement model [4] was proposed, namely,

$$\hat{\mathbf{W}}_k = A\hat{\mathbf{V}}_k + \Delta\hat{\mathbf{W}}_k\,, \qquad k = 1, \ldots, N \tag{8}$$

where the $\Delta\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$, the measurement noise vectors, were assumed to be mutually independent, zero-mean, and (approximately) Gaussian with covariance matrix

$$E\{\Delta\hat{\mathbf{W}}_k \Delta\hat{\mathbf{W}}_k^{\mathrm{T}}\} = \sigma_k^2(I_{3\times 3} - \hat{\mathbf{W}}_k^{\text{true}}\hat{\mathbf{W}}_k^{\text{true T}}) \equiv R_{\hat{\mathbf{W}}_k} \tag{9a}$$

$$\hat{\mathbf{W}}_k^{\text{true}} = A^{\text{true}}\hat{\mathbf{V}}_k \tag{9b}$$

Equation (9b) assumes that the reference vectors $\hat{\mathbf{V}}_k$, $k = 1, \ldots, N$, can be taken to be error-free, i.e., nonrandom. The assumption that $\Delta\hat{\mathbf{W}}_k$, $k = 1, \ldots, N$, are zero-mean and Gaussian can only be approximate [12]. The assumption that these quantities have circles of error rather than ellipses is an enormous convenience but also cannot be exact.[6]

Thus, the observed directions were assumed to have circles of error in the planes tangent to the observations. It could now be shown that $J'(\bar{q})$ would achieve its smallest value if the a_k were chosen as[7]

$$a_k = \lambda_o \sigma_{\text{tot}}^2/\sigma_k^2\,, \qquad k = 1, 2, \ldots, N \tag{10}$$

where

$$\frac{1}{\sigma_{\text{tot}}^2} \equiv \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \tag{11}$$

Generally, one chooses $\lambda_o = 1$ or $\lambda_o = 1/\sigma_{\text{tot}}^2$.

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the measurements (hence, $\lambda^{\text{true}} = \lambda_o$). The random $\lambda_{\max}$ is the characteristic value in the random equation $K\bar{q}^* = \lambda_{\max}\bar{q}^*$. We prefer to retain the notation $\lambda_{\max}$ for the random variable, because the random K-matrix bears no related mark. We will observe this nice distinction in the sequel. Our brief discussion of the Wahba problem and Davenport's q-method, equations (1) through (6) above, always used $\hat{\mathbf{W}}_k$ rather than $\hat{\mathbf{W}}'_k$, the general notation for the sampled value of a random direction measurement. This is alright, because the equations for the attitude estimator are homologous *mutatis mutandus* to the equations for the attitude estimate. See the glossary in Appendix A for a summary of the statistical notation in this paper.

[6]An examination of more complex models for the error distribution has been carried out recently [13] and showed as much as a 30 percent improvement in attitude estimation accuracy (standard deviation) for attitude sensors with large fields of view. The computational burden of this improvement over that of the QUEST measurement model, however, is high.

[7]$J(\bar{q}^*)$ is a function not only of the random variable $\bar{q}^*$ but also of the random measurements. Thus, in mission support, when we evaluate J at the attitude estimate, we must write $J'(\bar{q}^{*\prime})$, since we will use sampled values of the measurements as well. Likewise, when the true values of the attitude and the measurements are used, we write $J^{\text{true}}(\bar{q}^{\text{true}})$ (which vanishes, obviously). $J'(\bar{q})$, $J'(\bar{q}^{*\prime})$, and $J'(\bar{q}^{\text{true}})$ are all useful expressions. The term *estimate* applies correctly only to the sampled value $\bar{q}^{*\prime}$. The random variable $\bar{q}^*$ is the *estimator.* A, $\bar{q}$, and $\boldsymbol{\epsilon}$ without any marks simply denote free variables.

Based on the QUEST measurement model and the weights given by equation (10), the inverse attitude estimate-error covariance matrix corresponding to the QUEST attitude estimate (or to any solution to the Wahba problem) was shown to be [4]

$$P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \left(I_{3\times 3} - \hat{\mathbf{W}}_k^{\text{true}} \hat{\mathbf{W}}_k^{\text{true T}}\right) \tag{12}$$

where $\boldsymbol{\epsilon}$ denotes the attitude increment vector [2], a small rotation vector measured most generally in attitude estimation from predicted body axis. When the attitude increment vector is measured from the *true* body axes, it is denoted in this work by $\tilde{\boldsymbol{\epsilon}}$. Note that the inverse covariance matrix of equation (12) may not be invertible. Thus, the covariance matrix itself may not exist.

Another innovation of consequence for the present work is the TASTE test. TASTE is a random variable defined to be[8]

$$\text{TASTE} \equiv 2J(A^*)/\lambda_o \sigma_{\text{tot}}^2 = 2(\lambda_o - \lambda_{\max})/\lambda_o \sigma_{\text{tot}}^2 \tag{13}$$

It is easy to demonstrate [14] that TASTE is a χ^2 variable with $2N - 3$ degrees of freedom

$$\text{TASTE} \sim \chi^2(2N - 3) \tag{14}$$

if the data is correctly modeled by equations (8) and (9). Under this condition TASTE will have a mean of $2N - 3$ and a variance $2(2N - 3)$. If this is not the case, due, for example, to misidentification of stars, errors in the star catalogue, light interference in a Sun sensor or star tracker, or some other fault of a sensor, then TASTE will take on values very far from the expected mean, generally by enormous multiples of the confidence bound, and indicate the lack of validity of the data. In this manner, the TASTE test has been very valuable for data validation in mission support, beginning with the Magsat mission [15]. As we shall see below, some operations can disable the TASTE test.

