ATTITUDE DETERMINATION USING TWO VECTOR MEASUREMENTS

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ABSTRACT

Many spacecraft attitude determination methods use exactly two vector measurements. The two vectors are typically the unit vector to the Sun and the Earth's magnetic field vector for coarse "sun-mag" attitude determination or unit vectors to two stars tracked by two star trackers for fine attitude determination. TRIAD, the earliest published algorithm for determining spacecraft attitude from two vector measurements, has been widely used in both ground-based and onboard attitude determination. Later attitude determination methods have been based on Wahba's optimality criterion for *n* arbitrarily weighted observations. The solution of Wahba's problem is somewhat difficult in the general case, but there is a simple closed-form solution in the two-observation case. This solution reduces to the TRIAD solution for certain choices of measurement weights. This paper presents and compares these algorithms as well as sub-optimal algorithms proposed by Bar-Itzhack, Harman, and Reynolds. Some new results will be presented, but the paper is primarily a review and tutorial.

INTRODUCTION

Suppose that we have measured two unit vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ in the spacecraft body frame. These can be the unit vectors to an observed object like a star or the Sun, or some ambient vector field such as the Earth's magnetic field. We consider only unit vectors because the length of the vector has no information relevant to attitude determination. Each of these unit vectors thus contains two independent scalar pieces of attitude information. The spacecraft attitude is represented by a 3×3 orthogonal matrix *A*, *i.e.* $A^T A = I$, the 3×3 identity matrix. The attitude matrix must also be proper, *i.e.*, it must have unit determinant, so it is an element of the three-parameter group SO(3). Euler's Theorem states that the most general motion of a rigid body with one fixed point is a rotation about some axis. This shows explicitly that SO(3) is a three-parameter group, since the three parameters can be taken as the rotation angle and two parameters specifying a unit vector along the rotation axis. Thus two unit vector measurements determine the attitude matrix, in general; in fact they overdetermine it.

It is also necessary to know the components of the two measured vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in some reference frame. The reference frame is usually taken to be an inertial frame, but this is not necessary. One can use a rotating frame such as the frame referenced to the orbit normal vector and the local vertical. The attitude matrix to be determined is the matrix that rotates vectors from the reference frame to the spacecraft body frame. Thus we would like to find an attitude matrix such that

$$A\mathbf{r}_1 = \mathbf{b}_1 \tag{1a}$$

and

$$A\mathbf{r}_2 = \mathbf{b}_2 . \tag{1b}$$

This is not possible in general, however, for equation (1) implies that

$$\mathbf{b}_1 \cdot \mathbf{b}_2 = (A\mathbf{r}_1) \cdot (A\mathbf{r}_2) = \mathbf{r}_1^T A^T A \mathbf{r}_2 = \mathbf{r}_1^T \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_2 . \tag{2}$$

This equality is true for error-free measurements, but is not generally true in the presence of measurement errors. It will be seen in the following that all reasonable two-vector attitude determination schemes give the same estimate when equation (2) is valid.

It is clear from simple counting arguments that the two independent scalar pieces of information contained in a single vector measurement cannot determine the attitude uniquely. More concretely, if the attitude matrix *A* obeys equation (1a), then so does the matrix $R(\mathbf{b}_1, \phi_b) A R(\mathbf{r}_1, \phi_r)$, for any ϕ_b and ϕ_r, where $R(\mathbf{e}, \phi)$ denotes a rotation by angle ϕ about the axis $\mathbf{e}$. This line of argument also makes it clear that the attitude matrix is not uniquely determined if either the pair $\mathbf{b}_1$ and $\mathbf{b}_2$ or the pair $\mathbf{r}_1$ and $\mathbf{r}_2$ are parallel or antiparallel.

The earliest published algorithm for determining spacecraft attitude from two vector measurements was the TRIAD algorithm[1,2]. This algorithm has been widely used in both ground-based and onboard[3] attitude determination. The two vectors are typically the unit vector to the Sun and the Earth's magnetic field vector for coarse "sun-mag" attitude determination or unit vectors to two stars tracked by two star trackers for fine attitude determination. Recent developments in star tracker technology have produced star trackers that can track 5, 6, or even 50 stars at a time. For attitude determination using more than two vectors, optimal estimators based on a loss function introduced by Wahba are appropriate[4]. However, Bronzenac and Bender have shown that the *n* vectors from a small-field-of-view star tracker can be replaced by an average vector without significant loss of precision[5]. With this approximation, the two star tracker case, even with multiple stars tracked in each star tracker, can be treated as a two-vector-measurement problem.

With this motivation, we survey solutions to the two-vector measurement problem, beginning with TRIAD. We then consider the optimal solution of Wahba's problem. After this, we look at sub-optimal algorithms have been proposed by Bar-Itzhack and Harman[6] and by Reynolds[7,8]. We compare the various algorithms for both accuracy and computational effort, and finally present conclusions.

TRIAD

The TRIAD algorithm, introduced by Black in 1964[1,2], is based on the following idea. If we have an orthogonal right-handed triad of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in the reference frame, and a corresponding triad $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ in the spacecraft body frame, the the attitude matrix

$$A = [\mathbf{w}_1 \vdots \mathbf{w}_2 \vdots \mathbf{w}_3][\mathbf{v}_1 \vdots \mathbf{v}_2 \vdots \mathbf{v}_3]^T = \mathbf{w}_1\mathbf{v}_1^T + \mathbf{w}_2\mathbf{v}_2^T + \mathbf{w}_3\mathbf{v}_3^T \tag{3}$$

will transform the $\mathbf{v}_i$ to the $\mathbf{w}_i$ by

$$A\mathbf{v}_i = \mathbf{w}_i, \ \ i = 1, 2, 3. \tag{4}$$

The TRIAD algorithm forms the triad $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ from $\mathbf{r}_1$ and $\mathbf{r}_2$, and the triad $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ from $\mathbf{b}_1$ and $\mathbf{b}_2$. Incidentally, TRIAD can be considered either as the word "triad" or as an acronym for "TRIaxial Attitude Determination." The triads can be formed in three convenient ways. First, it is useful to define the normalized cross products

$$\mathbf{r}_3 \equiv (\mathbf{r}_1 \times \mathbf{r}_2) / |\mathbf{r}_1 \times \mathbf{r}_2| \tag{5a}$$

and

$$\mathbf{b}_3 \equiv (\mathbf{b}_1 \times \mathbf{b}_2) / |\mathbf{b}_1 \times \mathbf{b}_2|. \tag{5b}$$

We note that $\mathbf{r}_3$ or $\mathbf{b}_3$ is undefined if the reference vectors or the observed vectors, respectively, are parallel or antiparallel. This is the case noted above in which there is insufficient information to determine the attitude uniquely. If this is not the case, two of the TRIAD attitude estimates are

$$A_{T1} \equiv \mathbf{b}_1\mathbf{r}_1^T + \mathbf{b}_3\mathbf{r}_3^T + (\mathbf{b}_1 \times \mathbf{b}_3)(\mathbf{r}_1 \times \mathbf{r}_3)^T \tag{6}$$

and

$$A_{T2} \equiv \mathbf{b}_2\mathbf{r}_2^T + \mathbf{b}_3\mathbf{r}_3^T + (\mathbf{b}_2 \times \mathbf{b}_3)(\mathbf{r}_2 \times \mathbf{r}_3)^T. \tag{7}$$

These estimates treat the two measurements unsymmetrically. In fact $A_{T1}\mathbf{r}_1 = \mathbf{b}_1$ and $A_{T2}\mathbf{r}_2 = \mathbf{b}_2$, but

$$A_{T1}\mathbf{r}_2 \equiv \mathbf{b}_1(\mathbf{r}_1 \cdot \mathbf{r}_2) + (\mathbf{b}_1 \times \mathbf{b}_3)[(\mathbf{r}_1 \times \mathbf{r}_3) \cdot \mathbf{r}_2] = (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{b}_1 + [\mathbf{b}_2 - (\mathbf{b}_1 \cdot \mathbf{b}_2)\mathbf{b}_1]|\mathbf{r}_1 \times \mathbf{r}_2| / |\mathbf{b}_1 \times \mathbf{b}_2| \tag{8}$$

and

$$A_{T2}\mathbf{r}_1 \equiv \mathbf{b}_2(\mathbf{r}_1 \cdot \mathbf{r}_2) + (\mathbf{b}_2 \times \mathbf{b}_3)[(\mathbf{r}_2 \times \mathbf{r}_3) \cdot \mathbf{r}_1] = (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{b}_2 + [\mathbf{b}_1 - (\mathbf{b}_1 \cdot \mathbf{b}_2)\mathbf{b}_2]|\mathbf{r}_1 \times \mathbf{r}_2| / |\mathbf{b}_1 \times \mathbf{b}_2|. \tag{9}$$

Thus the estimate A_{T1} emphasizes the first measurement and A_{T2} emphasizes the second. It's not difficult to see, though, that both A_{T1} and A_{T2} satisfy equations (1a) and (1b) if $\mathbf{b}_1 \cdot \mathbf{b}_2 = \mathbf{r}_1 \cdot \mathbf{r}_2$.

