CHAPTER 12

THREE-AXIS ATTITUDE DETERMINATION METHODS

Chapter 11 described deterministic procedures for computing the orientation of a single spacecraft axis and estimating the accuracy of this computation. The methods described there may be used either to determine single-axis attitude or the orientation of any single axis on a three-axis stabilized spacecraft. However, when the three-axis attitude of a spacecraft is being computed, some additional formalism is appropriate. The attitude of a single axis can be parameterized either as a three-component unit vector or as a point on the unit celestial sphere, but three-axis attitude is most conveniently thought of as a coordinate transformation which transforms a set of reference axes in inertial space to a set in the spacecraft. The alternative parameterizations for this transformation are described in Section 12.1. Section 12.2 then describes three-axis attitude determination methods, and Section 12.3 introduces the covariance analysis needed to estimate the uncertainty in three-axis attitude.

12.1 Parameterization of the Attitude

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Let us consider a rigid body in space, either a rigid spacecraft or a single rigid component of a spacecraft with multiple components moving relative to each other. We assume that there exists an orthogonal, right-handed triad $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ of unit vectors fixed in the body, such that

$$\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}} \tag{12-1}$$

The basic problem is to specify the orientation of this triad, and hence of the rigid body, relative to some reference coordinate frame, as illustrated in Fig. 12-1.

It is clear that specifying the components of $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{w}}$ along the three axes of the coordinate frame will fix the orientation completely. This requires nine parameters, which can be regarded as the elements of a 3×3 matrix, A, called the *attitude matrix*:

$$A \equiv \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \tag{12-2}$$

where $\hat{\mathbf{u}} = (u_1, u_2, u_3)^{\mathrm{T}}$, $\hat{\mathbf{v}} = (v_1, v_2, v_3)^{\mathrm{T}}$, and $\hat{\mathbf{w}} = (w_1, w_2, w_3)^{\mathrm{T}}$. Each of these elements is the cosine of the angle between a body unit vector and a reference axis; u_1, for

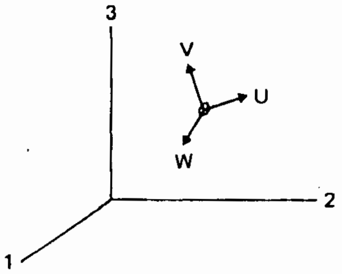

Fig. 12-1. The fundamental problem of three-axis attitude parameterization is to specify the orientation of the spacecraft axes $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ in the reference 1, 2, 3 frame.

example, is the cosine of the angle between $\hat{\mathbf{u}}$ and the reference 1 axis. For this reason, A is often referred to as the *direction cosine matrix*. The elements of the direction cosine matrix are not all independent. For example, the fact that $\hat{\mathbf{u}}$ is a unit vector requires

$$u_1^2+u_2^2+u_3^2=1$$

and the orthogonality of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ means that

$$u_1v_1+u_2v_2+u_3v_3=0$$

These relationships can be summarized by the statement that the product of A and its transpose is the identity matrix

$$AA^{\mathrm{T}}=\mathbf{1} \tag{12-3}$$

(See Appendix C for a review of matrix algebra.) This means that A is a *real orthogonal* matrix. Also, the definition of the determinant is equivalent to

$$\det A=\hat{\mathbf{u}}\cdot(\hat{\mathbf{v}}\times\hat{\mathbf{w}})$$

so the fact that $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ form a right-handed triad means that $\det A=1$. Thus, A is a *proper* real orthogonal matrix.

The direction cosine matrix is a coordinate transformation that maps vectors from the reference frame to the body frame. That is, if $\mathbf{a}$ is a vector with components a_1, a_2, a_3 along the reference axes, then

$$A\mathbf{a}=\begin{bmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}\hat{\mathbf{u}}\cdot\mathbf{a}\\ \hat{\mathbf{v}}\cdot\mathbf{a}\\ \hat{\mathbf{w}}\cdot\mathbf{a}\end{bmatrix}\equiv\begin{bmatrix}a_u\\ a_v\\ a_w\end{bmatrix} \tag{12-4}$$

The components of $A\mathbf{a}$ are the components of the vector $\mathbf{a}$ along the body triad $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$. As shown in Appendix C, a proper real orthogonal matrix transformation preserves the lengths of vectors and the angles between them, and thus represents a *rotation*. The product of two proper real orthogonal matrices $A''=A'A$ represents the results of successive rotations by A and A', in that order. Because the transpose and inverse of an orthogonal matrix are identical, A^{T} maps vectors from the body frame to the reference frame.