Perhaps, the most significant innovation in the study of the Wahba problem with relevance to the present work was the demonstration [12] that the maximum-likelihood estimate of the attitude given the QUEST measurement model was, in fact, the Wahba problem with the weights defined according to equations (10) and (11). Rather than just being an heuristically motivated attitude problem, the Wahba problem now entered the mainstream of Estimation Theory. One immediate consequence of this was that asymptotically (i.e., as $N \longrightarrow \infty$) the inverse attitude-error covariance matrix became the Fisher information matrix [16], which could be calculated much more easily than the attitude-error covariance matrix in reference [4]. Thus,

$$P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}^{-1} = F_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}} \equiv E\left\{ \frac{\partial^2 J(\tilde{\boldsymbol{\epsilon}})}{\partial \tilde{\boldsymbol{\epsilon}} \partial \tilde{\boldsymbol{\epsilon}}^{\text{T}}} \right\}\Bigg|_{\tilde{\boldsymbol{\epsilon}}=0} \tag{15}$$

which has immediate consequences for this work. For optimal attitude estimation, this result has led to the possibility of recasting the Wahba problem both as a Kalman filter and as a Rauch-Tung-Striebel smoother [17].

The Generalized Wahba Problem

The Wahba problem [1] estimates the attitude from direction measurements alone assuming effectively that these have random measurement errors described by the

[8] TASTE so defined is independent of the value chosen for λ_o.

QUEST measurement model, equations (8) and (9). We can extend the Wahba problem to use whole attitude measurements and their respective attitude error covariance matrices as well. We call this extension the *Generalized Wahba Problem.*[9]

The Generalized Attitude Profile Matrix

Consider equation (15) above. We may write

$$A = \delta A(\tilde{\boldsymbol{\epsilon}}) A^{\text{true}} \tag{16}$$

where

$$\begin{aligned}\delta A(\tilde{\boldsymbol{\epsilon}}) &= I_{3\times 3} + \frac{\sin(|\tilde{\boldsymbol{\epsilon}}|)}{|\tilde{\boldsymbol{\epsilon}}|}[[\tilde{\boldsymbol{\epsilon}}]] + \frac{1-\cos(|\tilde{\boldsymbol{\epsilon}}|)}{|\tilde{\boldsymbol{\epsilon}}|^2}[[\tilde{\boldsymbol{\epsilon}}]]^2 \\ &= I_{3\times 3} + [[\tilde{\boldsymbol{\epsilon}}]] + (1/2)\,[[\tilde{\boldsymbol{\epsilon}}]]^2 + \ldots\end{aligned} \tag{17}$$

with

$$[[\tilde{\boldsymbol{\epsilon}}]] \equiv \begin{bmatrix} 0 & \epsilon_3 & -\epsilon_2 \\ -\epsilon_3 & 0 & \epsilon_1 \\ \epsilon_2 & -\epsilon_1 & 0 \end{bmatrix} \tag{18}$$

$\delta A(\boldsymbol{\epsilon})$ is the direction-cosine matrix for a very small rotation [2].

At the true value of the measurements, equations (2) and (3) lead to

$$\begin{aligned}J^{\text{true}}(\tilde{\boldsymbol{\epsilon}}) &= \lambda_o - \text{tr}\left[B_{\text{true}}^{\text{T}}\,\delta A(\tilde{\boldsymbol{\epsilon}}) A^{\text{true}}\right] \\ &= \lambda_o - \text{tr}\left[A^{\text{true}} B_{\text{true}}^{\text{T}}\,\delta A(\tilde{\boldsymbol{\epsilon}})\right] \\ &\equiv \lambda_o - \text{tr}\left[D_{\text{true}}^{\text{T}}\,\delta A(\tilde{\boldsymbol{\epsilon}})\right]\end{aligned} \tag{19}$$

Carrying out the partial differentiations of equation (15) leads directly to

$$\begin{aligned}P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}^{-1} &= (\text{tr}\,D_{\text{true}}) I_{3\times 3} - \frac{1}{2}\left(D_{\text{true}} + D_{\text{true}}^{\text{T}}\right) \\ &= (\text{tr}\,D_{\text{true}}) I_{3\times 3} - D_{\text{true}}\end{aligned} \tag{20}$$

since D_{true} is symmetric, because $A^{\text{true}\,\text{T}}$ is the orthogonal matrix of the polar decomposition of B_{true} [18], and similarly for D and D'.

In general, we have no choice but to evaluate $P_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}}^{-1}$ at the sampled values of the measurements and the estimated value of the attitude. In practice, what we *usually* call the attitude-error covariance matrix is, in fact, equation (20) evaluated at the *sampled* values of the measurements, that is,

$$(P'_{\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}})^{-1} = (\text{tr}\,D') I_{3\times 3} - D' \tag{21}$$

with (for all three statistical possibilities)[10]

$$D = BA^{*\text{T}}, \qquad D' = B'A^{*\prime\text{T}} \qquad \text{and} \qquad D_{\text{true}} = B_{\text{true}} A_{\text{true}}^{\text{T}} \tag{22abc}$$

[9]The generalized Wahba problem first appeared without being so named in reference [12], where it was applied to the inclusion of an initial condition in the Wahba cost function. The exploration of the generalized Wahba problem in this work is far more extensive.