The third form of TRIAD treats the two measurements symmetrically. We define the unit vectors

$$\mathbf{r}_+ \equiv (\mathbf{r}_2 + \mathbf{r}_1) / |\mathbf{r}_2 + \mathbf{r}_1| = (\mathbf{r}_2 + \mathbf{r}_1) / \sqrt{2(1 + \mathbf{r}_1 \cdot \mathbf{r}_2)}, \tag{10}$$

$$\mathbf{r}_- \equiv (\mathbf{r}_2 - \mathbf{r}_1) / |\mathbf{r}_2 - \mathbf{r}_1| = (\mathbf{r}_2 - \mathbf{r}_1) / \sqrt{2(1 - \mathbf{r}_1 \cdot \mathbf{r}_2)}, \tag{11}$$

and $\mathbf{b}_+$ and $\mathbf{b}_-$ similarly. It is easy to see that $\mathbf{r}_+$ is perpendicular to $\mathbf{r}_-$, $\mathbf{b}_+$ is perpendicular to $\mathbf{b}_-$, and also that $\mathbf{r}_3 = \mathbf{r}_+ \times \mathbf{r}_-$ and $\mathbf{b}_3 = \mathbf{b}_+ \times \mathbf{b}_-$. Thus $\{\mathbf{r}_+, \mathbf{r}_-, \mathbf{r}_3\}$ and $\{\mathbf{b}_+, \mathbf{b}_-, \mathbf{b}_3\}$ are orthogonal triads, and the third TRIAD estimate is given by

$$A_{T3} \equiv \mathbf{b}_+\mathbf{r}_+^T + \mathbf{b}_-\mathbf{r}_-^T + (\mathbf{b}_+ \times \mathbf{b}_-)(\mathbf{r}_+ \times \mathbf{r}_-)^T. \tag{12}$$

This estimate treats the two observations symmetrically, and gives $A_{T3}\mathbf{r}_+ = \mathbf{b}_+$ and $A_{T3}\mathbf{r}_- = \mathbf{b}_-$, but

$$A_{T3}\mathbf{r}_1 \equiv \mathbf{b}_+(\mathbf{r}_+\cdot\mathbf{r}_1) + \mathbf{b}_-(\mathbf{r}_-\cdot\mathbf{r}_1) = \frac{1}{2}\left[\sqrt{\frac{1+\mathbf{r}_1\cdot\mathbf{r}_2}{1+\mathbf{b}_1\cdot\mathbf{b}_2}}(\mathbf{b}_1+\mathbf{b}_2) + \sqrt{\frac{1-\mathbf{r}_1\cdot\mathbf{r}_2}{1-\mathbf{b}_1\cdot\mathbf{b}_2}}(\mathbf{b}_1-\mathbf{b}_2)\right] \tag{13}$$

and

$$A_{T3}\mathbf{r}_2 \equiv \mathbf{b}_+(\mathbf{r}_+\cdot\mathbf{r}_2) + \mathbf{b}_-(\mathbf{r}_-\cdot\mathbf{r}_2) = \frac{1}{2}\left[\sqrt{\frac{1+\mathbf{r}_1\cdot\mathbf{r}_2}{1+\mathbf{b}_1\cdot\mathbf{b}_2}}(\mathbf{b}_1+\mathbf{b}_2) - \sqrt{\frac{1-\mathbf{r}_1\cdot\mathbf{r}_2}{1-\mathbf{b}_1\cdot\mathbf{b}_2}}(\mathbf{b}_1-\mathbf{b}_2)\right]. \tag{14}$$

Again, it's not difficult to see that A_{T3} satisfies equations (1a) and (1b) if $\mathbf{b}_1\cdot\mathbf{b}_2 = \mathbf{r}_1\cdot\mathbf{r}_2$.

All three TRIAD estimates satisfy $A_{Ti}\mathbf{r}_3 = \mathbf{b}_3$, for i = 1, 2, 3. From this and the above observations, it is clear that A_{T1}, A_{T2}, and A_{T3} give identical estimates if equation (2) is valid, since they provide the same mapping of a basis {$\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$} in the reference frame to a basis {$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$} in the spacecraft body frame.

THE OPTIMAL SOLUTION

In 1965, Grace Wahba, then a graduate student at Stanford University on a summer job with IBM, proposed the following problem[4]: Find the orthogonal matrix A with determinant +1 that minimizes the loss function

$$L(A) \equiv \tfrac{1}{2}\sum_i a_i\left|\mathbf{b}_i - A\mathbf{r}_i\right|^2 . \tag{15}$$

where {$\mathbf{b}_i$} is a set of n unit vectors measured in a spacecraft's body frame, {$\mathbf{r}_i$} are the corresponding unit vectors in a reference frame, and {a_i} are non-negative weights. We can rewrite equation (15), using the invariance of the trace under cyclic permutations, as

$$L(A) = \tfrac{1}{2}\sum_i a_i\left(\left|\mathbf{b}_i\right|^2 + \left|\mathbf{r}_i\right|^2\right) - \sum_i a_i\mathbf{b}_i^T A\mathbf{r}_i = \left(\sum_i a_i\right) - \mathrm{trace}(AB^T), \tag{16}$$

where

$$B \equiv \sum_i a_i\mathbf{b}_i\mathbf{r}_i^T . \tag{17}$$

It is obvious that the attitude matrix that minimizes the loss function is the proper orthogonal matrix that maximizes trace(AB^T). Almost all solutions of Wahba's problem are based on this observation. The original solutions solved for the attitude matrix A directly, but most practical applications have been based on Davenport's q-method[2,9], which solves for the attitude quaternion[10,11]. Shuster's QUEST algorithm, in particular, has been widely used[12]. Shuster showed a simplification in the two-observation Wahba problem, but the first explicit closed-form solution was presented in reference 13.

We begin by noting that the matrix B has the singular value decomposition[14-16]

$$B = USV^T , \tag{18}$$

where U and V are orthogonal matrices, and S is diagonal;

$$S = \mathrm{diag}(s_1, s_2, s_3) , \tag{19}$$

with

$$s_1 \ge s_2 \ge s_3 \ge 0 . \tag{20}$$

In the two-observation case, it is clear from equation (17) that B has rank at most 2, and therefore

$$\det B = s_1 s_2 s_3 = 0 . \tag{21}$$

Equations (20) and (21) show that

$$s_3 = 0 \tag{22}$$

in the two-observation case. We shall take advantage of some resulting simplifications in this case. The general n–observation case is treated in references 13 and 14.

Since s_3 = 0, we are free to choose the sign of the last column of U and of V so that both of these matrices have positive determinants. We shall assume that this is the case. Now

$$\mathrm{trace}(AB^T) = \mathrm{trace}(AVSU^T) = \mathrm{trace}(WS) = s_1W_{11} + s_2W_{22}, \tag{23}$$

where we have again used the invariance of the trace under cyclic permutations, and

$$W \equiv U^T A V. \tag{24}$$

Now using the Euler axis/angle parameterization for $W = R(\mathbf{e}, \phi)$ gives[10,11]

$$\text{trace}(AB^T) = s_1[\cos\varphi + e_1^2(1-\cos\varphi)] + s_2[\cos\varphi + e_2^2(1-\cos\varphi)] = s_1 e_1^2 + s_2 e_2^2 + \cos\varphi[s_1(1-e_1^2) + s_2(1-e_2^2)]. \tag{25}$$

This is clearly maximized for $\cos\phi = 1$, which means that $W = I$. Thus the optimal attitude is given by

$$A_{opt} \equiv U V^T. \tag{26}$$

Equation (25) shows that the minimum of trace(AB^T) is unique unless $s_2 = 0$. The vanishing of s_2 is the sign in the optimal algorithm that the observations are not sufficient to determine the attitude. We shall see below that this is related to the parallelism of the reference frame or body frame vectors.