It is also shown in Appendix C that a proper real orthogonal 3×3 matrix has at least one eigenvector with eigenvalue unity. That is, there exists a unit vector, $\hat{\mathbf{e}}$, that is unchanged by A:

$$A\hat{\mathbf{e}} = \hat{\mathbf{e}} \tag{12-5}$$

The vector $\hat{\mathbf{e}}$ has the same components along the body axes and along the reference axes. Thus, $\hat{\mathbf{e}}$ is a vector along the axis of rotation. The existence of $\hat{\mathbf{e}}$ demonstrates Euler's Theorem: *the most general displacement of a rigid body with one point fixed is a rotation about some axis.*

We regard the direction cosine matrix as the fundamental quantity specifying the orientation of a rigid body. However, other parameterizations, as summarized in Table 12-1 and discussed more fully below, may be more convenient for specific applications. In each case, we will relate the parameters to the elements of the direction cosine matrix. Our treatment follows earlier work by Sabroff, *et al.*, [1965].

Table 12-1. Alternative Representations of Three-Axis Attitude

PARAMETERIZATION	NOTATION	ADVANTAGES	DISADVANTAGES	COMMON APPLICATIONS
DIRECTION COSINE MATRIX	$A = [A_{ij}]$	NO SINGULARITIES NO TRIGONOMETRIC FUNCTIONS CONVENIENT PRODUCT RULE FOR SUCCESSIVE ROTATIONS	SIX REDUNDANT PARAMETERS	IN ANALYSIS, TO TRANSFORM VECTORS FROM ONE REFERENCE FRAME TO ANOTHER
EULER AXIS/ANGLE	$\mathbf{e}, \Phi$	CLEAR PHYSICAL INTERPRETATION	ONE REDUNDANT PARAMETER AXIS UNDEFINED WHEN $\sin\Phi = 0$ TRIGONOMETRIC FUNCTIONS	COMMANDING SLEW MANEUVERS
EULER SYMMETRIC PARAMETERS (QUATERNION)	q_1, q_2, q_3, q_4 (q)	NO SINGULARITIES NO TRIGONOMETRIC FUNCTIONS CONVENIENT PRODUCT RULE FOR SUCCESSIVE ROTATIONS	ONE REDUNDANT PARAMETER NO OBVIOUS PHYSICAL INTERPRETATION	ONBOARD INERTIAL NAVIGATION
GIBBS VECTOR	$\mathbf{g}$	NO REDUNDANT PARAMETERS NO TRIGONOMETRIC FUNCTIONS CONVENIENT PRODUCT RULE FOR SUCCESSIVE ROTATIONS	INFINITE FOR 180-DEG ROTATION	ANALYTIC STUDIES
EULER ANGLES	ϕ, θ, ψ	NO REDUNDANT PARAMETERS PHYSICAL INTERPRETATION IS CLEAR IN SOME CASES	TRIGONOMETRIC FUNCTIONS SINGULARITY AT SOME θ NO CONVENIENT PRODUCT RULE FOR SUCCESSIVE ROTATIONS	ANALYTIC STUDIES INPUT/OUTPUT ONBOARD ATTITUDE CONTROL OF 3-AXIS STABILIZED SPACECRAFT

Euler Axis/Angle. A particularly simple rotation is one about the 3 axis by an angle Φ, in the positive sense, as illustrated in Fig. 12-2. The direction cosine matrix for this rotation is denoted by $A_3(\Phi)$; its explicit form is

$$A_3(\Phi) = \begin{bmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12-6a}$$

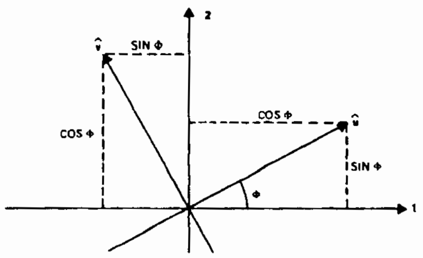

Fig. 12-2. Rotation About the Three-Axis by the Angle Φ

The direction cosine matrices for rotations by an angle Φ about the 1 or 2 axis, denoted by $A_1(\Phi)$ and $A_2(\Phi)$, respectively, are

$$A_1(\Phi)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \tag{12-6b}$$

$$A_2(\Phi)=\begin{bmatrix} \cos\Phi & 0 & -\sin\Phi \\ 0 & 1 & 0 \\ \sin\Phi & 0 & \cos\Phi \end{bmatrix} \tag{12-6c}$$