[10]When an equation can be written entirely in terms of random variables (assumed to be the true values plus zero-mean random error), then obviously it holds separately for true values and sampled values as well. A glossary of statistical variable types can be found in Appendix A.

We may solve equations (20) and (21) for B' and write, in general,[11]

$$B = DA^* , \qquad B' = D'A^{*\prime} \qquad \text{and} \qquad B_{\text{true}} = D_{\text{true}} A_{\text{true}} \tag{23abc}$$

with, recalling equation (12),

$$D = \frac{1}{2}\left[\text{tr}\,(P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{r.v.}})^{-1}\right] I_{3\times3} - (P_{\tilde{\epsilon}\tilde{\epsilon}}^{\text{r.v.}})^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k \hat{\mathbf{W}}_k^{\text{T}} \tag{24a}$$

$$D' = \frac{1}{2}\left[\text{tr}\,(P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1}\right] I_{3\times3} - (P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}'_k \hat{\mathbf{W}}_k^{\prime\text{T}} \tag{24b}$$

$$D_{\text{true}} = \frac{1}{2}\left[\text{tr}\,(P_{\tilde{\epsilon}\tilde{\epsilon}})^{-1}\right] I_{3\times3} - (P_{\tilde{\epsilon}\tilde{\epsilon}})^{-1} = \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \hat{\mathbf{W}}_k^{\text{true}} \hat{\mathbf{W}}_k^{\text{true}\,\text{T}} \tag{24c}$$

The matrix D will appear often in this work and deserves a name. Since it is closely related to $P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1}$, the *attitude information matrix,* we call D the *attitude co-information matrix.*

Thus, we may include the estimate of the star-tracker quaternion and attitude-error covariance matrix of the previous section in the Wahba problem by including in the (cumulative) attitude profile matrix B' of the spacecraft a term B'_{ST}, which is[12]

$$B'_{\text{ST}} = \left[\frac{1}{2}\left(\text{tr}\,(R'_{\text{ST}})^{-1}\right) I_{3\times3} - (R'_{\text{ST}})^{-1}\right] C_{\text{ST}}^{*\prime} \tag{25}$$

where $C_{\text{ST}}^{*\prime}$ denotes the star-tracker attitude matrix computed from star-tracker data alone, and R'_{ST} is the corresponding covariance matrix computed from the star-tracker data.

B'_{ST} of equation (25), alone (if there is more than a single star observation) or in conjunction with the attitude profile matrix from other attitude measurements conforming to the QUESTmeasurement model, is sufficient to determine the spacecraft attitude estimate and associated attitude-error covariance matrix. However, to calculate the attitude estimate efficiently using the QUEST algorithm or other fast algorithms deriving from Davenport's q-method, we also need to know the contribution of the star-tracker measurements to λ_o or to an equally good approximation of λ'_{max}. This is because those algorithms compute λ'_{max} iteratively and require a good initial value for the iteration.

To accomplish this, we remark that were the star-tracker estimate to have been determined from the QUEST algorithm [4] or from another solution method [3] for the Wahba problem, then the value of $\lambda_{o\text{ST}}$ from the star-tracker data alone would have been, noting equations (10) and (11),

$$\lambda_{o\text{ST}} = \frac{1}{\sigma_{\text{tot}}^2} = \sum_{k=1}^{N_{\text{ST}}} \frac{1}{\sigma_k^2} = \frac{N_{\text{ST}}}{\sigma^2} \tag{26}$$

[11]Equations (22) through (24) illustrate the nonuniform (but not inconsistent) notation in this paper, an unavoidable consequence if we wish to retain the traditional notation for well-known quantities. Appendix A clarifies our conventions.

[12]When we write only one statistical version of an equation, we will usually choose the sampled value, since that is the one which will generally enter into our calculations. We shall assume that by using the methods of an earlier section we have made the necessary transformations so that R_{ST} is always the attitude-error covariance matrix of the star tracker in spacecraft body coordinates and C_{ST} transforms from inertial (or any other "space" reference frame) to the spacecraft body axes.

if the variance is uniform over the star-tracker field of view and known. Thus, the cost function with incorporated star-tracker attitude becomes

$$J(A) = \lambda_{o\,\text{other sensors}} + \lambda_{o\,\text{ST}} - \text{tr}\left[(B'_{\text{other sensors}} + B'_{\text{ST}})^{\text{T}} A\right] \tag{27}$$

from which the attitude estimate and attitude-error covariance matrix using all of the data is readily calculated.

Now

$$R_{\text{ST}}^{-1} = \sum_{k=1}^{N_{\text{ST}}} \frac{1}{\sigma_k^2} \left(I_{3\times 3} - \hat{\mathbf{W}}_k^{\text{true}} \hat{\mathbf{W}}_k^{\text{true}\,\text{T}}\right) \tag{28}$$

so that

$$\lambda_{o\text{ST}} = \frac{1}{2} \text{tr}\, R^{-1} = \text{tr}\left[B_{\text{ST}}^{\text{true}} (C_{B\leftarrow I}^{\text{true}})^{\text{T}}\right] = \text{tr}\, D_{\text{ST}}^{\text{true}} \tag{29}$$

The similar calculation of "λ" from the *sampled* version of equation (29), on the other hand, will yield not $\lambda_{o\text{ST}}$ but $\lambda'_{\max\,\text{ST}}$. To see this we note that

$$\frac{1}{2} \text{tr}\left[(R'_{\text{ST}})^{-1}\right] = \text{tr}\left[B'_{\text{ST}} C_{\text{ST}}^{*\prime}\right] = \bar{\eta}_{\text{ST}}^{*\prime\text{T}} K'_{\text{ST}} \bar{\eta}_{\text{ST}}^{*\prime} = \lambda'_{\max\,\text{ST}} \tag{30}$$

where $\bar{\eta}_{\text{ST}}^{*\prime}$ is the star-tracker quaternion estimate from star-tracker data.