The singular value decomposition is rather expensive computationally, so we look for a simpler way to compute A_{opt}. We note that the classical adjoint, or adjugate, (the transposed matrix of cofactors) of B^T is given in terms of the SVD by[16]

$$\text{adj}\, B^T = U[\text{diag}(0, 0, s_1 s_2)]V^T. \tag{27}$$

We also note that

$$B B^T B = U[\text{diag}(s_1^3, s_2^3, 0)]V^T. \tag{28}$$

These allow us to write

$$(\lambda^2 - s_1 s_2)B + \lambda\,\text{adj}\, B^T - BB^T B = \lambda s_1 s_2 U V^T = \lambda s_1 s_2 A_{opt}, \tag{29}$$

where

$$\lambda \equiv s_1 + s_2 = \text{trace}(AB^T). \tag{30}$$

We can compute the optimal attitude without actually performing the expensive SVD of B if we can find an alternative means of computing the quantities appearing in equation (29). Direct computation from equation (17) gives

$$\text{adj}\, B^T = a_1 a_2 (\mathbf{b}_1 \times \mathbf{b}_2)(\mathbf{r}_1 \times \mathbf{r}_2)^T = a_1 a_2 |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2| \mathbf{b}_3 \mathbf{r}_3^T. \tag{31}$$

Then we see from equation (27) that

$$s_1 s_2 = \|\text{adj}\, B^T\|_F = a_1 a_2 |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2|, \tag{32}$$

where $\|M\|_F$ denotes the Frobenius (or Euclidean, or Schur, or Hilbert-Schmidt) norm[15,16]

$$\|M\|_F \equiv [\text{trace}(MM^T)]^{1/2}. \tag{33}$$

We note from equation (32) that $s_2 = 0$ if either of the cross products vanishes, as was mentioned above. A little effort is required to show that

$$\lambda^2 = s_1^2 + s_2^2 + 2 s_1 s_2 = \|B\|_F^2 + 2 a_1 a_2 |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2| = a_1^2 + a_2^2 + 2 a_1 a_2 \left[(\mathbf{b}_1 \cdot \mathbf{b}_2)(\mathbf{r}_1 \cdot \mathbf{r}_2) + |\mathbf{b}_1 \times \mathbf{b}_2||\mathbf{r}_1 \times \mathbf{r}_2|\right]. \tag{34}$$

In the two-observation case, λ is just the positive square root of the quantity on the right side of equation (34); finding λ in the case of more than two observations requires solving a quartic equation. To complete the analytic derivation, we need to evaluate

$$B B^T B = \sum_{i,j,k=1}^{2} a_i a_j a_k \mathbf{b}_i (\mathbf{r}_i \cdot \mathbf{r}_j)(\mathbf{b}_j \cdot \mathbf{b}_k) \mathbf{r}_k^T \tag{35}$$

Combining all these intermediate results with much vector algebra gives the final equation for the optimal attitude estimate:

$$A_{opt} \equiv (a_1/\lambda)[\mathbf{b}_1 \mathbf{r}_1^T + (\mathbf{b}_1 \times \mathbf{b}_3)(\mathbf{r}_1 \times \mathbf{r}_3)^T] + (a_2/\lambda)[\mathbf{b}_2 \mathbf{r}_2^T + (\mathbf{b}_2 \times \mathbf{b}_3)(\mathbf{r}_2 \times \mathbf{r}_3)^T] + \mathbf{b}_3 \mathbf{r}_3^T. \tag{36}$$

It is interesting to note that this expression has a unique limit as either a_1 or a_2 goes to zero, with λ equal to the non-zero weight in the limit. This is true even though Wahba's loss function of equation (15) does not have a unique minimum in either limit, since it effectively only includes a single observation. In fact, the limit of the optimal estimate is the TRIAD estimate A_{T1} as a_2 goes to zero, and A_{T2} as a_1 goes to zero. It is also true, but more difficult to see, that the optimal estimate is equal to A_{T3} for equal weights, $a_1 = a_2$.

The optimal estimate maps the two reference vectors as

$$A_{opt}\mathbf{r}_1 = (a_1/\lambda)A_{T1}\mathbf{r}_1 + (a_2/\lambda)A_{T2}\mathbf{r}_1 = (a_1/\lambda)\mathbf{b}_1 + (a_2/\lambda)\left\{(\mathbf{r}_1\cdot\mathbf{r}_2)\mathbf{b}_1 + [\mathbf{b}_2 - (\mathbf{b}_1\cdot\mathbf{b}_2)\mathbf{b}_1]|\mathbf{r}_1\times\mathbf{r}_2|/|\mathbf{b}_1\times\mathbf{b}_2|\right\} \tag{37}$$

and

$$A_{opt}\mathbf{r}_2 = (a_1/\lambda)A_{T1}\mathbf{r}_2 + (a_2/\lambda)A_{T2}\mathbf{r}_2 = (a_2/\lambda)\mathbf{b}_2 + (a_1/\lambda)\left\{(\mathbf{r}_1\cdot\mathbf{r}_2)\mathbf{b}_2 + [\mathbf{b}_1 - (\mathbf{b}_1\cdot\mathbf{b}_2)\mathbf{b}_2]|\mathbf{r}_1\times\mathbf{r}_2|/|\mathbf{b}_1\times\mathbf{b}_2|\right\}. \tag{38}$$

The main point to note about these equations is that the optimal attitude estimate maps both $\mathbf{r}_1$ and $\mathbf{r}_2$ into the plane spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$. It's clear from the loss function of equation (15) that this has to be the case; any out-of-plane component would be non-optimal.

In the case that $\mathbf{b}_1\cdot\mathbf{b}_2 = \mathbf{r}_1\cdot\mathbf{r}_2$, equation (34) for λ simplifies to $\lambda = a_1 + a_2$, and the optimal estimate is

$$A_{opt} = (a_1A_{T1} + a_2A_{T2})/(a_1 + a_2). \tag{39}$$

Since A_{T1} and A_{T2} are equal in this case, we see that A_{opt} is equal to their common value, also.

Mortari has found an alternative representation of the closed-form solution to the two-observation Wahba problem that is equivalent to the solution found here [17].

OPTIMIZED TRIAD

Bar-Itzhack and Harman[6] have proposed using equation (39) even when $\mathbf{b}_1\cdot\mathbf{b}_2 \neq \mathbf{r}_1\cdot\mathbf{r}_2$. In general, this estimator is not optimal, nor is the resulting attitude estimate exactly orthogonal. In order to produce a more nearly orthogonal attitude matrix, they employ the first-order orthogonalization step

$$A_{OT} = \tfrac{1}{2}[(a_1 + a_2)^{-1}(a_1A_{T1} + a_2A_{T2}) + (a_1 + a_2)(a_1A_{T1}^T + a_2A_{T2}^T)^{-1}] \tag{40}$$

They call the resulting estimator "Optimized TRIAD." This estimate has the correct limits of A_{T1} and A_{T2} as a_1 or a_2 tends to zero, respectively, but is not the same as A_{T3} for equal weights. It avoids the computation of λ that is required for the optimal estimate, but requires the inverse of a 3×3 matrix.

There is an alternative way to orthogonalize the matrix computed by equation (37) when $\mathbf{b}_1\cdot\mathbf{b}_2 \neq \mathbf{r}_1\cdot\mathbf{r}_2$. This is to extract a quaternion from the attitude matrix and then normalize the resulting quaternion. It is well known that the attitude matrix computed from a normalized quaternion is guaranteed to be orthogonal[10,11,18]. The extraction of the quaternion requires a square root, but it is often desirable to compute a quaternion for data transmission or storage, because it stores complete attitude information in four components instead of the nine required for the attitude matrix.

DIRECT QUATERNION METHOD

All the methods considered so far compute the attitude matrix. If a quaternion is desired, it can be extracted from the attitude matrix. However, it would be desirable to avoid this indirect and somewhat costly procedure. Reynolds has proposed a very simple estimation algorithm that computes a quaternion directly[7,8].