The matrices $A_1(\Phi)$, $A_2(\Phi)$, and $A_3(\Phi)$ all have the trace

$$\mathrm{tr}(A(\Phi))=1+2\cos\Phi \tag{12-6d}$$

The trace of a direction cosine matrix representing a rotation by the angle Φ about an arbitrary axis takes the same value. This result, which will be used without proof below, follows from the observation that the rotation matrices representing rotations by the same angle about different axes can be related by an orthogonal transformation, which leaves the trace invariant (see Appendix C).

In general, the axis of rotation will not be one of the reference axes. In terms of the unit vector along the rotation axis, $\hat{\mathbf{e}}$, and angle of rotation, Φ, the most general direction cosine matrix is

$$A=\begin{bmatrix} \cos\Phi+e_1^2(1-\cos\Phi) & e_1e_2(1-\cos\Phi)+e_3\sin\Phi & e_1e_3(1-\cos\Phi)-e_2\sin\Phi \\ e_1e_2(1-\cos\Phi)-e_3\sin\Phi & \cos\Phi+e_2^2(1-\cos\Phi) & e_2e_3(1-\cos\Phi)+e_1\sin\Phi \\ e_1e_3(1-\cos\Phi)+e_2\sin\Phi & e_2e_3(1-\cos\Phi)-e_1\sin\Phi & \cos\Phi+e_3^2(1-\cos\Phi) \end{bmatrix} \tag{12-7a}$$

$$=\cos\Phi\mathbf{1}+(1-\cos\Phi)\hat{\mathbf{e}}\hat{\mathbf{e}}^{\mathrm{T}}-\sin\Phi E \tag{12-7b}$$

where $\hat{\mathbf{e}}\hat{\mathbf{e}}^{\mathrm{T}}$ is the outer product (see Appendix C) and E is the skew-symmetric matrix

$$E\equiv\begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \tag{12-8}$$

This representation of the spacecraft orientation is called the *Euler axis and angle* parameterization. It appears to depend on four parameters, but only three are independent because $|\hat{\mathbf{e}}|=1$. It is a straightforward exercise to show that A defined by Eq. (12-7) is a proper real orthogonal matrix and that $\hat{\mathbf{e}}$ is the axis of rotation, that is, $A\hat{\mathbf{e}}=\hat{\mathbf{e}}$. The rotation angle is known to be Φ because the trace of A satisfies Eq. (12-6d).

It is also easy to see that Eq. (12-7) reduces to the appropriate one of Eqs. (12-6) when $\hat{\mathbf{e}}$ lies along one of the reference axes. The Euler rotation angle, Φ, can be expressed in terms of direction cosine matrix elements by

$$\cos\Phi=\frac{1}{2}[\mathrm{tr}(A)-1] \tag{12-9}$$

If $\sin\Phi \neq 0$, the components of $\hat{\mathbf{e}}$ are given by

$$e_1 = (A_{23} - A_{32})/(2\sin\Phi) \tag{12-10a}$$

$$e_2 = (A_{31} - A_{13})/(2\sin\Phi) \tag{12-10b}$$

$$e_3 = (A_{12} - A_{21})/(2\sin\Phi) \tag{12-10c}$$

Equation (12-9) has two solutions for Φ, which differ only in sign. The two solutions have axis vectors $\hat{\mathbf{e}}$ in opposite directions, according to Eq. (12-10). This expresses the fact that a rotation about $\hat{\mathbf{e}}$ by an angle Φ is equivalent to a rotation about $-\hat{\mathbf{e}}$ by $-\Phi$.