If, instead of the typical star tracker, the spacecraft has a whole-attitude sensor which uses data from which the attitude cannot be calculated in a statistically reliable manner using the Wahba cost function (for example, because the error ellipses are so eccentric that they cannot be represented adequately by circles of error), then one has no choice but to calculate an effective λ_o for that sensor using the sampled version of equation (29), namely, equation (30), assuming that the sensor outputs both the optimal attitude and the attitude-error covariance matrix or information matrix. One can still incorporate the data from that sensor into the Wahba problem via equation (27) with λ_o replaced by $\lambda'_{\max}$ *for that data set,* but, as apparent from equation (13), the TASTE test for data from that sensor will be disabled, because effectively TASTE$' = 0$ (since λ_o has been replaced by $\lambda'_{\max}$ in equation (13)).

Combining Attitude Matrix Estimates

Although the fastest way to combine estimates of the attitude matrix is via the generalized Wahba problem, as developed above, it is noteworthy to add that the optimal attitude matrix, assuming that we have been given $A_i^{*\prime}$ and P'_i, $i = 1, \ldots, n$, can be written equivalently as the value of A which minimizes the cost function

$$J'(A) = \frac{1}{2} \sum_{i=1}^{n} \text{tr}\left[(A_i^{*\prime} - A)^{\text{T}} D'_i \; (A_i^{*\prime} - A)\right] \tag{31}$$

with D' defined by equation (24b). This result is derived most easily by writing

$$A_i^{*\prime} = (I + [[\tilde{\boldsymbol{\epsilon}}_i^{*\prime}]])\, A^{\text{true}} + O(|\tilde{\boldsymbol{\epsilon}}_i^{*\prime}|^2) \tag{32a}$$

$$A = (I + [[\tilde{\boldsymbol{\epsilon}}]])\, A^{\text{true}} + O(|\tilde{\boldsymbol{\epsilon}}|^2) \tag{32b}$$

linearizing equation (31), and comparing the result with the usual least-square cost function for $\tilde{\boldsymbol{\epsilon}}$, namely,

$$J'(\tilde{\boldsymbol{\epsilon}}) = \frac{1}{2}\sum_{i=1}^{N} (\tilde{\boldsymbol{\epsilon}}_i^{*\prime} - \tilde{\boldsymbol{\epsilon}})^{\mathrm{T}} (P'_{\tilde{\epsilon}\tilde{\epsilon}\, i})^{-1} (\tilde{\boldsymbol{\epsilon}}_i^{*\prime} - \tilde{\boldsymbol{\epsilon}}) \tag{33}$$

From a practical standpoint, of course, it is much more convenient to estimate the attitude via a construction similar to that in equation (27). In fact, clearing the parentheses in equation (31) leads directly to

$$J'(A) = \sum_{i=1}^{n} (\mathrm{tr}[D_i'] - \mathrm{tr}[A_i^{*\prime \mathrm{T}} D_i' A]) = \sum_{i=1}^{n} (\lambda'_{\max\, i} - \mathrm{tr}[B_i'^{\mathrm{T}} A]) \tag{34}$$

which should be compared with equation (27).

The Generalized Davenport *K* Matrix

Consider again the calculation of the attitude-error covariance matrix, this time from the Davenport form of the cost function. From equations (2) and (4),

$$J^{\mathrm{true}}(\bar{q}) = \lambda_o - \bar{q}^{\mathrm{T}} K^{\mathrm{true}} \bar{q} \tag{35}$$

we write, in analogy with equation (16),

$$\bar{q} = \delta\bar{q}(\tilde{\boldsymbol{\epsilon}}) \circ \bar{q}^{\mathrm{true}} = \{\bar{q}^{\mathrm{true}}\}_R\, \delta\bar{q}(\tilde{\boldsymbol{\epsilon}}) \tag{36}$$

with $\{\bar{q}^{\mathrm{true}}\}_R$ defined as in reference [2]. Thus,

$$J^{\mathrm{true}}(\tilde{\boldsymbol{\epsilon}}) = \lambda_o - \delta\bar{q}^{\mathrm{T}}(\tilde{\boldsymbol{\epsilon}})\, M^{\mathrm{true}}\, \delta\bar{q}(\tilde{\boldsymbol{\epsilon}}) \tag{37}$$

with

$$M^{\mathrm{true}} \equiv \{\bar{q}^{\mathrm{true}}\}_R^{\mathrm{T}}\, K^{\mathrm{true}}\, \{\bar{q}^{\mathrm{true}}\}_R \tag{38}$$

and

$$\delta\bar{q}(\tilde{\boldsymbol{\epsilon}}) = \begin{bmatrix} \tilde{\boldsymbol{\epsilon}}/2 \\ 1 - |\tilde{\boldsymbol{\epsilon}}|^2/8 \end{bmatrix} + O(|\tilde{\boldsymbol{\epsilon}}|^3) \tag{39}$$

