12.2 Three-Axis Attitude Determination

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Three-axis attitude determination, which is equivalent to the complete specification of the attitude matrix, A, is accomplished either by an extension of the geometric techniques described in Chapter 11 or by a direct application of the concept of attitude as a rotation matrix. If the spacecraft has a preferred axis, such as the angular momentum vector of a spinning spacecraft or the boresight of a payload sensor, it is usually convenient to specify three-axis attitude in terms of the attitude of the preferred axis plus a phase angle about that axis. This asymmetric treatment of the attitude angles is usually justified by the attitude sensor configuration and the attitude accuracy requirements, which are generally more severe for the preferred axis. We refer to this method as *geometric* three-axis attitude determination because the phase angle is computed most conveniently using spherical trigonometry. Alternatively, in the *algebraic* method, the attitude matrix is determined directly from two vector observations without resorting to any angular representation. Finally, the ***q** method* provides a means for computing an optimal three-axis attitude from many vector observations. In this section we describe these methods for the computation of three-axis attitude.

12.2.1 Geometric Method

The geometric method is normally used when there is a body axis—such as the spin axis of a momentum wheel, a wheel-mounted sensor, or the spacecraft itself, about which there is preferential attitude data. Either deterministic techniques, as described in Chapter 11, or differential correction techniques, as will be described in Chapter 13, may be used to compute the attitude of the preferred axis. The phase angle about the preferred axis is then computed from any measurement which provides an angle about that axis.

In many cases, the geometric method is required because the sensor measurements themselves (e.g., spinning Sun sensors or horizon scanners) define a preferred spacecraft axis and provide only poor azimuthal information about that axis.

Figure 12-5 illustrates the geometric method. The reference axes are the celestial coordinates axes, $\hat{\mathbf{X}}_I$, $\hat{\mathbf{Y}}_I$, and $\hat{\mathbf{Z}}_I$. We wish to compute the 3-1-3 Euler angles, ϕ, θ, and ψ, which define the transformation from the celestial to the body

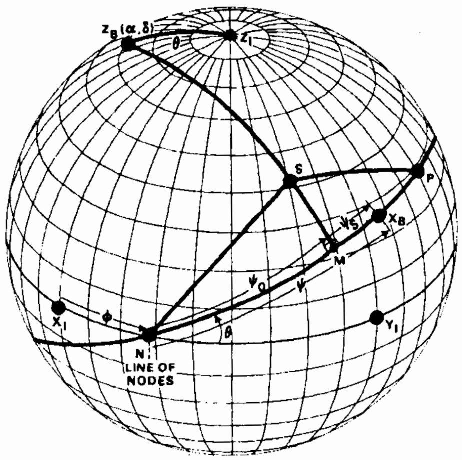

Fig. 12-5. Determination of the Phase Angle, ψ

coordinates, $\hat{\mathbf{X}}_B$, $\hat{\mathbf{Y}}_B$, and $\hat{\mathbf{Z}}_B$. The Euler angles ϕ and θ are related to the attitude (α,δ) of the preferred body axis, $\hat{\mathbf{Z}}_B$, by

$$\phi=90°+\alpha \tag{12-25a}$$

$$\theta=90°-\delta \tag{12-25b}$$

where the right ascension, α, and the declination, δ, are obtained by using any one-axis attitude determination method. ϕ defines the orientation of the node, $\hat{\mathbf{N}}$. The phase angle, ψ, is computed from the azimuth, ψ_S, of the projection of a measured vector, $\hat{\mathbf{S}}$ (e.g., the Sun or magnetic field) on the plane normal to $\hat{\mathbf{Z}}_B$. Let $\hat{\mathbf{M}}$ be the projection of $\hat{\mathbf{S}}$ on the plane normal to $\hat{\mathbf{Z}}_B$ and $\hat{\mathbf{P}}=\hat{\mathbf{Z}}_B\times\hat{\mathbf{N}}$. Application of Napier's rules (Appendix A) to the right spherical triangles SMN and SMP yields

$$\hat{\mathbf{N}}\cdot\hat{\mathbf{S}}=\hat{\mathbf{M}}\cdot\hat{\mathbf{S}}\cos\psi_0 \tag{12-26a}$$