We first present some background information on quaternions to establish our conventions. A more complete discussion can be found in reference 11. A quaternion q has a vector part $\mathbf{q}$ and a scalar part q_s, which we write as

$$q = [\mathbf{q}, q_s]. \tag{41}$$

This is similar to Reynolds's notation except that we use square brackets rather than parentheses. A unit quaternion (*i.e.*, a quaternion with $|\mathbf{q}|^2 + q_s^2 = 1$) can be used to represent an attitude matrix, which rotates a vector by

$$A(q)\mathbf{v} = (q_s^2 - |\mathbf{q}|^2)\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v})\mathbf{q} - 2q_s(\mathbf{q}\times\mathbf{v}). \tag{42}$$

We will follow Shuster's convention for quaternion products[11], writing

$$p\otimes q = [\mathbf{p}, p_s]\otimes[\mathbf{q}, q_s] = [q_s\mathbf{p} + p_s\mathbf{q} - \mathbf{p}\times\mathbf{q},\ p_sq_s - \mathbf{p}\cdot\mathbf{q}]. \tag{43}$$

This differs from the historical convention in the sign of the cross-product, and has the advantage that the order of quaternion multiplication is the same as the order of attitude matrix multiplication:

$$A(p\otimes q) = A(p)A(q). \tag{44}$$

The quaternion corresponding to the rotation matrix $R(\mathbf{e},\phi)$ is

$$q = \left[\mathbf{e}\sin\frac{\phi}{2}, \cos\frac{\phi}{2}\right]. \tag{45}$$

The derivation of the direct quaternion method begins with the observation that the quaternion that maps the reference vector $\mathbf{r}_1$ into the body frame vector $\mathbf{b}_1$, using the minimum-angle rotation, is

$$q_{\min 1} \equiv \frac{1}{\sqrt{2(1+\mathbf{b}_1\cdot\mathbf{r}_1)}}[\mathbf{b}_1\times\mathbf{r}_1, 1+\mathbf{b}_1\cdot\mathbf{r}_1] \tag{46}$$

The most general matrix that maps $\mathbf{r}_1$ into $\mathbf{b}_1$ is $R(\mathbf{b}_1,\phi_b)A(q_{\min 1})R(\mathbf{r}_1,\phi_r)$, where ϕ_b and ϕ_r are arbitrary angles of rotation about $\mathbf{b}_1$ and $\mathbf{r}_1$, respectively. This general rotation has the quaternion representation

$$\begin{aligned} q_1 &= \frac{1}{\sqrt{2(1+\mathbf{b}_1\cdot\mathbf{r}_1)}}\left[\mathbf{b}_1\sin\frac{\phi_b}{2},\cos\frac{\phi_b}{2}\right]\otimes[\mathbf{b}_1\times\mathbf{r}_1, 1+\mathbf{b}_1\cdot\mathbf{r}_1]\otimes\left[\mathbf{r}_1\sin\frac{\phi_r}{2},\cos\frac{\phi_r}{2}\right] \\ &= \frac{1}{\sqrt{2(1+\mathbf{b}_1\cdot\mathbf{r}_1)}}\left[\cos\frac{\phi}{2}(\mathbf{b}_1\times\mathbf{r}_1)+\sin\frac{\phi}{2}(\mathbf{b}_1+\mathbf{r}_1),\ (1+\mathbf{b}_1\cdot\mathbf{r}_1)\cos\frac{\phi}{2}\right], \end{aligned} \tag{47}$$

where $\phi \equiv \phi_b + \phi_r$. By parallel reasoning, the most general quaternion that maps $\mathbf{r}_2$ into $\mathbf{b}_2$ is given by

$$q_2 = \frac{1}{\sqrt{2(1+\mathbf{b}_2\cdot\mathbf{r}_2)}}\left[\cos\frac{\psi}{2}(\mathbf{b}_2\times\mathbf{r}_2)+\sin\frac{\psi}{2}(\mathbf{b}_2+\mathbf{r}_2),\ (1+\mathbf{b}_2\cdot\mathbf{r}_2)\cos\frac{\psi}{2}\right] \tag{48}$$

for some angle ψ. The vector part of q_1 is perpendicular to $\mathbf{b}_1 - \mathbf{r}_1$, and the vector part of q_2 is perpendicular to $\mathbf{b}_2 - \mathbf{r}_2$. Based on this observation, Reynolds proposed to look for a quaternion whose vector part is perpendicular to both $\mathbf{b}_1 - \mathbf{r}_1$ and $\mathbf{b}_2 - \mathbf{r}_2$. The vector part of q_1 will be perpendicular to $\mathbf{b}_2 - \mathbf{r}_2$ if we choose

$$\cos(\phi/2) = \pm\{[(\mathbf{b}_1\times\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2)]^2+[(\mathbf{b}_1+\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2)]^2\}^{-1/2}(\mathbf{b}_1+\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2) \tag{49a}$$

and

$$\sin(\phi/2) = \mp\{[(\mathbf{b}_1\times\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2)]^2+[(\mathbf{b}_1+\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2)]^2\}^{-1/2}(\mathbf{b}_1\times\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2). \tag{49b}$$

Substituting this into equation (47) gives

$$q_1 = c_1^{-1/2}[(\mathbf{b}_1-\mathbf{r}_1)\times(\mathbf{b}_2-\mathbf{r}_2),\ (\mathbf{b}_1+\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2)], \tag{50}$$

where c_1 is the normalization factor

$$c_1 = |(\mathbf{b}_1-\mathbf{r}_1)\times(\mathbf{b}_2-\mathbf{r}_2)|^2+[(\mathbf{b}_1+\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2)]^2. \tag{51}$$

We have ignored the ambiguous overall sign of the quaternion, which has no significance, since the attitude matrix is a homogeneous quadratic function of the quaternion. The appearance of the cross product $(\mathbf{b}_1-\mathbf{r}_1)\times(\mathbf{b}_2-\mathbf{r}_2)$ is not at all surprising, since this vector is guaranteed to be orthogonal to both $\mathbf{b}_1 - \mathbf{r}_1$ and $\mathbf{b}_2 - \mathbf{r}_2$.

Similarly, choosing ψ so that the vector part of q_2 will be perpendicular to $\mathbf{b}_1 - \mathbf{r}_1$ gives

$$q_2 = c_2^{-1/2}[(\mathbf{b}_1-\mathbf{r}_1)\times(\mathbf{b}_2-\mathbf{r}_2),\ (\mathbf{b}_2+\mathbf{r}_2)\cdot(\mathbf{r}_1-\mathbf{b}_1)], \tag{52}$$

The vector parts of q_1 and q_2 are equal up to the normalization constant. However, the scalar part of q_1 is

$$q_{1s} = c_1^{-1/2}(\mathbf{b}_1+\mathbf{r}_1)\cdot(\mathbf{b}_2-\mathbf{r}_2) = c_1^{-1/2}[(\mathbf{b}_2\cdot\mathbf{r}_1-\mathbf{b}_1\cdot\mathbf{r}_2)+(\mathbf{b}_1\cdot\mathbf{b}_2-\mathbf{r}_1\cdot\mathbf{r}_2)] \tag{53}$$

and the scalar part of q_2 is

$$q_{2s} = c_2^{-1/2}(\mathbf{b}_2+\mathbf{r}_2)\cdot(\mathbf{r}_1-\mathbf{b}_1) = c_2^{-1/2}[(\mathbf{b}_2\cdot\mathbf{r}_1-\mathbf{b}_1\cdot\mathbf{r}_2)-(\mathbf{b}_1\cdot\mathbf{b}_2-\mathbf{r}_1\cdot\mathbf{r}_2)]. \tag{54}$$

Thus, q_1 and q_2 are identical if $\mathbf{b}_1\cdot\mathbf{b}_2 = \mathbf{r}_1\cdot\mathbf{r}_2$, and they are equal to

$$q_3 \equiv c_3^{-1/2}[(\mathbf{b}_1-\mathbf{r}_1)\times(\mathbf{b}_2-\mathbf{r}_2),\ \mathbf{b}_2\cdot\mathbf{r}_1-\mathbf{b}_1\cdot\mathbf{r}_2]. \tag{55}$$

We see that q_1, q_2, and q_3 all have the same rotation axis, and the rotation angle of q_3 is intermediate between those of q_1 and q_2. The quaternion q_3, which treats the two measurements symmetrically, is the estimate preferred by Reynolds; but we will also consider the asymmetrical estimates q_1 and q_2.

COMPARISON OF THE DIRECT QUATERNION METHOD WITH TRIAD

It would seem that the quaternion q_1 should correspond to the TRIAD estimate A_{T1}, q_2 to A_{T2}, and q_3 to A_{T3}. As evidence for this, we note that the direct quaternion estimation methods have $A(q_1)\mathbf{r}_1 = A_{T1}\mathbf{r}_1 = \mathbf{b}_1$, $A(q_2)\mathbf{r}_2 = A_{T2}\mathbf{r}_2 = \mathbf{b}_2$, and q_3 symmetric in the measurements, as A_{T3} is. However, we shall now show that this correspondence is not exact. The algebra in the general case becomes rather messy, so we consider a simple example. Assume that we have two reference vectors

$$\mathbf{r}_1 = [1, 0, 0]^T \quad \text{and} \quad \mathbf{r}_2 = [0, 1, 0]^T \tag{56}$$

and two observation vectors

$$\mathbf{b}_1 = [0, 0, 1]^T \quad \text{and} \quad \mathbf{b}_2 = [\cos\vartheta, 0, \sin\vartheta]^T . \tag{57}$$

We note that $\mathbf{b}_1 \cdot \mathbf{b}_2 = \mathbf{r}_1 \cdot \mathbf{r}_2$ only if $\sin\vartheta = 0$, in which case all algorithms should give the same estimate.