Euler Symmetric Parameters. A parameterization of the direction cosine matrix in terms of *Euler symmetric parameters* q_1, q_2, q_3, q_4 has proved to be quite useful in spacecraft work. These parameters are not found in many modern dynamics textbooks, although Whittaker [1937] does introduce them and they are discussed by Sabroff, *et al.*, [1965]. They are defined by

$$q_1 \equiv e_1 \sin\frac{\Phi}{2} \tag{12-11a}$$

$$q_2 \equiv e_2 \sin\frac{\Phi}{2} \tag{12-11b}$$

$$q_3 \equiv e_3 \sin\frac{\Phi}{2} \tag{12-11c}$$

$$q_4 \equiv \cos\frac{\Phi}{2} \tag{12-11d}$$

The four Euler symmetric parameters are not independent, but satisfy the constraint equation

$$q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1 \tag{12-12a}$$

These four parameters can be regarded as the components of a quaternion,

$$\boldsymbol{q} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \equiv \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} \tag{12-12b}$$

Quaternions are discussed in more detail in Appendix D. The Euler symmetric parameters are also closely related to the *Cayley-Klein parameters* [Goldstein, 1950].

The direction cosine matrix can be expressed in terms of the Euler symmetric parameters by

$$A(\boldsymbol{q}) = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \tag{12-13a}$$

$$= (q_4^2 - \mathbf{q}^2)\mathbf{1} + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} - 2q_4 Q \tag{12-13b}$$

where Q is the skew-symmetric matrix

$$Q \equiv \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{12-13c}$$

These equations can be verified by substituting Eqs. (12-11) into them, using some trigonometric identities, and comparing them with Eq. (12-7).

The Euler symmetric parameters corresponding to a given direction cosine matrix, A, can be found from

$$q_4 = \pm \frac{1}{2}(1 + A_{11} + A_{22} + A_{33})^{1/2} \tag{12-14a}$$

$$q_1 = \frac{1}{4q_4}(A_{23} - A_{32}) \tag{12-14b}$$

$$q_2 = \frac{1}{4q_4}(A_{31} - A_{13}) \tag{12-14c}$$

$$q_3 = \frac{1}{4q_4}(A_{12} - A_{21}) \tag{12-14d}$$

Note that there is a sign ambiguity in the calculation of these parameters. Inspection of Eq. (12-13) shows that changing the signs of all the Euler symmetric parameters simultaneously does not affect the direction cosine matrix. Equations (12-14) express one of four possible ways of computing the Euler symmetric parameters. We could also compute

$$q_1 = \pm \frac{1}{2}(1 + A_{11} - A_{22} - A_{33})^{1/2}$$

$$q_2 = \frac{1}{4q_1}(A_{12} + A_{21})$$

and so forth. All methods are mathematically equivalent, but numerical inaccuracy can be minimized by avoiding calculations in which the Euler symmetric parameter appearing in the denominator is close to zero. Other algorithms for computing Euler symmetric parameters from the direction cosine matrix are given by Klumpp [1976].

Euler symmetric parameters provide a very convenient parameterization of the attitude. They are more compact than the direction cosine matrix, because only four parameters, rather than nine, are needed. They are more convenient than the Euler axis and angle parameterization (and the Euler angle parameterizations to be considered below) because the expression for the direction cosine matrix in terms of Euler symmetric parameters does not involve trigonometric functions, which require time-consuming computer operations. Another advantage of Euler symmetric parameters is the relatively simple form for combining the parameters for two individual rotations to give the parameters for the product of the two rotations. Thus, if

$$A(\mathbf{q}'') = A(\mathbf{q}')A(\mathbf{q}) \tag{12-15a}$$

then

$$q'' = \begin{bmatrix} q'_4 & q'_3 & -q'_2 & q'_1 \\ -q'_3 & q'_4 & q'_1 & q'_2 \\ q'_2 & -q'_1 & q'_4 & q'_3 \\ -q'_1 & -q'_2 & -q'_3 & q'_4 \end{bmatrix} \mathbf{q} \qquad (12\text{-}15b)$$

Equation (12-15b) can be verified by direct substitution of Eq. (12-13) into Eq. (12-15a), but the algebra is exceedingly tedious. The relationship of Eq. (12-15b) to the quaternion product is given in Appendix D. Note that the evaluation of Eq. (12-15b) involves 16 multiplications and the computation of Eq. (12-15a) requires 27; this is another advantage of Euler symmetric parameters.