$$\hat{\mathbf{P}}\cdot\hat{\mathbf{S}}=\hat{\mathbf{M}}\cdot\hat{\mathbf{S}}\cos(90°-\psi_0)=\hat{\mathbf{M}}\cdot\hat{\mathbf{S}}\sin\psi_0 \tag{12-26b}$$

which may be rewritten as*

$$\tan\psi_0=\hat{\mathbf{P}}\cdot\hat{\mathbf{S}}/\hat{\mathbf{N}}\cdot\hat{\mathbf{S}} \tag{12-27}$$

where

$$\hat{\mathbf{N}}=(\cos\phi,\sin\phi,0)^{\mathrm{T}} \tag{12-28a}$$

$$\hat{\mathbf{P}}=(-\cos\theta\sin\phi,\cos\theta\cos\phi,\sin\theta)^{\mathrm{T}} \tag{12-28b}$$

The phase angle, ψ, is then given by

$$\psi=\psi_0+\psi_S \tag{12-29}$$

As a more complex example of the geometric technique, we consider the three-axis attitude determination for the CTS spacecraft during attitude acquisition as illustrated in Fig. 12-6. The spacecraft Z axis is along the sunline and the spacecraft Y axis (the spin axis of a momentum wheel) is fixed in inertial space on a great circle 90 deg from the Sun. An infrared Earth horizon sensor has its boresight along the spacecraft Z axis and measures both the rotation angle, Ω_E, from the Sun to the nadir about the spacecraft Y axis and the nadir angle, η, from the spacecraft Y axis to the Earth's center. We wish to compute the rotation angle, Φ_S, about the sunline required to place the spacecraft Y axis into the celestial X-Y plane as a function of the following angles: the Sun declination in celestial coordinates, δ_S; the *clock angle*† or difference between the Earth and Sun azimuth in celestial coordinates, $\Delta\alpha=\alpha_E-\alpha_S$; and either measurement Ω_E or η. As shown in Fig. 12-6, Φ_S is 180 deg minus the sum of three angles:

$$\Phi_S=180°-(\angle YSR+\Lambda+\Phi_E) \tag{12-30}$$

*Note that $\hat{\mathbf{M}}\cdot\hat{\mathbf{S}}\geq 0$ by the definition of $\hat{\mathbf{M}}$. If $\hat{\mathbf{M}}\cdot\hat{\mathbf{S}}=0$, ψ_0 is indeterminate because $\hat{\mathbf{S}}$ provides no phase information about $\hat{\mathbf{Z}}_B$. If $\hat{\mathbf{M}}\cdot\hat{\mathbf{S}}>0$, ψ_0 is obtained unambiguously because the quadrants of both $\sin\psi_0$ and $\cos\psi_0$ are known.

†For the synchronous CTS orbit, the azimuthal difference or clock angle is zero at local midnight and decreases by 15 deg/hour

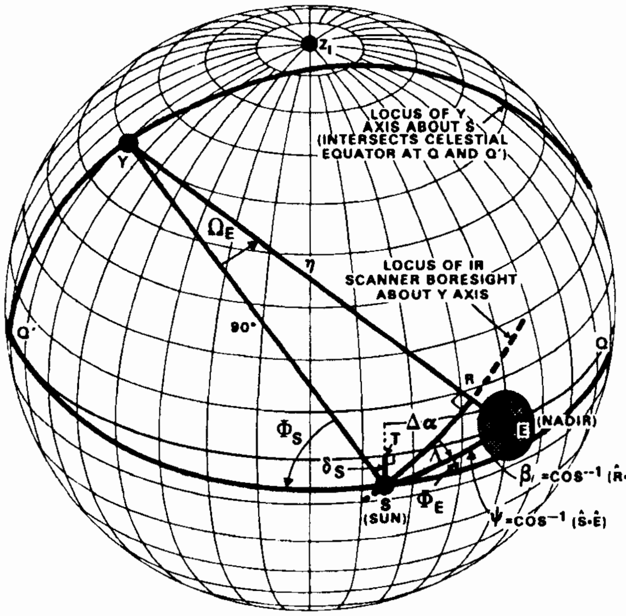

Fig. 12-6. Attitude Determination Geometry for CTS

Applying Napier's rules to the right spherical triangles ETS, ERS, and QTS, the arc length SE is

$$\psi \equiv \arccos(\hat{\mathbf{S}} \cdot \hat{\mathbf{E}}) = \arccos(\cos\delta_S \cos\Delta\alpha) \tag{12-31}$$

and the rotation angle, Λ, is

$$\Lambda \equiv \angle RSE = \arcsin(\sin\beta_E / \sin\psi) \tag{12-32}$$

where $\beta_E = \eta - 90°$ and the arc length, TQ, is 90 deg. Next, the quadrantal spherical triangle, QES, is solved for the angle ESQ:

$$\Phi_E \equiv \angle ESQ = \arccos(\cos(90° - \Delta\alpha)/\sin\psi) \tag{12-33}$$