We first compute the TRIAD estimates. Straightforward algebra results in

$$A_{T1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{T2} = \begin{bmatrix} -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \\ \cos\vartheta & \sin\vartheta & 0 \end{bmatrix}, \quad \text{and} \quad A_{T3} = \begin{bmatrix} -\sin(\vartheta/2) & \cos(\vartheta/2) & 0 \\ 0 & 0 & 1 \\ \cos(\vartheta/2) & \sin(\vartheta/2) & 0 \end{bmatrix}. \tag{58}$$

We note that $A_{T1}\mathbf{r}_1 = \mathbf{b}_1$, $A_{T2}\mathbf{r}_2 = \mathbf{b}_2$, $A_{T3}\mathbf{r}_+ = \mathbf{b}_+$, and $A_{T3}\mathbf{r}_- = \mathbf{b}_-$, as expected. However,

$$|A_{T1}\mathbf{r}_2 - \mathbf{b}_2| = |A_{T2}\mathbf{r}_1 - \mathbf{b}_1| = 2|\sin(\vartheta/2)|, \tag{59a}$$

and

$$|A_{T3}\mathbf{r}_1 - \mathbf{b}_1| = |A_{T3}\mathbf{r}_2 - \mathbf{b}_2| = 2|\sin(\vartheta/4)| . \tag{59b}$$

These results are not surprising, since the vectors $A_{T1}\mathbf{r}_2$, $A_{T2}\mathbf{r}_1$, $A_{T3}\mathbf{r}_1$, and $A_{T3}\mathbf{r}_2$ are all in the plane spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$, as we argued was necessary for an optimal estimator. For comparison with the direct quaternion method, it is interesting to present the quaternions extracted from A_{T1}, A_{T2}, A_{T3}:

$$q_{T1} = \tfrac{1}{2}[1, 1, 1, 1], \tag{60a}$$

$$q_{T2} = \tfrac{1}{2}\left[\sqrt{1-\sin\vartheta}, \sqrt{1+\sin\vartheta}, \sqrt{1+\sin\vartheta}, \sqrt{1-\sin\vartheta}\right], \tag{60b}$$

and

$$q_{T3} = \tfrac{1}{2}\left[\sqrt{1-\sin(\vartheta/2)}, \sqrt{1+\sin(\vartheta/2)}, \sqrt{1+\sin(\vartheta/2)}, \sqrt{1-\sin(\vartheta/2)}\right], \tag{60c}$$

where we have written out the three components of the vector part of each quaternion explicitly.

The estimates produced by the direct quaternion method embodied in equations (50), (52), and (55) are

$$q_1 = \tfrac{1}{2}(1+\cos\vartheta\sin\vartheta)^{-1/2}[1, \cos\vartheta+\sin\vartheta, 1, \cos\vartheta+\sin\vartheta], \tag{61a}$$

$$q_2 = \tfrac{1}{2}[1, \cos\vartheta+\sin\vartheta, 1, \cos\vartheta-\sin\vartheta], \tag{61b}$$

and

$$q_3 = (4+2\cos\vartheta\sin\vartheta-\sin^2\vartheta)^{-1/2}[1, \cos\vartheta+\sin\vartheta, 1, \cos\vartheta]. \tag{61c}$$

It is immediately apparent that the quaternions in equation (61) do not correspond to those in equation (60), unless $\sin\vartheta = 0$ and all reasonable estimators agree. The attitude matrices computed from q_1, q_2, and q_3 lead to further insights:

$$A(q_1) = \frac{1}{1+\cos\vartheta\sin\vartheta}\begin{bmatrix} 0 & \cos\vartheta+\sin\vartheta & -\cos\vartheta\sin\vartheta \\ 0 & \cos\vartheta\sin\vartheta & \cos\vartheta+\sin\vartheta \\ 1+\cos\vartheta\sin\vartheta & 0 & 0 \end{bmatrix}, \tag{62a}$$

$$A(q_2) = \begin{bmatrix} -\cos\vartheta\sin\vartheta & \cos\vartheta & \sin^2\vartheta \\ \sin\vartheta & 0 & \cos\vartheta \\ \cos^2\vartheta & \sin\vartheta & -\cos\vartheta\sin\vartheta \end{bmatrix}, \tag{62b}$$

and

$$A(q_3) = \frac{1}{4 + 2\cos\vartheta\sin\vartheta - \sin^2\vartheta}\begin{bmatrix} -\sin\vartheta(2\cos\vartheta + \sin\vartheta) & 2(2\cos\vartheta + \sin\vartheta) & -2\sin\vartheta(\cos\vartheta - \sin\vartheta) \\ 2\sin\vartheta & \sin\vartheta(2\cos\vartheta - \sin\vartheta) & 2(2\cos\vartheta + \sin\vartheta) \\ 4 + 2\sin\vartheta(\cos\vartheta - \sin\vartheta) & 2\sin\vartheta & -\sin\vartheta(2\cos\vartheta + \sin\vartheta) \end{bmatrix}. \quad (62c)$$

We note that $A(q_1)\mathbf{r}_1 = \mathbf{b}_1$ and $A(q_2)\mathbf{r}_2 = \mathbf{b}_2$, as expected. However, in the general case,

$$\sqrt{1+\cos\vartheta\sin\vartheta}\,\left|A(q_1)\mathbf{r}_2 - \mathbf{b}_2\right| = \left|A(q_2)\mathbf{r}_1 - \mathbf{b}_1\right| = \sqrt{2}\,|\sin\vartheta| \quad (63a)$$

and

$$\left|A(q_3)\mathbf{r}_1 - \mathbf{b}_1\right| = \left|A(q_3)\mathbf{r}_2 - \mathbf{b}_2\right| = \sqrt{2}\,(4 + 2\cos\vartheta\sin\vartheta - \sin^2\vartheta)^{-1/2}|\sin\vartheta|. \quad (63b)$$

These residuals are all larger than the corresponding residuals in equation (59), because the vectors $A(q_1)\mathbf{r}_2$, $A(q_2)\mathbf{r}_1$, $A(q_3)\mathbf{r}_1$, and $A(q_3)\mathbf{r}_2$ all have components along the *y* axis in the body frame, which is perpendicular to the plane spanned by $\mathbf{b}_1$ and $\mathbf{b}_2$. According to our previous argument, they can't correspond to optimal estimates for any choice of weights. We may be prepared to give up optimality for computational simplicity, however.

SINGULARITY OF THE DIRECT QUATERNION METHOD

The direct quaternion method has the disadvantage of being ill determined whenever both the vector part and the scalar part of the estimated quaternion take the indeterminate value 0/0. We can easily see from equation (50) that q_1 is undefined if $\mathbf{b}_2 = \mathbf{r}_2$, which is when the axis of the attitude rotation is along $\mathbf{r}_2$ (and therefore is also along $\mathbf{b}_2$). Similarly, equation (52) shows that q_2 is undefined if $\mathbf{b}_1 = \mathbf{r}_1$, which is when the axis of the attitude rotation is along $\mathbf{r}_1$ and $\mathbf{b}_1$. These estimators are identical in the absence of measurement noise, and we certainly don't want to depend on measurement noise to avoid a singular condition. Thus we see that the direct quaternion method is singular whenever the attitude rotation axis is along $\mathbf{r}_1$ or $\mathbf{r}_2$ (or along $\mathbf{b}_1$ or $\mathbf{b}_2$). We will now show that the direct quaternion method is singular whenever the attitude rotation axis is in the $\mathbf{r}_1$, $\mathbf{r}_2$ plane, which means that it is also in the $\mathbf{b}_1$, $\mathbf{b}_2$ plane.

If neither $\mathbf{b}_1 - \mathbf{r}_1$ nor $\mathbf{b}_2 - \mathbf{r}_2$ is zero, the vector part of the quaternion estimate vanishes if they are parallel, that is, if

$$\mathbf{b}_2 - \mathbf{r}_2 = \beta(\mathbf{b}_1 - \mathbf{r}_1) \quad (64)$$

for some scalar β. The vector $\mathbf{b}_2 = \mathbf{r}_2 + \beta(\mathbf{b}_1 - \mathbf{r}_1)$ has unit norm, which means that

$$1 = 1 + 2\beta\,\mathbf{r}_2\cdot(\mathbf{b}_1 - \mathbf{r}_1) + 2\beta^2(1 - \mathbf{b}_1\cdot\mathbf{r}_1). \quad (65)$$

Solving this for β (the zero root is not allowed since $\mathbf{b}_2 - \mathbf{r}_2 \neq 0$) and substituting into equation (64) gives

$$\mathbf{b}_2 = \mathbf{r}_2 - \left[\mathbf{r}_2\cdot(\mathbf{b}_1 - \mathbf{r}_1)/(1 - \mathbf{b}_1\cdot\mathbf{r}_1)\right](\mathbf{b}_1 - \mathbf{r}_1). \quad (66)$$

It is now straightforward to show that equation (2) is obeyed and that

$$\mathbf{b}_2\cdot\mathbf{r}_1 = \mathbf{b}_1\cdot\mathbf{r}_2. \quad (67)$$

Thus the vanishing of the vector part of the quaternion estimates of equations (50), (52), and (55) ensures that the scalar parts vanish automatically.