Partitioning M^{true} as

$$M^{\mathrm{true}} = \begin{bmatrix} M_{3\times3}^{\mathrm{true}} & M_{3\times1}^{\mathrm{true}} \\ M_{1\times3}^{\mathrm{true}} & M_{1\times1}^{\mathrm{true}} \end{bmatrix} \tag{40}$$

and carrying out the partial differentiations of equation (15) yields

$$P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} = -\frac{1}{2} M_{3\times3}^{\mathrm{true}} + \frac{1}{2} M_{1\times1}^{\mathrm{true}} I_{3\times3} \tag{41}$$

Noting

$$M_{3\times3}^{\mathrm{true}} = \Xi^{\mathrm{T}}(\bar{q}^{\mathrm{true}})\, K^{\mathrm{true}}\, \Xi(\bar{q}^{\mathrm{true}}) \tag{42a}$$

$$M_{3\times1}^{\mathrm{true}} = \Xi^{\mathrm{T}}(\bar{q}^{\mathrm{true}})\, K^{\mathrm{true}}\, \bar{q}^{\mathrm{true}} = M_{1\times3}^{\mathrm{true}\,\mathrm{T}} \tag{42b}$$

$$M_{1\times1}^{\mathrm{true}} = \bar{q}^{\mathrm{true}\,\mathrm{T}}\, K^{\mathrm{true}}\, \bar{q}^{\mathrm{true}} \tag{42c}$$

and also $K^{\mathrm{true}}\, \bar{q}^{\mathrm{true}} = \lambda_o \bar{q}^{\mathrm{true}}$, we obtain

$$P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} = -\frac{1}{2}\, \Xi^{\mathrm{T}}(\bar{q}^{\mathrm{true}})\, K^{\mathrm{true}}\, \Xi(\bar{q}^{\mathrm{true}}) + \frac{1}{2}\, \lambda_o I_{3\times3} \tag{43}$$

Note that $M_{3\times 1}^{\text{true}} = \mathbf{0}$. Using sampled values of K and $\bar{q}^{*\prime}$, an analogous development leads to[13]

$$(P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1} = -\frac{1}{2}\,\Xi^{\mathrm{T}}(\bar{q}^{*\prime})\,K'\,\Xi(\bar{q}^{*\prime}) + \frac{1}{2}\,\lambda'_{\max}\,I_{3\times 3} \tag{44}$$

Writing K' in terms of its four characteristic values and characteristic vectors

$$K' = \sum_{i=1}^{4} \lambda'_i\,\bar{q}'_i\,\bar{q}'^{\mathrm{T}}_i \tag{45}$$

with $\lambda'_4 = \lambda'_{\max}$ and $\bar{q}'_4 = \bar{q}^{*\prime}$, leads to

$$(P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1} = \frac{1}{2}\sum_{i=1}^{3} (\Xi^{\mathrm{T}}(\bar{q}^{*\prime})\bar{q}'_i)\,(\lambda'_{\max} - \lambda'_i)(\Xi^{\mathrm{T}}(\bar{q}^{*\prime})\bar{q}'_i)^{\mathrm{T}} \tag{46}$$

Thus, $\Xi^{\mathrm{T}}(\bar{q}^{*\prime})\bar{q}'_i$, $i = 1, 2, 3$, are the three characteristic vectors of the inverse attitude-error covariance matrix.

Equation (44) can be solved together with equations (42abc) for K' to obtain

$$\begin{aligned} K' &= \lambda'_{\max}\,I_{4\times 4} - 2\Xi(\bar{q}^{*\prime})\,(P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1}\Xi^{\mathrm{T}}(\bar{q}^{*\prime}) \\ &= \{\bar{q}^{*\prime}\}_R \begin{bmatrix} \lambda'_{\max}\,I_{3\times 3} - 2(P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \lambda'_{\max} \end{bmatrix} \{\bar{q}^{*\prime}\}_R^{\mathrm{T}} \end{aligned} \tag{47}$$

Note that we can write equation (47) as

$$K' = 2\Xi(\bar{q}^{*\prime}) \left[\frac{1}{2}\,(\mathrm{tr}\,(P'_{\epsilon\epsilon})^{-1})I_{4\times 4} - (P'_{\epsilon\epsilon})^{-1} \right] \Xi^{\mathrm{T}}(\bar{q}^{*\prime}) \tag{48}$$

in analogy to[14]

$$B' = \left[\frac{1}{2}\,(\mathrm{tr}\,(P'_{\theta\theta})^{-1})I_{3\times 3} - (P'_{\theta\theta})^{-1} \right] A^{*\prime} \tag{49}$$

Since K' is traceless, we must have from equation (46), (47), or (48) that

$$\lambda'_{\max} = \frac{1}{2}\,\mathrm{tr}\,(P'_{\tilde{\epsilon}\tilde{\epsilon}})^{-1} \tag{50}$$

which corresponds to our earlier result for the star tracker.

Clearly, with respect to the estimated body frame

$$\bar{q}^{*\prime} = \bar{1} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \tag{51}$$

with characteristic value $\lambda'_{\max}$. The other three characteristic values of M' (and, hence, of K') are

$$\lambda'_i = \lambda'_{\max} - 2\,(P'_{\tilde{\epsilon}\tilde{\epsilon}})_i^{-1}, \quad i = 1, 2, 3 \tag{52}$$

or

[13] λ_o in equation (43) is not a carryover of the λ_o which appeared in equation (2) but the result of equation (42c). Thus, the corresponding quantity in equation (44) is $\lambda'_{\max}$ and not λ_o.