Gibbs Vector. The direction cosine matrix can also be parameterized by the *Gibbs vector,** which is defined by

$$g_1 \equiv q_1/q_4 = e_1 \tan\frac{\Phi}{2} \qquad (12\text{-}16a)$$

$$g_2 \equiv q_2/q_4 = e_2 \tan\frac{\Phi}{2} \qquad (12\text{-}16b)$$

$$g_3 \equiv q_3/q_4 = e_3 \tan\frac{\Phi}{2} \qquad (12\text{-}16c)$$

The direction cosine matrix is given in terms of the Gibbs vector by

$$A = \frac{1}{1+g_1^2+g_2^2+g_3^2} \begin{bmatrix} 1+g_1^2-g_2^2-g_3^2 & 2(g_1g_2+g_3) & 2(g_1g_3-g_2) \\ 2(g_1g_2-g_3) & 1-g_1^2+g_2^2-g_3^2 & 2(g_2g_3+g_1) \\ 2(g_1g_3+g_2) & 2(g_2g_3-g_1) & 1-g_1^2-g_2^2+g_3^2 \end{bmatrix} \qquad (12\text{-}17a)$$

$$= \frac{(1-\mathbf{g}^2)\mathbf{1} + 2\mathbf{g}\mathbf{g}^{\mathrm{T}} - 2G}{1+\mathbf{g}^2} \qquad (12\text{-}17b)$$

where G is the skew-symmetric matrix

$$G \equiv \begin{bmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{bmatrix} \qquad (12\text{-}17c)$$

Expressions for the Gibbs vector components in terms of the direction cosine matrix elements can be found by using Eqs. (12-16) and (12-14). Thus,

$$g_1 = \frac{A_{23}-A_{32}}{1+A_{11}+A_{22}+A_{33}} \qquad (12\text{-}18a)$$

*Gibbs [1901, p. 340] named this vector the "vector semitangent of version." Cayley [1899] used the three quantities g_1, g_2, g_3 in 1843 (before the introduction of vector notation), and he credits their discovery to Rodriguez.

$$g_2 = \frac{A_{31} - A_{13}}{1 + A_{11} + A_{22} + A_{33}} \tag{12-18b}$$

$$g_3 = \frac{A_{12} - A_{21}}{1 + A_{11} + A_{22} + A_{33}} \tag{12-18c}$$

Note that there is no sign ambiguity in the definition of the Gibbs vector and that the components are independent parameters. The product law for Gibbs vectors analogous to Eq. (12-15b) can be found from that equation and Eq. (12-16), and takes the convenient vector form

$$\mathbf{g}'' = \frac{\mathbf{g} + \mathbf{g}' - \mathbf{g}' \times \mathbf{g}}{1 - \mathbf{g} \cdot \mathbf{g}'} \tag{12-19}$$

The Gibbs vector has not been widely used because it becomes infinite when the rotation angle is an odd multiple of 180 deg.

Euler Angles. It is clear from the above discussion that three independent parameters are needed to specify the orientation of a rigid body in space. The only parameterization considered so far that has the minimum number of parameters is the Gibbs vector. We now turn to a class of parameterizations in terms of three rotation angles, commonly known as Euler angles. These are not as convenient for numerical computations as the Euler symmetric parameters, but their geometrical significance is more apparent (particularly for small rotations) and they are often used for computer input/output. They are also useful for analysis, especially for finding closed-form solutions to the equations of motion in simple cases. Euler angles are also commonly employed for three-axis stabilized spacecraft for which small angle approximations can be used.

To define the Euler angles precisely, consider four orthogonal triads of unit vectors, which we shall denote by

$$\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$$
$$\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$$
$$\hat{\mathbf{x}}'', \hat{\mathbf{y}}'', \hat{\mathbf{z}}''$$
$$\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$$

The initial triad $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ is parallel to the reference 1,2,3 axes. The triad $\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$ differs from $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ by a rotation about the i axis ($i = 1$, 2, or 3 depending on the particular transformation) through an angle ϕ.* Thus, the orientation of the $\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$ triad relative to the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ triad is given by $A_i(\phi)$ for $i = 1$, 2, or 3, one of the simple direction cosine matrices given by Eq. (12-6). Similarly, the $\hat{\mathbf{x}}'', \hat{\mathbf{y}}'', \hat{\mathbf{z}}''$ triad orientation relative to the $\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$ triad is a rotation about a coordinate axis in the $\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$ system by an angle θ, specified by $A_j(\theta)$, $j = 1$, 2, or 3, $j \neq i$. Finally, the orientation of $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ relative to $\hat{\mathbf{x}}'', \hat{\mathbf{y}}'', \hat{\mathbf{z}}''$ is a third rotation, by an angle ψ, with the direction cosine matrix $A_k(\psi)$, $k = 1$, 2, or 3, $k \neq j$. The final $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ triad is the body-fixed triad considered previously, so the overall sequence of three rotations specifies the orientation of the body relative to the reference coordinate axes.