Combining Eqs. (12-30), (12-32), and (12-33) with $\angle YSR = 90$ deg gives the result

$$\Phi_S = 90° - \arcsin(\sin\beta_E/\sin\psi) - \arccos(\sin\Delta\alpha/\sin\psi) \tag{12-34}$$

or

$$\Phi_S = \arccos(\sin\beta_E/\sin\psi) - \arccos(\sin\Delta\alpha/\sin\psi) \tag{12-35}$$

where

$$\sin\psi = \left(\cos^2\Delta\alpha \ \sin^2\delta_S + \sin^2\Delta\alpha\right)^{1/2} \tag{12-36}$$

Finally, Ω_E and β_E are related through the quadrantal spherical triangle, YSE, by

$$\beta_E = \arccos(\cos\delta_S \cos\Delta\alpha / \cos\Omega_E) \tag{12-37}$$

One problem with the geometric method is apparent from the proliferation of inverse trigonometric functions in Eqs. (12-31) to (12-37), which results in quadrant and consequent attitude ambiguities. Ambiguity is a frequent problem when dealing with inverse trigonometric functions and must be carefully considered in mission analysis. Although from Fig. 12-6, Φ_S and all the rotation angles in Eq. (12-30) are in the first quadrant by inspection, the generalization of Eq. (12-35) for arbitrary angles is not apparent. From the form of Eq. (12-35), it would appear that there is a fourfold ambiguity in Φ_S; however, some of these ambiguities may be resolved by applying the rules for quadrant specification given in Appendix A. There is, however, a true ambiguity in the sign of Λ which may be seen by redrawing Fig. 12-6 for $\Phi_S \approx -70$ deg and noting that, in this case,

$$\Phi_S = 180° - (\angle YSR - \Lambda + \Phi_E) \tag{12-38}$$

The ambiguity between Eqs. (12-30) and (12-38) is real if only pitch or roll measurements are available and must be differentiated from apparent ambiguities which may be resolved by proper use of the spherical triangle relations. However, if both Ω_E and η measurements are available, the ambiguity may be resolved by the sign of Ω_E because Ω_E is positive for Eq. (12-30) and negative for Eq. (12-38).

12.2.2 Algebraic Method

The algebraic method is based on the rotation matrix representation of the attitude. Any two vectors, **u** and **v**, define an orthogonal coordinate system with the basis vectors, $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$, and $\hat{\mathbf{s}}$ given by

$$\hat{\mathbf{q}} = \hat{\mathbf{u}} \tag{12-39a}$$

$$\hat{\mathbf{r}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}} / |\hat{\mathbf{u}} \times \hat{\mathbf{v}}| \tag{12-39b}$$

$$\hat{\mathbf{s}} = \hat{\mathbf{q}} \times \hat{\mathbf{r}} \tag{12-39c}$$

provided that $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are not parallel, i.e.,

$$|\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}| < 1 \tag{12-40}$$

At a given time, two measured vectors in the spacecraft body coordinates (denoted by the subscript B) $\hat{\mathbf{u}}_B$ and $\hat{\mathbf{v}}_B$, determine the body matrix, M_B:

$$M_B = \left[\hat{\mathbf{q}}_B \,\vdots\, \hat{\mathbf{r}}_B \,\vdots\, \hat{\mathbf{s}}_B\right] \tag{12-41}$$

For example, the measured vectors may be the Sun position from two-axis Sun sensor data, an identified star position from a star tracker, the nadir vector from an infrared horizon scanner, or the Earth's magnetic field vector from a magnetometer. These vectors may also be obtained in an appropriate reference frame (denoted by the subscript R) from an ephemeris, a star catalog, and a magnetic field model. The reference matrix, M_R, is constructed from $\hat{\mathbf{u}}_R$ and $\hat{\mathbf{v}}_R$ by

$$M_R = \left[\hat{\mathbf{q}}_R \,\vdots\, \hat{\mathbf{r}}_R \,\vdots\, \hat{\mathbf{s}}_R\right] \tag{12-42}$$