[14] Analogously, we may speak of a generalized q-method.

$$(P'_{\tilde{\epsilon}\tilde{\epsilon}})_i^{-1} = (\lambda'_{\max} - \lambda'_i)/2\,, \quad i = 1, 2, 3 \tag{53}$$

where here $(P'_{\tilde{\epsilon}\tilde{\epsilon}})_i$ is the i-th characteristic value of $P'_{\tilde{\epsilon}\tilde{\epsilon}}$. Thus, we see clearly, that the observability of the attitude is related directly to the separation of the other characteristic values of K' from $\lambda'_{\max}$.

From our decomposition of K above we can readily extract B from K.

$$B = \frac{1}{2}\Big[K_{3\times 3} + K_{1\times 1}\, I_{3\times 3} + [[K_{3\times 1}]]\Big] \tag{54a}$$

or

$$B = \frac{1}{2}\Big[(S - s\, I_{3\times 3}) + s\, I_{3\times 3} + [[\,Z\,]]\Big] = \frac{1}{2}\,(S + [[\,Z\,]]) \tag{54b}$$

For the star-tracker contribution to the K-Matrix we have then

$$\begin{aligned} K'_{\mathrm{ST}} &= \lambda'_{\max\,\mathrm{ST}}\; I_{4\times 4} - 2\Xi(\bar{\eta}^{*\prime}_{B\leftarrow I})\,(R'_{\mathrm{ST}})^{-1}\,\Xi^{\mathrm{T}}(\bar{\eta}^{*\prime}_{B\leftarrow I}) \\ &= \{\bar{\eta}^{*\prime}_{B\leftarrow I}\}_R \begin{bmatrix} \lambda'_{\max\,\mathrm{ST}}\; I_{3\times 3} - 2\,(R'_{\mathrm{ST}})^{-1} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \lambda'_{\max\,\mathrm{ST}} \end{bmatrix} \{\bar{\eta}^{*\prime}_{B\leftarrow I}\}_R^{\mathrm{T}} \end{aligned} \tag{55a}$$

and

$$\lambda'_{\max\,\mathrm{ST}} = \frac{1}{2}\,\mathrm{tr}\,(R'_{\mathrm{ST}})^{-1} \tag{55b}$$

Extension to Scalar Measurements

The generalized Wahba problem can accommodate more complete measurements of the attitude than direction measurements but not less complete measurements. To see this let us reexamine the direction measurement from the standpoint of an effective attitude profile matrix.

Examine first the QUEST measurement model [3, 4]. For a single direction measurement described by equations (8) and (9) we obtain readily from equation (15)

$$P^{-1}_{\tilde{\epsilon}\tilde{\epsilon}} = \frac{1}{\sigma^2}\,(I - \hat{\mathbf{W}}^{\mathrm{true}}\hat{\mathbf{W}}^{\mathrm{true\,T}}) \tag{56}$$

and from equations (23) and (24)

$$B^{\mathrm{true}} = \left[\frac{1}{\sigma^2}\,\hat{\mathbf{W}}^{\mathrm{true}}\hat{\mathbf{W}}^{\mathrm{true\,T}}\right] A^{\mathrm{true}} = \frac{1}{\sigma^2}\,\hat{\mathbf{W}}^{\mathrm{true}}\hat{\mathbf{W}}^{\mathrm{T}} \tag{57}$$

from which we infer[15]

$$B' = \frac{1}{\sigma^2}\,\hat{\mathbf{W}}'\,\hat{\mathbf{V}}^{\mathrm{T}} \tag{58}$$

as one would wish.

If, on the other hand, we consider a scalar measurement, namely, the component of $\hat{\mathbf{W}}$ along the (non-random) direction $\hat{\mathbf{U}}$

[15]It is this very simple form of the attitude profile matrix for a single direction measurement which permits the recasting of the Wahba problem as a Kalman filter and a Rauch-Tung-Striebel smoother [17].

$$z \equiv \hat{\mathbf{U}}^{\mathrm{T}}\hat{\mathbf{W}} = \hat{\mathbf{U}}^{\mathrm{T}} A\hat{\mathbf{V}} + \Delta z = \hat{\mathbf{U}}^{\mathrm{T}}\,\delta A(\tilde{\boldsymbol{\epsilon}})\hat{\mathbf{W}}^{\mathrm{true}} + \Delta z \tag{59}$$

with $\Delta z \sim \mathcal{N}(0, \sigma_z^2)$, with $\sigma_z^2 = \hat{\mathbf{U}}^{\mathrm{T}}R\hat{\mathbf{U}}$, the resulting inverse covariance matrix is

$$P_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} = \frac{1}{\sigma_z^2}[[\hat{\mathbf{W}}^{\mathrm{true}}]]\,\hat{\mathbf{U}}\hat{\mathbf{U}}^{\mathrm{T}}[[\hat{\mathbf{W}}^{\mathrm{true}}]]^{\mathrm{T}} \tag{60}$$

which leads directly to

$$B^{\mathrm{true}} = \frac{1}{\sigma_z^2}\left[\frac{1}{2}|\hat{\mathbf{U}} \times \hat{\mathbf{W}}^{\mathrm{true}}|^2 I_{3\times3} - [[\hat{\mathbf{W}}^{\mathrm{true}}]]\,\hat{\mathbf{U}}\hat{\mathbf{U}}^{\mathrm{T}}[[\hat{\mathbf{W}}^{\mathrm{true}}]]^{\mathrm{T}}\right]A^{\mathrm{true}} \tag{61}$$