Now let us see what these conditions imply about the attitude quaternion, which certainly exists even if it cannot be computed by the direct quaternion method. Equation (42) requires

$$\mathbf{b}_i = (q_s^2 - |\mathbf{q}|^2)\mathbf{r}_i + 2(\mathbf{q}\cdot\mathbf{r}_i)\mathbf{q} - 2q_s(\mathbf{q}\times\mathbf{r}_i) \quad \text{for } i = 1, 2. \quad (68)$$

From this equation, we can see that

$$\mathbf{b}_2\cdot\mathbf{r}_1 - \mathbf{b}_1\cdot\mathbf{r}_2 = 4q_s\mathbf{q}\cdot(\mathbf{r}_1\times\mathbf{r}_2). \quad (69)$$

This means that the scalar part of the direct quaternion estimate vanishes either if $\mathbf{q}$ is perpendicular to $\mathbf{r}_1\times\mathbf{r}_2$, which is to say that it is in the $\mathbf{r}_1$, $\mathbf{r}_2$ plane, or else if q_s is zero, which indicates a 180° rotation. We still have to investigate the requirement that $\mathbf{b}_1 - \mathbf{r}_1$ is parallel to $\mathbf{b}_2 - \mathbf{r}_2$. If $\mathbf{q}$ is in the $\mathbf{r}_1$, $\mathbf{r}_2$ plane, we can write

$$\mathbf{q} = \alpha_1\mathbf{r}_1 + \alpha_2\mathbf{r}_2. \quad (70)$$

With equation (68), this gives

$$\mathbf{b}_1 - \mathbf{r}_1 = 2\alpha_2[-(\alpha_2 + \alpha_1 \mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{r}_1 + (\alpha_1 + \alpha_2 \mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{r}_2 + q_s(\mathbf{r}_1 \times \mathbf{r}_2)] \tag{71a}$$

and

$$\mathbf{b}_2 - \mathbf{r}_2 = -2\alpha_1[-(\alpha_2 + \alpha_1 \mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{r}_1 + (\alpha_1 + \alpha_2 \mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{r}_2 + q_s(\mathbf{r}_1 \times \mathbf{r}_2)]. \tag{71b}$$

These two vectors are clearly parallel. On the other hand, equation (68) for a 180° rotation gives

$$\mathbf{b}_i - \mathbf{r}_i = -2\mathbf{r}_i + 2(\mathbf{q} \cdot \mathbf{r}_i)\mathbf{q} = 2\mathbf{q} \times (\mathbf{q} \times \mathbf{r}_i), \tag{72}$$

and a straightforward but tedious calculation gives

$$(\mathbf{b}_1 - \mathbf{r}_1) \times (\mathbf{b}_2 - \mathbf{r}_2) = 4[\mathbf{q} \cdot (\mathbf{r}_1 \times \mathbf{r}_2)]\mathbf{q}. \tag{73}$$

Thus the attitude rotation axis is required to be in the $\mathbf{r}_1$, $\mathbf{r}_2$ plane for the 180° rotation case to be singular, also. Thus we have completely characterized the singular cases of the direct quaternion method as those cases for which the attitude rotation axis is in the $\mathbf{r}_1$, $\mathbf{r}_2$ plane, and therefore in the $\mathbf{b}_1$, $\mathbf{b}_2$ plane, also.

The direct quaternion estimate method is singular if the attitude matrix is the identity, giving $\mathbf{r}_1 = \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{b}_2$. We can say that the rotation axis is in the $\mathbf{r}_1$, $\mathbf{r}_2$ plane in this case, also, because the rotation axis can be arbitrarily assigned for zero rotation angle. Reynolds has proposed a method to avoid the singular condition in most cases, but it does not avoid the singularity for attitude matrices close to the identity[7,8]. The singular condition can be avoided in all cases by applying Shuster's method of sequential rotations[10,19]. This method solves for the attitude with respect to reference coordinate frames rotated from the original frame by 180° about the x, y, or z coordinate axis. That is, we solve for the quaternions

$$q^i \equiv q \otimes [\mathbf{e}_i, 0] = [\mathbf{q}, q_s] \otimes [\mathbf{e}_i, 0] = [q_s\mathbf{e}_i - \mathbf{q} \times \mathbf{e}_i, -\mathbf{q} \cdot \mathbf{e}_i] \qquad \text{for } i = 1, 2, 3, \tag{74}$$

where $\mathbf{e}_i$ is the unit vector along the i^{th} coordinate axis. These rotations are easy to implement on the reference vectors, since they simply change the signs of the components perpendicular to $\mathbf{e}_i$. Merely permuting and changing signs of the components of the rotated quaternion recovers the unrotated quaternion. For example

$$q^1 = [q_1, q_2, q_3, q_s] \otimes [1, 0, 0, 0] = [q_s, -q_3, q_2, -q_1]. \tag{75}$$

The method of sequential rotations always avoids the singularity, since the 3×4 matrix

$$[\mathbf{q} \vdots q_s\mathbf{e}_1 - \mathbf{q} \times \mathbf{e}_1 \vdots q_s\mathbf{e}_2 - \mathbf{q} \times \mathbf{e}_2 \vdots q_s\mathbf{e}_3 - \mathbf{q} \times \mathbf{e}_3] \tag{76}$$

always has rank three, as can be seen with some effort. Thus the rotation axes produced by the method of sequential rotations span the entire three-dimensional space, which means that they cannot all be coplanar with $\mathbf{r}_1$ and $\mathbf{r}_2$.

To use Shuster's rotations to avoid the singularity, we compute the reference vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ in all four reference frames, unrotated and rotated about the x, y, and z axes. We compute the magnitude squared of the cross product $(\mathbf{b}_1 - \mathbf{r}_1) \times (\mathbf{b}_2 - \mathbf{r}_2)$ in each frame, and evaluate the quaternion in the frame where the cross product has the largest magnitude. The above analysis shows that this should provide the most robust estimate. If the optimal reference frame is not the unrotated frame, we recover the desired quaternion that transforms the unrotated reference frame to the spacecraft body frame by using equation (75) or its equivalent for other rotations.

COMPUTATIONAL EFFORT

The speed comparison is based on the floating point operation (flop) counts in MATLAB implementations of the algorithms, which have the advantage of being platform-independent. There are some caveats to make with regard to timing comparisons. First, for ground computations, absolute speed isn't all that important, since the estimation algorithm is only a part of the overall attitude determination data processing effort. Speed was more important in the past, when thousands of attitude solutions had to be computed by slower machines. Second, for real-time processing, as for an attitude control system onboard a spacecraft, the longest time is more important than the average time, because the attitude control system processor has to finish its task in a limited amount of time. This works against methods that may require sequential rotations.

Four methods for computing the attitude matrix are compared in Table 1: asymmetric TRIAD of equation (8), symmetric TRIAD of equation (12), the optimal two-measurement estimator of equation (36), and Optimized TRIAD of equation (40). An "asymmetric" estimator maps one of the two reference vectors into the corresponding observed vector exactly, throwing all the measurement errors into the other vector. A "symmetric" estimator, on the other hand, treats the two measurements symmetrically. The cost of using these four estimators to produce a quaternion is also presented. Every algorithm except

Optimized TRIAD computes the quaternion by extracting it from the corresponding attitude matrix, a process that costs 29 flops (see the Appendix). The quaternion output of Optimized TRIAD is cheaper than the attitude matrix output because it is extracted from the estimate of equation (39) rather than from equation (40). In addition to these four estimators, three other estimators are included for quaternion output only: the asymmetric direct quaternion estimator of equation (50), the symmetric direct quaternion estimator of equation (55), and QUEST, for comparison[12]. The computational effort for the direct quaternion estimation methods is given both with and without the use of rotations to avoid singular configurations. The computational effort of QUEST does not include the cost of sequential rotations. No special efforts have been made to achieve the most efficient possible implementation of any of the algorithms.