*Although Euler angles are rotation angles, we follow the usual convention of denoting them by lower-case Greek letters.

A specific example of Euler angle rotations is shown in Fig. 12-3. Here, the first rotation is through an angle ϕ about the $\hat{\mathbf{z}}$ axis, so that the $\hat{\mathbf{z}}$ and $\hat{\mathbf{z}}'$ axes coincide. The second rotation is by θ about the $\hat{\mathbf{x}}'$ axis, which thus is identical with $\hat{\mathbf{x}}''$. The third rotation is by ψ about the $\hat{\mathbf{z}}''$ (or $\hat{\mathbf{w}}$) axis. This sequence of rotations is called a 3-1-3 sequence, because the rotations are about the 3, 1, and 3 axes, in that order. The labeled points in the figure are the locations of the ends of the unit vectors on the unit sphere. The circles containing the numbers 1, 2, and 3 are the first, second, and third rotation axes, respectively. The solid lines are the great circles containing the unit vectors of the reference coordinate system, $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$. The cross-hatched lines are the great circles containing the unit vectors of the body coordinate system, $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$. The dotted and dashed lines are the great circles defined by intermediate coordinate systems.

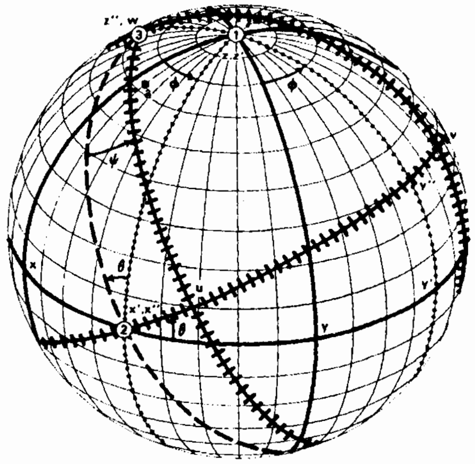

Fig. 12-3. 3-1-3 Sequence of Euler Rotations. (See text for explanation.)

The direction cosine matrix for the overall rotation sequence is the matrix product of the three matrices for the individual rotations, with the first rotation matrix on the right and the last on the left:

$$A_{313}(\phi,\theta,\psi) = A_3(\psi)A_1(\theta)A_3(\phi) =$$

$$\begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\sin\psi\cos\phi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \quad (12\text{-}20)$$

The Euler axis corresponding to $A_{313}(\phi,\theta,\psi)$ can be found from Eq. (12-10); it is denoted by $\hat{\mathbf{e}}$ in Fig. 12-3.

The 3-1-3 Euler angles can be obtained from the direction cosine matrix elements by

$$\theta = \arccos A_{33} \quad (12\text{-}21a)$$

$$\phi = -\arctan(A_{31}/A_{32}) \quad (12\text{-}21b)$$

$$\psi = \arctan(A_{13}/A_{23}) \quad (12\text{-}21c)$$

Note that Eq. (12-21a) leaves a twofold ambiguity in θ, corresponding to $\sin\theta$ being positive or negative. Once this ambiguity is resolved, ϕ and ψ are determined uniquely (modulo 360 deg) by the signs and magnitudes of A_{13}, A_{23}, A_{31}, and A_{32}, with the exception that when θ is a multiple of 180 deg, only the sum or difference of ϕ and ψ is determined, depending on whether θ is an even or an odd multiple of 180 deg. The origin of this ambiguity is apparent in Fig. 12-3. The usual resolution of this ambiguity is to choose $\sin\theta \geqslant 0$, or $0 \leqslant \theta < 180$ deg.