and, after further reduction and substitution,

$$B' = \frac{1}{\sigma_z^2}\left[\frac{1}{2}|\hat{\mathbf{U}} \times \hat{\mathbf{W}}'|^2 A^{*\prime} - [[\hat{\mathbf{W}}']]\,\hat{\mathbf{U}}\hat{\mathbf{U}}^{\mathrm{T}}A^{*\prime}[[\hat{\mathbf{V}}]]^{\mathrm{T}}\right] \tag{62}$$

which cannot be calculated from available data, because the unknown $\hat{\mathbf{W}}'$ and the unknown $A^{*\prime}$ appear explicitly. In equations (57) and (58) these quantities also appear, but $\hat{\mathbf{W}}'$ is known from the measurement and $A^{*\prime}$ is absorbed formally into $\hat{\mathbf{V}}^{\mathrm{T}}$, whose value is known. Thus, we conclude that the generalized Wahba problem cannot be extended to include scalar measurements. Likewise, the Wahba problem cannot be generalized to include direction measurements whose errors are not described by the QUEST measurement model, since such a measurement model must be constructed from scalar measurements.

Summary of the Generalized Wahba Problem

In the generalized Wahba problem for a mixture of N simultaneous unit-vector and complete-attitude sensors (or suites of sensors) we have for the generalized cost function

$$J'(A) = \lambda - \mathrm{tr}[B'^{\mathrm{T}} A] \quad \text{with} \quad \lambda = \sum_{i=1}^{N} \lambda_i \quad \text{and} \quad B' = \sum_{i=1}^{N} B_i' \tag{63abc}$$

For a unit-vector sensor

$$\lambda_i = \lambda_{oi} = \frac{1}{\sigma_i^2} \quad \text{and} \quad B_i' = \frac{1}{\sigma_i^2}\hat{\mathbf{W}}_i'\hat{\mathbf{V}}_i^{\mathrm{T}} \tag{64ab}$$

and for a complete-attitude sensor

$$\lambda_i = \lambda_{oi} \quad \text{and} \quad B_i' = \left[\frac{1}{2}(\mathrm{tr}(R_i')^{-1})\, I_{3\times3} - (R_i')^{-1}\right]C_i^{*\prime} \tag{65ab}$$

and

$$\lambda'_{\max i} = \frac{1}{2}\,\mathrm{tr}(R_i')^{-1} \tag{65c}$$

The quantity λ_{oi} may not be available. The use of $\lambda'_{\max i}$ in place of λ'_{oi} in equation (65a) will disable the TASTE test for the data from sensor i.

By trivial transformation one can write equivalent relations for the Davenport K-matrix. For a complete attitude sensor (or suite of sensors) the K-matrix analogue of equation (65ab) is

$$\lambda'_{\max i} = \frac{1}{2}\,\mathrm{tr}\,(R'_i)^{-1} \quad \text{and} \quad K'_i = \lambda'_{\max i}\, I_{4\times 4} - 2\Xi(\bar{\eta}^{*\prime}_{B\leftarrow I\,i})\,(R'_i)^{-1}\Xi^{\mathrm{T}}(\bar{\eta}^{*\prime}_{B\leftarrow I\,i}) \tag{65de}$$

The Treatment of Star-Tracker Attitude Estimates in General Attitude Estimation

The use of attitude estimates from star trackers in calculating more complete estimates of the attitude has been treated before [19, 20], but we repeat that general treatment here for comparison with that in the generalized Wahba problem. We assume for simplicity that the fiducial body axes coincide with those of the star tracker and the space coordinate system is that of the star-tracker star catalogue. Thus, we write

$$\bar{\eta}^*_{\mathrm{ST}} = \delta\bar{q}(\mathbf{v}_{\mathrm{ST}}) \circ \bar{q} \quad \text{and} \quad \mathbf{v}_{\mathrm{ST}} \sim \mathcal{N}(\mathbf{0}, R_{\mathrm{ST}}) \tag{66ab}$$

where $\bar{\eta}^*_{\mathrm{ST}}$ is the measured star-tracker attitude quaternion and $\bar{q}$ is the attitude quaternion of the spacecraft. The quantity $\mathbf{v}_{\mathrm{ST}}$ denotes the star-tracker measurement noise. The form of equations (66) guarantees that $\bar{\eta}^*_{\mathrm{ST}}$ is always a quaternion of rotation, i.e., that it always has unit norm. If $\bar{q}^{*\prime}(-)$ is the *a priori* estimate of the quaternion, then

$$\bar{\eta}^*_{\mathrm{ST}} \circ \bar{q}^{*\prime}(-)^{-1} = \delta\bar{q}(\mathbf{v}_{\mathrm{ST}}) \circ \bar{q} \circ \bar{q}^{*\prime}(-)^{-1} \tag{67}$$

Defining

$$\delta\bar{q}(\boldsymbol{\zeta}_{\mathrm{ST}}) \equiv \bar{\eta}^*_{\mathrm{ST}} \circ \bar{q}^{*\prime}(-)^{-1} \quad \text{and} \quad \delta\bar{q}(\boldsymbol{\epsilon}) = \bar{q} \circ \bar{q}^{*\prime}(-)^{-1} \tag{68ab}$$

leads to

$$\delta\bar{q}(\boldsymbol{\zeta}_{\mathrm{ST}}) = \delta\bar{q}(\mathbf{v}_{\mathrm{ST}}) \circ \delta\bar{q}(\boldsymbol{\epsilon}) \tag{69}$$

in which all three quaternions represent very small rotations. Equation (69) can be linearized as [20]

$$\boldsymbol{\zeta}_{\mathrm{ST}} = \boldsymbol{\epsilon} + \mathbf{v}_{\mathrm{ST}} \quad \text{and} \quad \mathbf{v}_{\mathrm{ST}} \sim \mathcal{N}(\mathbf{0}, R_{\mathrm{ST}}) \tag{70ab}$$

The estimation of $\boldsymbol{\epsilon}$ from $\boldsymbol{\zeta}_{\mathrm{ST}}$ in either a batch estimator [21] or in a Kalman filter [22] is straightforward.[16] Equation (70ab) has been implemented in our deep-space missions for the past decade.