Table 1: Computational Effort of Estimation Algorithms in Flops

Algorithm	A output	q output
Asymmetric TRIAD	143	172
Symmetric TRIAD	166	195
Optimal Two-Measurement Estimator	265	294
Optimized TRIAD	335	273
Asymmetric Direct Quaternion	——	46
Asymmetric Direct Quaternion with Singularity Avoidance	——	108
Symmetric Direct Quaternion	——	50
Symmetric Direct Quaternion with Singularity Avoidance	——	112
QUEST	——	190

Several conclusions are apparent from these results. Symmetric estimators are a little more expensive than asymmetric estimators, in general. Optimized TRIAD with the approximate matrix orthogonalization of equation (40) is significantly more expensive than the optimal two-measurement estimator. If quaternion output is desired, Optimized TRIAD is slightly less expensive than the optimal two-measurement estimator; but the savings are less than 10%. However, the optimal two-measurement estimator and Optimized TRIAD (and even symmetric TRIAD) require more computational effort than QUEST to produce a quaternion. Asymmetric TRIAD is only slightly less expensive than QUEST, but the direct quaternion estimation methods developed by Reynolds are significantly faster. The implementation of rotations to avoid singularities in the direct quaternion estimation methods more than doubles their computational cost, but they are faster than other methods even with this modification. None of the three algorithms faster than QUEST is optimal, though; and QUEST also has the advantage of being a general-purpose algorithm applicable to any number of measurements, which avoids the need to develop and test a special-purpose two-observation algorithm.

ACCURACY

We will analyze two test scenarios, using the nine estimators with quaternion output that were used in the timing tests. The first scenario simulates two star trackers with narrow fields of view and orthogonal boresights at $[1, 0, 0]^T$ and $[0, 1, 0]^T$. We shall assume that the first tracker is tracking five stars at

$$\mathbf{b}_{11} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_{12} = \begin{bmatrix} 0.99712 \\ 0.07584 \\ 0 \end{bmatrix}, \quad \mathbf{b}_{13} = \begin{bmatrix} 0.99712 \\ -0.07584 \\ 0 \end{bmatrix}, \quad \mathbf{b}_{14} = \begin{bmatrix} 0.99712 \\ 0 \\ 0.07584 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_{15} = \begin{bmatrix} 0.99712 \\ 0 \\ -0.07584 \end{bmatrix}, \tag{77a}$$

and the second tracker is tracking three stars at

$$\mathbf{b}_{21} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_{22} = \begin{bmatrix} 0 \\ 0.99712 \\ 0.07584 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_{23} = \begin{bmatrix} 0 \\ 0.99712 \\ -0.07584 \end{bmatrix}, \tag{77b}$$

We simulate 1000 test cases with random attitude matrices. We use the attitude matrices to map the eight observation vectors to the reference frame, add Gaussian random noise with equal standard deviations of 6 arcseconds per axis to the reference vectors, and then normalize them. The errors are unconventionally applied to the reference vectors rather than the observation vectors so that equation (77) will remain valid in the presence of noise. The two-observation estimators use averages of the multiple vectors observed by each tracker, as suggested by Bronzenac and Bender[5]. In this example the two averaged vectors in the body frame are along the star tracker boresights. The optimal estimator weights for these estimators are proportional to the inverse measurement variances, or to the number of vectors included in the average, so we use $a_2 = 0.6\, a_1$ for the optimal two-measurement estimator and Optimized TRIAD.

We treat the eight star measurements independently in QUEST, rather than averaging them. QUEST requires 316 flops for eight measurements, but avoids the cost of averaging the vectors, which is 108 flops. Thus the computational effort of QUEST should be taken as 208 flops for comparison with the other estimators in this eight-measurement example, making it more expensive than the direct quaternion estimator and TRIAD, but faster than the optimal two-vector estimator and Optimized TRIAD. In these tests, QUEST always used information about the true quaternion to determine the optimally rotated reference frame for estimation, without the need to perform sequential rotations.

Table 2 shows that symmetric TRIAD, the optimal two-measurement estimator, and Optimized TRIAD perform almost as well as QUEST. This justifies Bronzenac and Bender's procedure of using average observation and measurement vectors for the two star trackers. It should be noted, however, that the choice of orthogonal tracker boresights is optimal for this approximation, and that symmetric TRIAD is the only one of these algorithms that is computationally less expensive than QUEST, requiring 13 fewer flops. The symmetric direct quaternion estimator with singularity avoidance provides average and maximum errors within 10% of those of the best estimators with less computational effort, though.

Table 2: Average (Maximum) Estimation Errors (arcseconds) for Star Tracker Scenario

Algorithm	All Cases	$\lvert q_3 \rvert \geq 1/2$	$\lvert q_3 \rvert < 1/2$
Asymmetric TRIAD	4.6 (12.1)	4.5 (11.3)	4.7 (12.1)
Asymmetric Direct Quaternion	13.6 (2562)	5.2 (17.8)	20.1 (2562)
Asymmetric Direct Quaternion with Singularity Avoidance	5.1 (16.9)	5.1 (14.5)	5.1 (16.9)
Symmetric TRIAD	4.4 (12.2)	4.3 (11.6)	4.5 (12.2)
Symmetric Direct Quaternion	14.2 (4763)	4.7 (14.6)	21.6 (4763)
Symmetric Direct Quaternion with Singularity Avoidance	4.7 (12.9)	4.6 (12.1)	4.8 (12.9)
Optimized TRIAD	4.6 (12.1)	4.5 (11.3)	4.7 (12.1)
Optimal Two-Measurement Estimator	4.6 (12.1)	4.5 (11.3)	4.7 (12.1)
QUEST	4.4 (11.8)	4.3 (11.5)	4.4 (11.8)

The results also show that symmetric estimators perform slightly better than asymmetric estimators in this scenario. This was expected, since the number of stars tracked in the two trackers and thus the measurement weights are nearly equal. A symmetric estimator would be preferred in a real star tracker application, since there would be no way of predicting *a priori* which tracker would view more stars.

Table 2 also shows inferior performance of the direct quaternion estimators without singularity avoidance. The performance is not so bad in the 436 simulated cases with $\lvert q_3 \rvert \geq 1/2$ as in the 564 cases with $\lvert q_3 \rvert < 1/2$. The latter are the cases in which we would expect singularities to occur, since they have either small rotation angles or rotation axes close to the *x*-*y* plane, the $\mathbf{b}_1$, $\mathbf{b}_2$ plane in this scenario. This shows the importance of avoiding singular cases in an application of these estimators. We note that the performance with singularity avoidance, as well as the performance of all the other estimators, is independent of q_3.

The second scenario that we consider is a sun-mag system, similar to that on SAMPEX[3], assuming a digital sun sensor with accuracy of 0.1° and a magnetometer with effective accuracy of 1°. We assume that the Sun is at the center of view of the digital sun sensor at $\mathbf{b}_1 = [1, 0, 0]^T$, but the orientation of the magnetic field vector is not fixed in the spacecraft body frame. We simulate 1000 random attitude matrices and random magnetic field vector orientations, except that we do not allow the magnetic field direction to be within 5° of the $\pm y$ axis. These are the cases with coaligned vectors that the SAMPEX onboard attitude determination system rejects. We use the attitude matrices to map the Sun and magnetic field observation vectors to the reference frame, add Gaussian random noise with standard deviations as specified above, and then normalize the reference vectors. In this case the optimal estimator weights have $a_2 = 0.01\, a_1$.

The estimation errors for this scenario are presented in Table 3. The roll (x axis) and pitch/yaw (root-sum-squared of y and z axes) errors are presented separately, since the estimate of pitch and yaw provided by the digital sun sensor on the x axis is more precise than the roll angle estimate provided by the magnetometer. We note from these tables that QUEST and the optimal two-measurement estimator give identical errors, as they must since this scenario has two vector measurements. Since the weight assigned to the magnetometer measurement is so much less than the weight of the sun sensor measurement, Optimized TRIAD and asymmetric TRIAD give virtually the same results as the optimal estimators. The asymmetric direct quaternion estimator with singularity avoidance provides equivalent pitch and yaw errors, and average and maximum roll errors within 20% of those of the best estimators, with less computational effort.

Symmetric estimators are inferior to asymmetric estimators in the sun-mag scenario, since they allow the magnetometer errors to corrupt the sun sensor determination of pitch and yaw. Table 3a shows that the direct quaternion estimation method must be modified to provide acceptable roll estimation in the 551 cases with $|\mathbf{q}_\perp| < 1/2$, where $\mathbf{q}_\perp$ is the component of $\mathbf{q}$ perpendicular to the $\mathbf{b}_1$, $\mathbf{b}_2$ plane. Table 3b shows that pitch and yaw estimates provided by the asymmetric direct quaternion estimator are insensitive to these singular configurations, since this estimator maps $\mathbf{r}_1$ into $\mathbf{b}_1$ exactly.