As defined in Section 12.1, the *attitude matrix*, or *direction cosine matrix*, A, is given by the coordinate transformation,

$$AM_R = M_B \tag{12-43}$$

because it carries the column vectors of M_R into the column vectors of M_B. This equation may be solved for A to give

$$A = M_B M_R^{-1} \tag{12-44}$$

Because M_R is orthogonal, $M_R^{-1} = M_R^{\mathrm{T}}$ and, hence (see Appendix C),

$$A = M_B M_R^{\mathrm{T}} \tag{12-45}$$

Nothing in the development thus far has limited the choice of the reference frame or the form of the attitude matrix. The only requirement is that M_R possess an inverse, which follows because the vectors $\hat{\mathbf{q}}$, $\hat{\mathbf{r}}$, and $\hat{\mathbf{s}}$ are linearly independent provided that Eq. (12-40) holds. The simplicity of Eq. (12-45) makes it particularly attractive for onboard processing. Note that inverse trigonometric functions are not required; a unique, unambiguous attitude is obtained; and computational requirements are minimal.

The preferential treatment of the vector $\hat{\mathbf{u}}$ over $\hat{\mathbf{v}}$ in Eq. (12-39) suggests that $\hat{\mathbf{u}}$ should be the more accurate measurement;* this ensures that the attitude matrix transforms $\hat{\mathbf{u}}$ from the reference frame to the body frame exactly and $\hat{\mathbf{v}}$ is used only to determine the phase angle about $\hat{\mathbf{u}}$. The four measured angles that are required to specify the two basis vectors are used to compute the attitude matrix which is parameterized by only three independent angles. Thus, some information is implicitly discarded by the algebraic method. The discarded quantity is the measured component of $\hat{\mathbf{v}}$ parallel to $\hat{\mathbf{u}}$, i.e., $\hat{\mathbf{u}}_B \cdot \hat{\mathbf{v}}_B$. This measurement is coordinate independent, equals the known scalar $\hat{\mathbf{u}}_R \cdot \hat{\mathbf{v}}_R$, and is therefore useful for data validation as described in Section 9.3. All of the error in $\hat{\mathbf{u}}_B \cdot \hat{\mathbf{v}}_B$ is assigned to the less accurate measurement $\hat{\mathbf{v}}_B$, which accounts for the lost information.

Three reference coordinate systems are commonly used: celestial, ecliptic, and orbital (see Section 3.2). The *celestial reference system*, M_C, is particularly convenient because it is obtained directly from standard ephemeris and magnetic field model subroutines such as EPHEMX and MAGFLD in Section 20.3. An *ecliptic reference system*, M_E, defined by the Earth-to-Sun vector, $\hat{\mathbf{S}}$, and the ecliptic north pole, $\hat{\mathbf{P}}_E$, is obtained by the transformation

$$M_E = \left[\hat{\mathbf{S}} \,\vdots\, \hat{\mathbf{P}}_E \times \hat{\mathbf{S}} \,\vdots\, \hat{\mathbf{P}}_E\right]^{\mathrm{T}} M_C \tag{12-46}$$

where $\hat{\mathbf{S}}$ and $\hat{\mathbf{P}}_E$ are in celestial coordinates,

$$\hat{\mathbf{P}}_E \approx (0, -\sin\epsilon, \cos\epsilon)^{\mathrm{T}} \tag{12-47}$$

and $\epsilon \approx 23.44$ deg is the obliquity of the ecliptic.

An *orbital reference system*, M_O, is defined by the nadir vector, $\hat{\mathbf{E}}$, and the negative orbit normal, $-\hat{\mathbf{n}}$, in celestial coordinates,

$$M_O = \left[-\hat{\mathbf{n}} \times \hat{\mathbf{E}} \,\vdots\, -\hat{\mathbf{n}} \,\vdots\, \hat{\mathbf{E}}\right]^{\mathrm{T}} M_C \tag{12-48}$$

*If both measurements are of comparable accuracy, basis vectors constructed from $\hat{\mathbf{u}} + \hat{\mathbf{v}}$ and $\hat{\mathbf{u}} - \hat{\mathbf{v}}$ would provide the advantage of symmetry.