Other sequences of Euler angle rotations are possible, and several are used. Figure 12-4 illustrates a 3-1-2 sequence: a rotation by ϕ about $\hat{\mathbf{z}}$ followed by a rotation by θ about $\hat{\mathbf{x}}'$ and then by a rotation by ψ about $\hat{\mathbf{y}}''$. This is often referred to as the yaw, roll, pitch sequence, but the meaning of these terms and the order of rotations implied is not standard. The direction cosine matrix illustrated in Fig. 12-4 is

$$A_{312}(\phi,\theta,\psi) = A_2(\psi)A_1(\theta)A_3(\phi) =$$

$$\begin{bmatrix} \cos\psi\cos\phi - \sin\theta\sin\psi\sin\phi & \cos\psi\sin\phi + \sin\theta\sin\psi\cos\phi & -\cos\theta\sin\psi \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\psi\cos\phi + \sin\theta\cos\psi\sin\phi & \sin\psi\sin\phi - \sin\theta\cos\psi\cos\phi & \cos\theta\cos\psi \end{bmatrix} \quad (12\text{-}22)$$

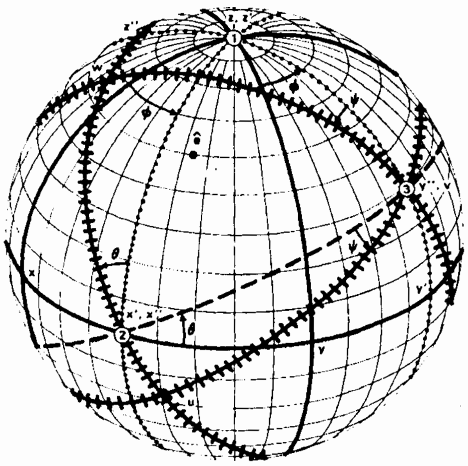

Fig. 12-4. 3-1-2 Sequence of Euler Rotations. (See text for explanation.)

The expressions for the rotation angles in terms of the elements of the direction cosine matrix are

$$\theta = \arcsin A_{23} \quad (12\text{-}23a)$$

$$\phi = -\arctan(A_{21}/A_{22}) \quad (12\text{-}23b)$$

$$\psi = -\arctan(A_{13}/A_{33}) \quad (12\text{-}23c)$$

As in the 3-1-3 case, the angles are determined up to a twofold ambiguity except at certain values of the intermediate angle θ. In this case, the singular values of θ are odd multiples of 90 deg. The usual resolution of the ambiguity is to choose $-90\text{ deg} < \theta \leqslant 90$ deg, which gives $\cos\theta \geqslant 0$.

If ϕ, θ, and ψ are all small angles, we can use small-angle approximations to the trigonometric functions, and Eq. (12-22) reduces to

$$A_{312}(\phi,\theta,\psi) \simeq \begin{bmatrix} 1 & \phi & -\psi \\ -\phi & 1 & \theta \\ \psi & -\theta & 1 \end{bmatrix} \tag{12-24a}$$

where the angles are measured in radians.

It is not difficult to enumerate all the possible sequences of Euler rotations. We cannot allow two successive rotations about a single axis, because the product of these rotations is equivalent to a single rotation about this axis. Thus, there are only 12 possible axis sequences:

313, 212, 121, 323, 232, 131,
312, 213, 123, 321, 231, 132.

Because of the twofold ambiguity in the angle θ mentioned above, there are 24 possible sequences of rotations, counting rotations through different angles as different rotations and ignoring rotations by multiples of 360 deg. The axis sequences divide naturally into two classes, depending on whether the third axis index is the same as or different from the first. Equation (12-20) is an example of the first class, and Eq. (12-22) is an example of the second. It is straightforward, using the techniques of this section, to write down the transformation equations for a given rotation sequence; these equations are collected in Appendix E. In the small-angle approximation, the 123, 132, 213, 231, 312, and 321 rotation sequences all have direction cosine matrices given by Eq. (12-24a) with the proviso that ϕ, θ, and ψ are the rotation angles about the 3, 1, 2 axes, respectively. Comparison with Eq. (12-13) shows that in the small-angle approximation, the Euler symmetric parameters are related to the Euler angles by

$$q_1 \simeq \frac{1}{2}\theta \tag{12-24b}$$

$$q_2 \simeq \frac{1}{2}\psi \tag{12-24c}$$

$$q_3 \simeq \frac{1}{2}\phi \tag{12-24d}$$

$$q_4 \simeq 1 \tag{12-24e}$$

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