Discussion

The Wahba problem has been extended to include measurements in the form of an attitude matrix or attitude quaternion. This permits us to include an initial condition in the Wahba problem [12] as

$$B'(-) = \left[\frac{1}{2}\,\mathrm{tr}\big((P'_{\boldsymbol{\epsilon\epsilon}})^{-1}\big) I_{3\times 3} - (P'_{\boldsymbol{\epsilon\epsilon}})^{-1}\right] A^{*\prime}(-) \tag{71}$$

[16]Note that we have taken care to insure that our quaternions have unit norm. The violation of this constraint can lead to ambiguous and even silly results [21, 23]. In a batch estimator we will generally wish to compute $\bar{q}^{*\prime}$ iteratively, replacing $\bar{q}^{*\prime}(-)$ by $\bar{q}^{*\prime}(-) \equiv \delta\bar{q}(\boldsymbol{\epsilon}^{*\prime}_i) \circ \bar{q}^{*\prime}_i(-)$ after each iteration until convergence. (Here, i is the iteration index.)

or to include attitude measurements from a star tracker as

$$B'_{\mathrm{ST}} = \left[\frac{1}{2}\,\mathrm{tr}\Big((R_{\mathrm{ST}})^{-1}\Big)\, I_{3\times3} - (R_{\mathrm{ST}})^{-1} \right] C^{*\prime}_{\mathrm{ST}} \tag{72}$$

An important question, seldom posed by researchers on the most recent subject of their attention, is whether or not their result is useful. While the formal importance of these results is obvious, it must be admitted that the practical utility of this work is limited. Nowadays, attitude determination systems consist usually of (1) the usual suite of coarse attitude sensors (vector magnetometer, vector Sun sensor, and infrared horizon scanner), (2) a star-tracker-gyro system, or (3) a GPS attitude determination system. For none of these does the present algorithm finds application. In some cases, a star-tracker-gyro system may not be adequate to meet mission requirements because of the limited attitude accuracy about the star-tracker boresight and may be supplemented by a precise Sun sensor. Such an attitude determination system was the case for the Wilkinson Microwave Anisotropy Probe (WMAP) [24], launched in June 2001. That spacecraft, however, employed an extended Kalman filter [22] to estimate the attitude. For high-accuracy missions, the chief use of the Wahba problem is as a preprocessor of star directions in the estimation of the star-tracker attitude [19, 20], sometimes as part of the star-tracker firmware. The Wahba problem, because of its limitations with regard to the assumed sensor error models and the need for simultaneous data, does not otherwise find extensive use in high-accuracy missions.

The general differential correction method (equation (70)) is not very burdensome computationally, since one begins with $C^{*\prime}_{\mathrm{ST}}$ and R'_{ST}, to which the Sun sensor can provide only a very small rotation as a correction. The general differential correction is fairly simple also, because it has been shown [20, 25] that the QUEST-like measurement for the Sun sensor can be treated in the Kalman filter in certain cases as if $R_{\mathrm{Sun}} = \sigma^2_{\mathrm{Sun}}\, I_{3\times3}$.

The formal results of the present work, especially with regard to the new results for the Davenport matrix, K, are the more interesting.

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Appendix A: Statistical Glossary

The following table lists the notation for variables in this paper which may be random. The choice of whether a random variable should bear the mark "r.v." or the true value should bear the marking "true" is not arbitrary. In general, covariance matrices, for which the basic definition is always at the true values of the arguments, do not bear the superscript "true" for the value "at truth." (Also, one often estimates the covariance matrix of an initial state and it seems silly to write P'_{true}.) Generally, if a random variable is "unmarked" the true value carries the marking "true"; if a true value is unmarked, the random variable carries the marking "r.v." State vectors *in this work* are not random by nature and only their estimators are random. For these last we must distinguish free variables from the first three categories. Random variables whose true values vanish are listed in the table by null values instead of by the obvious marking. The notation for sampled values is uniform.

TABLE 1. Statistical Variables

Random Variable	Sampled Value	True Value	Free Variable
$\hat{\mathbf{W}}_k$	$\hat{\mathbf{W}}'_k$	$\hat{\mathbf{W}}_k^{\text{true}}$	—
$J(A)$	$J'(A)$	$J^{\text{true}}(A)$	—
B	B'	B^{true}	—
D	D'	D^{true}	—
K	K'	K^{true}	—
$\lambda_{\max}$	$\lambda'_{\max}$	λ_o	λ
$P^{\text{r.v.}}$	P'	P	—
$R^{\text{r.v.}}$	R'	R	—
A^*	$A^{*\prime}$	A^{true}	A
$\bar{q}^*$	$\bar{q}^{*\prime}$	$\bar{q}^{\text{true}}$	$\bar{q}$
$\tilde{\boldsymbol{\epsilon}}^*$	$\tilde{\boldsymbol{\epsilon}}^{*\prime}$	$\mathbf{0}$	$\tilde{\boldsymbol{\epsilon}}$