Any convenient representation may be used to parameterize the attitude matrix. Quaternions and various Euler angle sequences are commonly used as described in the previous section.

The construction of vector measurements from sensor data is generally straightforward, particularly for magnetometers (Section 7.5), Sun sensors (Section 7.1), and star sensors (Section 7.6). For Earth-oriented spacecraft using horizon scanners, the nadir vector may be derived from the measured quantities by reference to the orbital coordinate system defined in Fig. 12-7. The Z_O axis is along the nadir vector and the Y_O axis is along the negative orbit normal. The scanner measures both (1) the pitch angle, Ω_E, about the scanner axis (the spacecraft Y axis, $\hat{\mathbf{Y}}_B$) from the spacecraft Z axis, $\mathbf{Z}_B$, to the $Y_B Z_O$ plane, and (2) β_E, the angle from the scanner axis to the nadir minus 90 deg.*

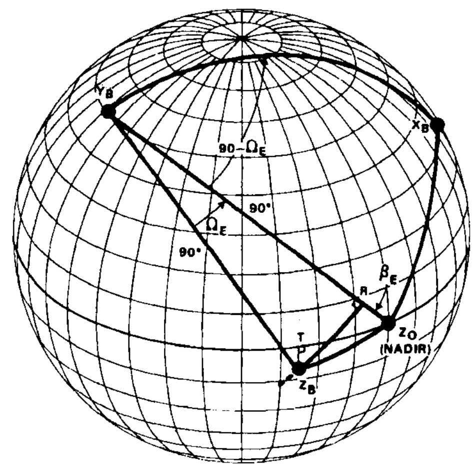

Fig. 12-7. Three-Axis Attitude From IR Scanner Plus Sun Sensor Data

Solving the quadrantal spherical triangles, $X_B Y_B Z_O$ and $Y_B Z_B Z_O$, gives

$$\hat{\mathbf{Z}}_B \cdot \hat{\mathbf{Z}}_O = \cos\Omega_E \cos\beta_E \tag{12-49}$$

$$\hat{\mathbf{X}}_B \cdot \hat{\mathbf{Z}}_O = \sin\Omega_E \cos\beta_E \tag{12-50}$$

Hence, the nadir vector in body coordinates is

$$\hat{\mathbf{E}}_B = (\sin\Omega_E \cos\beta_E, -\sin\beta_E, \cos\Omega_E \cos\beta_E)^{\mathrm{T}} \tag{12-51}$$

12.2.3 *q* Method

A major disadvantage of the attitude determination methods described thus far is that they are basically ad hoc. That is, the measurements are combined to provide an attitude estimate but the combination is not optimal in any statistical

*The angles Ω_E and β_E are analogous to pitch and roll, respectively, as they are defined in Chapter 2. Because standard definitions of pitch, roll, and yaw do not exist, the sign of the quantities here may differ from that used on some spacecraft. (See Section 2.2.)

sense. Furthermore, the methods are not easily applied to star trackers or combinations of sensors which provide many simultaneous vector measurements. Given a set of $n \geqslant 2$ vector measurements, $\hat{\mathbf{u}}_B^i$, in the body system, one choice for an optimal attitude matrix, A, is that which minimizes the loss function

$$J(A) = \sum_{i=1}^{n} w_i |\hat{\mathbf{u}}_B^i - A\hat{\mathbf{u}}_R^i|^2 \tag{12-52}$$

where w_i is the weight of the ith vector measurement and $\hat{\mathbf{u}}_R^i$ is the vector in the reference coordinate system. The loss function is the weighted sum squared of the difference between the measured and transformed vectors.

The attitude matrix may be computed by an elegant algorithm derived by Davenport [1968] and based in part on earlier work by Wahba [1965] and Stuelpnagel [1966]. This algorithm was used for the HEAO-1 attitude determination system [Keat, 1977].