Table 3a: Average (Maximum) Roll Estimation Errors (degrees) for Sun-Mag Test Case

Algorithm	All Cases	$\lvert\mathbf{q}_\perp\rvert \geq 1/2$	$\lvert\mathbf{q}_\perp\rvert < 1/2$
Asymmetric TRIAD	0.88 (3.06)	0.93 (2.92)	0.84 (3.06)
Asymmetric Direct Quaternion	2.82 (114)	1.07 (4.78)	4.24 (114)
Asymmetric Direct Quaternion with Singularity Avoidance	1.01 (3.58)	1.05 (3.36)	0.98 (3.58)
Symmetric TRIAD	0.88 (3.06)	0.93 (2.92)	0.84 (3.06)
Symmetric Direct Quaternion	1.97 (86.5)	0.98 (3.78)	2.77 (86.5)
Symmetric Direct Quaternion with Singularity Avoidance	0.92 (3.23)	0.96 (3.16)	0.89 (3.23)
Optimized TRIAD	0.88 (3.06)	0.93 (2.92)	0.84 (3.06)
Optimal Two-Measurement Estimator	0.88 (3.06)	0.93 (2.92)	0.84 (3.06)
QUEST	0.88 (3.06)	0.93 (2.92)	0.84 (3.06)

Table 3b: Average (Maximum) Pitch/Yaw Estimation Errors (degrees) for Sun-Mag Test Case

Algorithm	All Cases	$\lvert\mathbf{q}_\perp\rvert \geq 1/2$	$\lvert\mathbf{q}_\perp\rvert < 1/2$
Asymmetric TRIAD	0.13 (0.36)	0.13 (0.35)	0.13 (0.36)
Asymmetric Direct Quaternion	0.13 (0.36)	0.13 (0.35)	0.13 (0.36)
Asymmetric Direct Quaternion with Singularity Avoidance	0.13 (0.36)	0.13 (0.35)	0.13 (0.36)
Symmetric TRIAD	0.43 (1.60)	0.42 (1.54)	0.43 (1.60)
Symmetric Direct Quaternion	1.53 (96.3)	0.50 (1.91)	2.37 (96.3)
Symmetric Direct Quaternion with Singularity Avoidance	0.48 (1.92)	0.49 (1.91)	0.48 (1.92)
Optimized TRIAD	0.13 (0.37)	0.13 (0.35)	0.13 (0.37)
Optimal Two-Measurement Estimator	0.13 (0.37)	0.13 (0.35)	0.13 (0.37)
QUEST	0.13 (0.37)	0.13 (0.35)	0.13 (0.37)

CONCLUSIONS

We have analyzed four spacecraft attitude determination methods using exactly two vector measurements: the well-known TRIAD algorithm, an optimal closed-form two-measurement of Wahba's optimization problem, the Optimized TRIAD algorithm of Bar-Itzhack and Harman, and the direct quaternion estimation method of Reynolds. These methods are applicable to a variety of problems, including coarse "sun-mag" attitude estimation using the unit vector to the Sun and the Earth's magnetic field vector and precise estimation using unit vectors to stars tracked by two star trackers. For TRIAD and the direct quaternion estimation method, we investigate both "asymmetric" forms that map one of the two reference vectors into the corresponding observed vector exactly, and "symmetric" forms that distribute the errors symmetrically between the two measurements. We also include the well-known QUEST algorithm for comparison,

The computational speed of the algorithms was compared using floating point operation (flop) counts in MATLAB. These show that Optimized TRIAD and the optimal two-measurement estimator are more expensive than QUEST, which has the additional advantage of being a general-purpose algorithm applicable to any number of measurements. The direct quaternion estimation methods are significantly faster than QUEST, however. Both QUEST and the direct quaternion estimation methods have the disadvantage of sometimes requiring special computations to avoid singular cases, but the direct quaternion estimation methods are faster than any other methods even with these modifications.

We analyzed the accuracy of the estimators in two test scenarios. The first scenario simulated two star trackers with narrow fields of view and orthogonal boresights, using average vectors for five stars in the first tracker and three in the second. The second scenario simulated a digital sun sensor with accuracy of 0.1° and a magnetometer with effective accuracy of 1°. Symmetric estimators outperformed asymmetric estimators in the first scenario, and asymmetric estimators were superior in the second, as was expected. With this proviso, all the estimators had comparable errors. The one exception is that the direct quaternion estimators had larger errors if not modified to avoid singularities, showing the need for these modifications.

This paper demonstrates the superiority of TRIAD, QUEST, and the direct quaternion estimation methods for attitude estimation from two vector measurements.

REFERENCES

1. Black, Harold D., "A Passive System for Determining the Attitude of a Satellite," *AIAA Journal*, Vol. 2, No. 7, July 1964, pp. 1350–1351.
2. Lerner, Gerald M., "Three-Axis Attitude Determination," in *Spacecraft Attitude Determination and Control*, James R. Wertz, ed., D. Reidel, Dordrecht, Holland 1978.
3. Flatley, Thomas W., Josephine K. Forden, Debra A. Henretty, E. Glenn Lightsey, and F. Landis Markley, "Onboard Attitude Determination and Control Algorithms for SAMPEX," Flight Mechanics/ Estimation Theory Symposium, Goddard Space Flight Center, Greenbelt, MD, May 1990, NASA Conference Publication 3102.
4. Wahba, Grace, "A Least Squares Estimate of Spacecraft Attitude," *SIAM Review*, Vol. 7, No. 3, July 1965, p. 409.
5. Bronzenac, Thomas F. and Douglas J. Bender, "A Simple Suboptimal Least-Squares Algorithm for Attitude Determination with Multiple Sensors," Flight Mechanics/ Estimation Theory Symposium, Goddard Space Flight Center, Greenbelt, MD, May 1994, NASA Conference Publication 3265.
6. Bar-Itzhack, Itzhack Y. and Richard R. Harman, "Optimized TRIAD Algorithm for Attitude Determination," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 1, January-February 1997, pp. 208-211.
7. Reynolds, R. G., "Quaternion Parameterization and Global Attitude Estimation," Flight Mechanics Symposium, Goddard Space Flight Center, Greenbelt, MD, May 1997, NASA Conference Publication 3345.
8. Reynolds, R. G., "Quaternion Parameterization and a Simple Algorithm for Global Attitude Estimation," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 4, July-August 1998, pp. 669-671.
9. Keat, J., "Analysis of Least Squares Attitude Determination Routine DOAOP," Computer Sciences Corporation, CSC/TM-77/6034, February 1977.
10. Markley, F. Landis, "Parameterization of the Attitude," in *Spacecraft Attitude Determination and Control*, James R. Wertz, ed., D. Reidel, Dordrecht, Holland 1978.
11. Shuster, Malcolm D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4 October-December 1993, pp. 439-517.

12. Shuster, M. D. and S. D. Oh, "Three-Axis Attitude Determination from Vector Observations," *Journal of Guidance and Control*, Vol. 4, No. 1, January-February 1981, pp. 70-77.

13. Markley, F. Landis, "Attitude Determination Using Vector Observations: A Fast Optimal Matrix Algorithm," *Journal of the Astronautical Sciences*, Vol. 41, No. 2, April-June 1993, pp. 261-280.

14. Markley, F. Landis, "Attitude Determination Using Vector Observations and the Singular Value Decomposition," *Journal of the Astronautical Sciences*, Vol. 36, No. 3, July-September 1988, pp. 245-258.

15. Golub, Gene H. and Charles F. Van Loan, *Matrix Computations*, the Johns Hopkins University Press, Baltimore, 1983.

16. Horn, Roger A. and Charles R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985

17. Mortari, Daniele, "*EULER*-2 and *EULER-n* Algorithms for Attitude Determination from Vector Observations," *Space Technology*, Vol. 16, No. 5/6, 1996, pp. 317-3121.

18. Giardina, C. R., R. Bronson, and L. Wallen, "An Optimal Normalization Scheme," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-11, No. 4, July 1975, pp. 443-446.

19. Shuster, Malcolm D. and Gregory A. Natanson, "Quaternion Computation from a Geometric Point of View," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, October-December 1993, pp. 545-556.

APPENDIX

The following MATLAB function was used to extract quaternions from attitude matrices[10,11].

```
function q = dcm2quat(a)

% finds the quaternion representation of a direction cosine matrix a

% find maximum of trace or diagonal element of direction cosine matrix
tra    = trace(a);
[mx,i] = max([a(1,1) a(2,2) a(3,3) tra]);

% compute unnormalized quaternion
if i==1, q = [2*mx+1-tra;a(1,2)+a(2,1);a(1,3)+a(3,1);a(2,3)-a(3,2)]; end;
if i==2, q = [a(2,1)+a(1,2);2*mx+1-tra;a(2,3)+a(3,2);a(3,1)-a(1,3)]; end;
if i==3, q = [a(3,1)+a(1,3);a(3,2)+a(2,3);2*mx+1-tra;a(1,2)-a(2,1)]; end;
if i==4, q = [a(2,3)-a(3,2);a(3,1)-a(1,3);a(1,2)-a(2,1);1+tra];      end;

% normalize the quaternion
q = q/norm(q);
```