The loss function may be rewritten as

$$J(A) = -2\sum_{i=1}^{n} \mathbf{W}_i A \mathbf{V}_i + \text{constant terms} \tag{12-53}$$

where the unnormalized vectors $\mathbf{W}_i$ and $\mathbf{V}_i$ are defined as

$$\mathbf{W}_i = \sqrt{w_i}\,\hat{\mathbf{u}}_B^i; \qquad \mathbf{V}_i = \sqrt{w_i}\,\hat{\mathbf{u}}_R^i \tag{12-54}$$

The loss function $J(A)$ is clearly a minimum when

$$J'(A) = \sum_{i=1}^{n} \mathbf{W}_i A \mathbf{V}_i \equiv \operatorname{tr}(W^{\mathrm{T}} A V) \tag{12-55}$$

is a maximum, where the $(3 \times \mathrm{n})$ matrices W and V are defined by

$$W \equiv \left[\mathbf{W}_1 \,\vdots\, \mathbf{W}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{W}_n\right]$$
$$V \equiv \left[\mathbf{V}_1 \,\vdots\, \mathbf{V}_2 \,\vdots\, \cdots \,\vdots\, \mathbf{V}_n\right] \tag{12-56}$$

To find the attitude matrix, A, which maximizes Eq. (12-55), we parameterize A in terms of the quaternion, $\boldsymbol{q}$, Eq. (12-13b),

$$A(\boldsymbol{q}) = (q_4^2 - \mathbf{q}\cdot\mathbf{q})\mathbf{1} + 2\mathbf{q}\mathbf{q}^{\mathrm{T}} - 2q_4 Q \tag{12-57}$$

where the quaternion has been written in terms of its vector and scalar parts,

$$\boldsymbol{q} = \begin{pmatrix} \mathbf{q} \\ q_4 \end{pmatrix} \tag{12-58}$$

$\mathbf{1}$ is the (3×3) identity matrix, $\mathbf{q}\mathbf{q}^{\mathrm{T}}$ is the (3×3) matrix outer product formed from the vector part of $\boldsymbol{q}$, and Q is the skew-symmetric matrix

$$Q = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{12-59}$$

Substitution of Eq. (12-57) into (12-55) and considerable matrix algebra [Keat, 1977] yields the following convenient form for the modified loss function:

$$J'(q) = q^{T}Kq \tag{12-60}$$

where the (4×4) matrix K is

$$K = \begin{pmatrix} S - \mathbf{1}\sigma & \mathbf{Z} \\ \mathbf{Z}^{T} & \sigma \end{pmatrix} \tag{12-61}$$

and the intermediate (3×3) matrices B and S, the vector $\mathbf{Z}$, and the scalar σ are given by

$$B \equiv WV^{T} \tag{12-62a}$$

$$S \equiv B^{T} + B \tag{12-62b}$$

$$\mathbf{Z} \equiv (B_{23} - B_{32}, B_{31} - B_{13}, B_{12} - B_{21})^{T} \tag{12-62c}$$

$$\sigma \equiv \mathrm{tr}(B) \tag{12-62d}$$

The extrema of J', subject to the normalization constraint $q^{T}q = 1$, can be found by the method of Lagrange multipliers [Hildebrand, 1964]. We define a new function

$$g(q) = q^{T}Kq - \lambda q^{T}q \tag{12-63}$$

where λ is the Lagrange multiplier, $g(q)$ is maximized without constraint, and λ is chosen to satisfy the normalization constraint. Differentiating Eq. (12-63) with respect to q^{T} and setting the result equal to zero, we obtain the eigenvector equation (see Appendix C)

$$Kq = \lambda q \tag{12-64}$$

Thus, the quaternion which parameterizes the optimal attitude matrix, in the sense of Eq. (12-52), is an eigenvector of K. Substitution of Eq. (12-64) into (12-60) gives

$$J'(q) = q^{T}Kq = q^{T}\lambda q = \lambda \tag{12-65}$$

Hence, J' is a maximum if the eigenvector corresponding to the largest eigenvalue is chosen. It can be shown that if at least two of the vectors $\mathbf{W}_i$ are not collinear, the eigenvalues of K are distinct [Keat, 1977] and therefore this procedure yields an unambiguous quaternion or, equivalently, three-axis attitude. Any convenient means, e.g., use of the subroutine EIGRS [IMSL, 1975], may be used to find the eigenvectors of K.

A major disadvantage of the method is that it requires constructing vector measurements, which is not always possible, and weighting the entire vector. Alternative, optimal methods which avoid these disadvantages are described in Chapter 13. Variations on the q-method which avoid the necessity for computing eigenvectors are described by Shuster [1978a, 1978b].

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